RESIDUE FORMULAE FOR VOLUMES AND NUMBER OF INTEGRAL POINTS OF CONVEX RATIONAL POLYTOPES

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Abstract

In this talk, we present results on vector partition functions. We explain formulae for volumes of convex polytopes, and for Ehrhart polynomials, based on the notion of total residue. Formulae for Ehrhart polynomials are shown to match on a neighborhood of each wall.

Résumé

Dans cet exposé, nous exposons des résultats sur la fonction de partition. Nous donnons des formules pour les volumes et les polynômes d'Ehrhart de polytopes convexes rationnels, basées sur la notion de résidu total. Les formules pour les polynômes d'Ehrhart se recollent au voisinage des murs.

1. Introduction

Let E be a r-dimensional real vector space. Let Δ^+ be a set of non-zero linear forms on E all lying in an open half space. Associated to Δ^+, there is a decomposition of E in chambers, generated by intersections of simplicial cones with edges in Δ^+. Let $\Phi = \{\alpha^1, \alpha^2, ...\alpha^N\}$ be a sequence of elements of Δ^+ (we do not assume the α^k to be distinct). Let $a \in E$ be an element in the cone $C(\Delta^+)$ generated by Δ^+. We denote by $P_\Phi(a) \subset \mathbb{R}_+^N$ the convex polytope consisting of all solutions $(x_1, x_2, ..., x_N)$, in non negative real numbers x_k, of the equation

$$\sum_{k=1}^{N} x_k \alpha^k = a$$

and by $volP_\Phi(a)$ its volume. We first give residue formulae for the volume of $P_\Phi(a)$. These formulae are polynomials on each of the chambers.

Assume that the set Δ^+ span a lattice Γ. Then, for $a \in \Gamma$, the polytope $P_\Phi(a)$ has rational vertices. Let $K_\Phi(a)$ be the number of integral points in $P_\Phi(a)$. The generating function for $K_\Phi(a)$ is the meromorphic function $1/\prod_{k=1}^{N}(1 - e^{\alpha_k})$. Due to a multidimensional residue theorem, quasi-polynomial formulae for $K_\Phi(a)$ are given on each chamber. These different formulae are shown to match on a neighborhood (depending of Φ) of the walls.

References

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