THE TUTTE POLYNOMIAL OF A HYPERPLANE ARRANGEMENT (EXTENDED ABSTRACT)

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Abstract. We define the Tutte polynomial of a hyperplane arrangement and study its properties. We show that it is the universal Tutte-Grothendieck invariant for the class of hyperplane arrangements. We also show that its coefficients are nonnegative. We introduce a new finite field method for computing Tutte polynomials, which generalizes several known results. We apply our method to several specific arrangements, thus relating the computation of Tutte polynomials to some problems in classical enumerative combinatorics. As a consequence, we obtain new formulas for the generating functions enumerating alternating trees, labeled trees, semiorders and Dyck paths.

Résumé. Nous définissons le polynôme de Tutte d'un arrangement d'hyperplans et étudions ses propriétés. Nous montrons que ce polynôme est l'invariant universel de Tutte-Grothendieck pour la classe des arrangements d'hyperplans, et aussi que ses coefficients sont non-négatifs. Une nouvelle méthode utilisant des corps finis est introduite, qui étend plusieurs résultats connus. Nous l'appliquons à plusieurs arrangements classiques, ce qui relie le calcul des polynômes de Tutte à divers problèmes classiques de combinatoire énumérative. Nous obtenons ainsi de nouvelles formules pour les séries génératrices dénombrant les arbres alternants, les arbres étiquetés, les semi-ordres et les chemins de Dyck.

1. Introduction

The aim of this paper is to define the Tutte polynomial of a hyperplane arrangement, and find out what we can say about it. Central arrangements inherit Tutte polynomial properties from their associated matroids; we want to know whether such properties hold for affine arrangements as well.

In Section 2 we introduce the basic notions that we will need in the paper. In Section 3 we define the Tutte polynomial of a hyperplane arrangement, and show that it is the universal Tutte-Grothendieck invariant on the class of hyperplane arrangements. Our first main result is that the coefficients of the Tutte polynomial of a hyperplane arrangement are nonnegative. In Section 4 we obtain our second main result, a finite field method for computing Tutte polynomials of hyperplane arrangements. This is done in terms of the coboundary polynomial, a simple transformation of the Tutte polynomial. We derive some consequences of this method. Finally, in Section 5, we compute the Tutte polynomials of several families of arrangements. In particular, for deformations of the braid arrangement, we relate the computation of Tutte polynomials to some enumeration problems in classical combinatorics. As a consequence, we obtain new formulas for the generating functions enumerating alternating trees, labeled trees, semiorders and Dyck paths.

2. Hyperplane arrangements

Given a field $\mathbb{k}$ and a positive integer n, an *affine hyperplane* in $\mathbb{k}^n$ is an $(n-1)$-dimensional affine subspace of $\mathbb{k}^n$. If we put a system of coordinates $x_1, \ldots, x_n$ on $\mathbb{k}^n$, a hyperplane can be seen as the set of points that satisfy a certain equation $c_1x_1 + \cdots + c_nx_n = c$, where $c_1, \ldots, c_n, c$ are constants in $\mathbb{k}$ with not all c_i's equal to 0. A *hyperplane arrangement* $\mathcal{A}$ in $\mathbb{k}^n$ is a finite collection of affine hyperplanes of $\mathbb{k}^n$. We will refer to hyperplane arrangements simply as *arrangements*. We will always assume that $\Bbbk = \mathbb{R}$ unless explicitly stated, although most of our results extend immediately to any field of characteristic zero.

We will say that an arrangement $\mathcal{A}$ is *central* if the hyperplanes in $\mathcal{A}$ have a nonempty intersection.[1] Similarly, we will say that a subset (or *subarrangement*) $\mathcal{B} \subseteq \mathcal{A}$ of hyperplanes is *central* if the hyperplanes in $\mathcal{B}$ have a nonempty intersection. The *rank function* $r_{\mathcal{A}}$ is defined for each central subset $\mathcal{B}$ by the equation $r_{\mathcal{A}}(\mathcal{B}) = n - \dim \cap \mathcal{B}$. The rank of a noncentral subset $\mathcal{B}$ is defined to be the largest rank of a central subset of $\mathcal{B}$. The *rank* of $\mathcal{A}$ is $r_{\mathcal{A}}(\mathcal{A})$, and it is denoted $r_{\mathcal{A}}$. We will usually omit the subscripts when the underlying arrangement is clear, and simply write $r(\mathcal{B})$ and r respectively.

To each hyperplane arrangement $\mathcal{A}$ we assign a partially ordered set called the *intersection poset* of $\mathcal{A}$, and denoted $L_{\mathcal{A}}$. It consists of the nonempty intersections $H_{i_1} \cap \cdots \cap H_{i_k}$, ordered by reverse inclusion. This poset is graded, with rank function $r(H_{i_1} \cap \cdots \cap H_{i_k}) = r_{\mathcal{A}}(\{H_{i_1}, \ldots, H_{i_k}\})$, and a unique minimal element $\hat{0} = \mathbb{R}^n$. We say that two hyperplane arrangements are *isomorphic* if their intersection posets are isomorphic.

We define the *characteristic polynomial* of $\mathcal{A}$ to be

$$\chi_{\mathcal{A}}(q) = \sum_{x \in L_{\mathcal{A}}} \mu(\hat{0}, x) q^{n - r(x)}.$$

where μ denotes the Möbius function [17, Section 3.7] of $L_{\mathcal{A}}$. The following theorem will be important for us in Section 5.

Theorem 2.1. *(Zaslavsky, [25]) Let $\mathcal{A}$ be a hyperplane arrangement in $\mathbb{R}^n$. The number of regions into which $\mathcal{A}$ dissects $\mathbb{R}^n$ is equal to $(-1)^n \chi_{\mathcal{A}}(-1)$.*

Even though it is not $(n-1)$-dimensional, we need to allow $\mathbb{R}^n$ as a possible "hyperplane" in $\mathcal{A}$. We will also allow $\mathcal{A}$ to contain repeated hyperplanes. We will say that a central subset $\mathcal{B}$ is *independent* if $r(\mathcal{B}) = |\mathcal{B}|$, and *dependent* otherwise. It is a *base* of $\mathcal{A}$ if $r(\mathcal{B}) = |\mathcal{B}| = r(\mathcal{A})$. A hyperplane $H \in \mathcal{A}$ is called a *loop* if it is $\mathbb{R}^n$, and an *isthmus* if $r(\mathcal{A} - H) = r(\mathcal{A}) - 1$. Here we are slightly abusing notation, writing $\mathcal{A} - H$ instead of $\mathcal{A} - \{H\}$. We will often do this for simplicity.

3. The Tutte polynomial

The *Tutte polynomial* of a hyperplane arrangement $\mathcal{A}$ is defined by

$$T_{\mathcal{A}}(q, t) = \sum_{\substack{\mathcal{B} \subseteq \mathcal{A} \\ \text{central}}} (q-1)^{r - r(\mathcal{B})} (t-1)^{|\mathcal{B}| - r(\mathcal{B})}, \tag{3.1}$$

where the sum is over all central subsets $\mathcal{B} \subseteq \mathcal{A}$.

The characteristic polynomial is essentially an evaluation of the Tutte polynomial: we have $\chi_{\mathcal{A}}(q) = (-1)^r q^{n-r} T_{\mathcal{A}}(1-q, 0)$. This is a consequence of the following result.

Theorem 3.1. *(Whitney [24], Postnikov and Stanley [13]) For any hyperplane arrangement $\mathcal{A}$,*

$$\chi_{\mathcal{A}}(q) = \sum_{\substack{\mathcal{B} \subseteq \mathcal{A} \\ \text{central}}} (-1)^{|\mathcal{B}|} q^{n - r(\mathcal{B})}.$$

It is worth remarking that the same procedure can be used to define the Tutte polynomial of an arbitrary subspace arrangement, where the subspaces do not have to be hyperplanes. We can still define the rank function of $\mathcal{A}$ by $r_{\mathcal{A}}(\mathcal{B}) = n - \dim(\cap \mathcal{B})$ for each subarrangement $\mathcal{B} \subseteq \mathcal{A}$, and then use equation (3.1) to define $T_{\mathcal{A}}(q, t)$. In this paper we shall focus on hyperplane arrangements. However, we briefly mention that our second main result,

[1] Sometimes we will call an arrangement *affine* to emphasize that it does not need to be central.

Theorem 4.1 also holds for subspace arrangements; our proof extends to this case without difficulty. The other main result of this paper, Theorem 3.5, does not hold. The Tutte polynomial of a subspace arrangement does not necessarily have nonnegative coefficients.

3.1. **Deletion-contraction.** Let $\mathcal{A}$ be an arrangement and let H be a hyperplane in $\mathcal{A}$. The arrangement $\mathcal{A} - H$ is called the *deletion* of H in $\mathcal{A}$. It is an arrangement in $\mathbb{R}^n$. The arrangement $\mathcal{A}^H = \{H_0 \cap H \mid H_0 \in \mathcal{A} - H, H_0 \cap H \neq \emptyset\}$ is called the *contraction* (or *restriction*) of $\mathcal{A}$ to H. It is an arrangement in H.

A function f on the class of arrangements is called an *invariant* if $f(\mathcal{A}_1) = f(\mathcal{A}_2)$ whenever $\mathcal{A}_1$ and A_2 are isomorphic arrangements. There is a very important type of invariant, known as a *Tutte-Grothendieck invariant*. The Tutte polynomial is the universal *Tutte-Grothendieck invariant* on the class of arrangements. The following theorem shows that any other *generalized Tutte-Grothendieck invariant*, that is, an invariant satisfying the conditions of Theorem 3.2, is an evaluation of the Tutte polynomial. Analogous results are essentially known for matroids [4], [12], and for framed configurations [23]. Framed configurations include hyperplane arrangements as particular cases, but their generality does not allow for a statement as explicit as this one.

Theorem 3.2. *Let $\mathbb{A}$ be the set of isomorphism classes of arrangements in real vector spaces. Let $\Bbbk$ be a field and let $a, b \in \Bbbk$. Let R be a commutative ring containing $\Bbbk$, and let $f : \mathbb{A} \to R$ be a function satisfying the following conditions.*

(i) *If a hyperplane H in an arrangement $\mathcal{A}$ is neither an isthmus nor a loop, then $f(\mathcal{A}) = af(\mathcal{A} - H) + bf(\mathcal{A}^H)$.*
(ii) *If H is an isthmus in $\mathcal{A}$, then $f(\mathcal{A}) = f(I)f(\mathcal{A} - H)$.*
(iii) *If H is a loop in $\mathcal{A}$, then $f(\mathcal{A}) = f(L)f(\mathcal{A} - H)$.*

Then the function f is given by $f(\mathcal{A}) = a^{|\mathcal{A}|-r(\mathcal{A})}b^{r(\mathcal{A})}T_{\mathcal{A}}(f(I)/b, f(L)/a)$. Here I denotes the arrangement consisting of a single isthmus, and L denotes the arrangement consisting of a single loop.

3.2. **Base activity.** We now show that the Tutte polynomial of a hyperplane arrangement has nonnegative coefficients, by giving a combinatorial interpretation of them. Crapo interpreted the coefficients of the Tutte polynomial of a matroid as enumerators of bases with a given internal and external activity [6]. Our interpretation for hyperplane arrangements is analogous.

Define a *circuit* of an arrangement to be a minimal set of hyperplanes which is central and dependent. Define a *bond* to be a minimal set of hyperplanes, the removal of which makes the rank of the arrangement decrease.

Lemma 3.3. *Let B be a base of $\mathcal{A}$, and let e be a hyperplane not in B such that $B \cup e$ is central. Then $B \cup e$ contains a unique circuit.*

Lemma 3.4. *Let B be a base of $\mathcal{A}$, and let i be a hyperplane in B. Then $\mathcal{A} - B \cup i$ contains a unique bond.*

From now on, we will fix a linear order on $\mathcal{A}$. Now each k-subset of $\mathcal{A}$ corresponds to a strictly increasing sequence of k integers between 1 and $|\mathcal{A}|$. For each $0 \leq k \leq |\mathcal{A}|$, order the k-subsets of $\mathcal{A}$ using the lexicographic order on these sequences.

Given a base B, we will say that a hyperplane e not in B is an *external activity hyperplane for B* if $B \cup e$ is central, and e is the smallest hyperplane[2] of the unique circuit in $B \cup e$. Let $E(B)$ be the set of external activity hyperplanes for B, and let $e(B) = |E(B)|$. We call $e(B)$ the *external activity* of B.

[2]according to the fixed linear order

We will say that a hyperplane i in B is an *internal activity hyperplane for* B if i is the smallest hyperplane of the unique bond in $\mathcal{A} - B \cup i$. Let $I(B)$ be the set of internal activity hyperplanes for B, and let $i(B) = |I(B)|$. We call $i(B)$ the *internal activity* of B.

Now we are in a position to state the main theorem of this section.

Theorem 3.5. *For any arrangement* $\mathcal{A}$*,*

$$T_{\mathcal{A}}(q,t) = \sum_{\substack{B \text{ base} \\ \text{of } \mathcal{A}}} q^{i(B)} t^{e(B)}.$$

Theorem 3.5 shows that the coefficients of the Tutte polynomial are nonnegative integers. The coefficient of $q^i t^e$ is equal to the number of bases of $\mathcal{A}$ with internal activity equal to i and external activity equal to e.

A useful ingredient in the proof of Theorem 3.5 is the following characterization of internal and external activity hyperplanes. Given a subarrangement $\mathcal{B} \subseteq \mathcal{A}$ and a hyperplane H, let $\mathcal{B}_{>H} = \{H_0 \in \mathcal{B} \,|\, H_0 > H\}$. Define $\mathcal{B}_{<H}$ analogously.

Lemma 3.6. *Let B be a base and e be a hyperplane not in B such that $B \cup e$ is central. Then e is an external activity hyperplane for B if and only if $r(\mathcal{B}_{>e} \cup e) = r(\mathcal{B}_{>e})$.*

Lemma 3.7. *Let B be a base and i be a hyperplane in B. Then i is an internal activity hyperplane for B if and only if $r(B - i \cup \mathcal{A}_{<i}) < r$.*[3]

Now we wish to present a different description of the central subarrangements of $\mathcal{A}$. To do it, we need two definitions. For each subarrangement $\mathcal{B}$, let $d\mathcal{B}$ be the lexicographically largest base of $\mathcal{B}$. For each independent central subarrangement $\mathcal{B}$, let $u\mathcal{B}$ be the lexicographically smallest base of $\mathcal{A}$ which contains $\mathcal{B}$. Notice that, for any subarrangement $\mathcal{B}$, $ud\mathcal{B}$ is a base of $\mathcal{A}$.

Let S_1 be the set of triples (B, I, E) such that B is a base of $\mathcal{A}$, $I \subseteq I(B)$ is a set of internal activity hyperplanes for B, and $E \subseteq E(B)$ is a set of external activity hyperplanes for $\mathcal{B}$. Let S_2 be the set of central subarrangements of $\mathcal{A}$. We want to establish a bijection between S_1 and S_2. Define two maps ϕ_1 and ϕ_2 as follows. Given $(B, I, E) \in S_1$, let $\phi_1(B, I, E) = B - I \cup E$. Given $\mathcal{C} \in S_2$, let $\phi_2(\mathcal{C}) = (ud\mathcal{C}, ud\mathcal{C} - d\mathcal{C}, \mathcal{C} - d\mathcal{C})$. The maps ϕ_1 and ϕ_2 give the desired bijection: every central subarrangement $\mathcal{B}$ of $\mathcal{A}$ can be written uniquely in the form $\mathcal{B} = B - I \cup E$ where B is a base of $\mathcal{A}$, $I \subseteq I(B)$ and $E \subseteq E(B)$.

Lemma 3.8. *The map ϕ_1 maps S_1 to S_2.*

Lemma 3.9. *The map ϕ_2 maps S_2 to S_1.*

Proposition 3.10. *The map ϕ_1 is a bijection from S_1 to S_2, and the map ϕ_2 is its inverse.*

Proposition 3.10 is the key ingredient of our proof of Theorem 3.5. It is a consequence of the following lemmas.

Lemma 3.11. *For all $(B, I, E) \in S_1$, we have $r(B - I \cup E) = r - |I|$.*

Lemma 3.12. *For all $(B, I, E) \in S_1$, we have $d(B - I \cup E) = B - I$.*

Lemma 3.13. *For all $(B, I, E) \in S_1$, we have $ud(B - I \cup E) = B$.*

Proving these lemmas requires some work; once we have done that, Theorem 3.5 follows easily from Proposition 3.10 and Lemma 3.11.

[3] In fact, this is true if and only if $r(B - i \cup \mathcal{A}_{<i}) = r - 1$.

4. A finite field method

In [14], Reiner asked whether it is possible to define the Tutte polynomial of a subspace arrangement. We have shown how to do this in the introduction to Section 3. Reiner also asked whether it is possible to use [14, Corollary 3] to compute explicitly the Tutte polynomials of some nontrivial families of arrangements. Compared to all the work that has been done on computing characteristic polynomials explicitly, very little is known about computing Tutte polynomials.

In this section, we introduce a new method for computing Tutte polynomials of hyperplane arrangements. This method also works for arbitrary subspace arrangements. Our approach does not use Reiner's result; it is closer to Athanasiadis's finite field method for computing characteristic polynomials. In fact, Athanasiadis's result [1, Theorem 2.2] can be obtained as a special case of the main result of this section, Theorem 4.1, by setting $t = 0$.

Using Crapo's terminology [6], define the *coboundary polynomial* $\overline{\chi}_{\mathcal{A}}(q,t)$ of an arrangement $\mathcal{A}$ by

$$\overline{\chi}_{\mathcal{A}}(q,t) = \sum_{\substack{\mathcal{B} \subseteq \mathcal{A} \\ \text{central}}} q^{r-r(\mathcal{B})}(t-1)^{|\mathcal{B}|}. \tag{4.1}$$

It is easy to see that the coboundary and Tutte polynomials are simple transformations of each other, and computing the coboundary polynomial of an arrangement is essentially equivalent to computing its Tutte polynomial. Our results can be presented more elegantly in terms of the coboundary polynomial.

Let $\mathcal{A}$ be a $\mathbb{Z}$-arrangement in $\mathbb{R}^n$; that is, an arrangement where the defining equations have integer coefficients. Let q be a prime power. The arrangement $\mathcal{A}$ induces an arrangement $\mathcal{A}_q$ in the vector space $\mathbb{F}_q^n$. If we consider the equations defining the hyperplanes of $\mathcal{A}$ and regard them as equations over $\mathbb{F}_q$, they define the hyperplanes of $\mathcal{A}_q$.

Theorem 4.1. *Let $\mathcal{A}$ be a $\mathbb{Z}-$arrangement in $\mathbb{R}^n$. Let q be a power of a large enough prime, and let $\mathcal{A}_q$ be the induced arrangement in $\mathbb{F}_q^n$. Then*

$$q^{n-r}\,\overline{\chi}(q,t) \;=\; \sum_{p \in \mathbb{F}_q^n} t^{h(p)}, \tag{4.2}$$

where $h(p)$ denotes the number of hyperplanes of $\mathcal{A}_q$ that p lies on.

We remark that Theorem 4.1 was also discovered independently by Welsh and Whittle [23, Theorem 7.4].

Theorem 4.1 generalizes two classical enumerative results. The first result concerns vertex colorings of graphs. Given a graph G on $[n]$, we can associate to it an arrangement $\mathcal{A}_G$ in $\mathbb{R}^n$. It consists of the hyperplanes $x_i = x_j$, for all $1 \leq i < j \leq n$ such that ij is an edge in the graph G. When we apply Theorem 4.1 to this arrangement, we obtain the following result.

Theorem 4.2. ([5, Proposition 6.3.26]) *Let G be a graph with n vertices and c connected components. Then*

$$q^c\,\overline{\chi}_{\mathcal{A}_G}(q,t) = \sum_{\substack{q-\text{colorings} \\ \kappa \text{ of } G}} t^{\text{mono}(\kappa)},$$

where $\text{mono}(\kappa)$ *is the number of monochromatic edges in κ.*

The second result concerns linear codes. An $[n,r]$ *linear code* C over $\mathbb{F}_q$ is an r-dimensional subspace of $\mathbb{F}_q^n$. A *generator matrix* for C is an $r \times n$ matrix U over $\mathbb{F}_q$, the rows of which form a basis for C. It is not difficult to see that the isomorphism class of the matroid on the columns of U depends only on C. We shall denote the corresponding matroid M_C.

The elements of C are called *codewords*. The *weight* $w(v)$ of a codeword is the cardinality of its support; that is, the number of nonzero coordinates of v.

The translation of Theorem 4.1 to this setting is the following.

Theorem 4.3. *(Greene, [7]) For any linear code C over $\mathbb{F}_q$,*

$$\sum_{v \in C} t^{w(v)} = t^n \, \overline{\chi}_{M_C}\left(q, \frac{1}{t}\right).$$

We conclude this section by presenting two new results, which are also relatively simple consequences of Theorem 4.1.

Theorem 4.4. *Let $\mathcal{A}$ be an arrangement and let $0 \leq t \leq 1$ be a real number. Let $\mathcal{B}$ be a random subarrangement of $\mathcal{A}$, obtained by independently removing each hyperplane from $\mathcal{A}$ with probability t. Then the expected characteristic polynomial $\chi_{\mathcal{B}}(q)$ of $\mathcal{B}$ is $q^{n-r}\overline{\chi}_{\mathcal{A}}(q,t)$.*

Theorem 4.5. *For an arrangement $\mathcal{A}$ and an affine subspace x in the intersection poset $L_{\mathcal{A}}$, let $h(x)$ be the number of hyperplanes of A containing x. Then*

$$\overline{\chi}_{\mathcal{A}}(q,t) = \sum_{\substack{x \leq y \\ \text{in } L_{\mathcal{A}}}} \mu(x,y)\, q^{r-r(y)} t^{h(x)}.$$

5. Computation of Tutte polynomials

In this section we use Theorem 4.1 to compute the coboundary polynomials of several families of arrangements. As remarked in Section 4, this is essentially the same as computing their Tutte polynomials.

5.1. Coxeter arrangements.

Let Φ be an irreducible crystallographic root system in $\mathbb{R}^n$, with the standard inner product, and let W be its associated Weyl group. The Coxeter arrangement of type W consists of the hyperplanes $(\alpha, x) = 0$ for each $\alpha \in \Phi^+$. See [8] for an introduction to root systems and Weyl groups, and [11, Chapter 6] or [3, Section 2.3] for more information on Coxeter arrangements.

In this section we present the coboundary polynomials of the Coxeter arrangements of type A_n, B_n and D_n. (The arrangement of type C_n is the same as the arrangement of type B_n.) The best way to state our results is to compute the exponential generating function for the coboundary polynomials of each family.

Theorem 5.1. *Let $\mathcal{A}_n$ be the Coxeter arrangement of type A_{n-1} in $\mathbb{R}^n$, consisting of the hyperplanes $x_i = x_j$ for $1 \leq i < j \leq n$.[4] We have*

$$1 + q\sum_{n \geq 1} \overline{\chi}_{\mathcal{A}_n}(q,t)\frac{x^n}{n!} = \left(\sum_{n \geq 0} t^{\binom{n}{2}} \frac{x^n}{n!}\right)^q.$$

Theorem 5.2. *Let $\mathcal{B}_n$ be the Coxeter arrangement of type B_n in $\mathbb{R}^n$, consisting of the hyperplanes $x_i = x_j$ and $x_i + x_j = 0$ for $1 \leq i < j \leq n$, and the hyperplanes $x_i = 0$ for*

[4] This arrangement is also known as the *braid arrangement.*

$1 \leq i \leq n$. *We have*

$$\sum_{n\geq 0} \overline{\chi}_{\mathcal{B}_n}(q,t)\frac{x^n}{n!} = \left(\sum_{n\geq 0} 2^n t^{\binom{n}{2}} \frac{x^n}{n!}\right)^{\frac{q-1}{2}} \left(\sum_{n\geq 0} t^{n^2} \frac{x^n}{n!}\right).$$

Theorem 5.3. *Let $\mathcal{D}_n$ be the Coxeter arrangement of type D_n in $\mathbb{R}^n$, consisting of the hyperplanes $x_i = x_j$ and $x_i + x_j = 0$ for $1 \leq i < j \leq n$. We have*

$$\sum_{n\geq 0} \overline{\chi}_{\mathcal{D}_n}(q,t)\frac{x^n}{n!} = \left(\sum_{n\geq 0} 2^n t^{\binom{n}{2}} \frac{x^n}{n!}\right)^{\frac{q-1}{2}} \left(\sum_{n\geq 0} t^{n(n-1)} \frac{x^n}{n!}\right).$$

These results follow fairly easily from Theorem 4.1. Theorem 5.1 is equivalent to a result of Tutte [20], who computed the coboundary polynomial of the complete graph. It is also an immediate consequence of a more general result of Stanley [19, equation (15)]. Theorems 5.2 and 5.3 have never been stated explicitly in the literature, but they are implicit in the work of Zaslavsky [26].

Setting $t = 0$ in Theorems 5.1, 5.2 and 5.3, it is easy to recover the well-known formulas for the characteristic polynomials of the above arrangements:

$$\begin{aligned}
\chi_{\mathcal{A}_n}(q) &= q(q-1)(q-2)\cdots(q-n+1),\\
\chi_{\mathcal{B}_n}(q) &= (q-1)(q-3)\cdots(q-2n+1),\\
\chi_{\mathcal{D}_n}(q) &= (q-1)(q-3)\cdots(q-2n+3)(q-n+1).
\end{aligned}$$

5.2. **Two more examples.**

Theorem 5.4. *Let $\mathcal{A}_n^{\#}$ be a generic deformation of the arrangement $\mathcal{A}_n$, consisting of the hyperplanes $x_i - x_j = a_{ij}$ $(1 \leq i < j \leq n)$, where the a_{ij} are generic real numbers* [5]. *For $n \geq 1$,*

$$q\,\overline{\chi}_{\mathcal{A}_n^{\#}}(q,t) = \sum_F q^{n-e(F)}(t-1)^{e(F)}$$

where the sum is over all forests F on $[n]$, and $e(F)$ denotes the number of edges of F. Also,

$$1 + q\sum_{n\geq 1} \overline{\chi}_{\mathcal{A}_n^{\#}}(q,t)\frac{x^n}{n!} = \left(\sum_{n\geq 0} f(n)\frac{x^n(t-1)^n}{n!}\right)^{\frac{q}{t-1}},$$

where $f(n)$ is the number of forests on $[n]$.

Theorem 5.5. *The threshold arrangement $\mathcal{T}_n$ in $\mathbb{R}^n$ consists of the hyperplanes $x_i + x_j = 0$, for $1 \leq i < j \leq n$. For all $n \geq 0$ we have*

$$\overline{\chi}_{\mathcal{T}_n}(q,t) = \sum_G q^{\mathrm{bc}(G)}(t-1)^{e(G)},$$

where the sum is over all graphs G on $[n]$. Here $\mathrm{bc}(G)$ is the number of connected components of G which are bipartite, and $e(G)$ is the number of edges of G. Also,

$$\sum_{n\geq 0} \overline{\chi}_{\mathcal{T}_n}(q,t)\frac{x^n}{n!} = \left(\sum_{n\geq 0}\sum_{k=0}^{n} \binom{n}{k} t^{k(n-k)} \frac{x^n}{n!}\right)^{\frac{q-1}{2}} \left(\sum_{n\geq 0} t^{\binom{n}{2}} \frac{x^n}{n!}\right).$$

[5] The a_{ij} are "generic" if no n of the hyperplanes have a nonempty intersection, and any nonempty intersection of k hyperplanes has rank k. This can be achieved, for example, by requiring that the a_{ij}'s are linearly independent over the rational numbers. Almost all choices of a_{ij}'s are generic.

5.3. **Deformations of the braid arrangement.** A deformation of the braid arrangement is an arrangement in $\mathbb{R}^n$ consisting of the hyperplanes $x_i - x_j = a_{ij}^{(1)}, \ldots, a_{ij}^{(k_{ij})}$ for $1 \leq i < j \leq n$, where the k_{ij} are nonnegative integers, and the $a_{ij}^{(r)}$ are real numbers. Such arrangements have been studied extensively by Athanasiadis [2] and Postnikov and Stanley [13]. In this section we study their coboundary polynomials.

The most natural deformations of the braid arrangement are the following. Fix a set A of k distinct integers $a_1 < \ldots < a_k$. Let $\mathcal{E}_n$ be the arrangement in $\mathbb{R}^n$ consisting of the hyperplanes

$$x_i - x_j = a_1, \ldots, a_k, \qquad 1 \leq i < j \leq n. \tag{5.1}$$

A *graded graph* is a triple $G = (V, E, h)$, where V is a linearly ordered set of vertices (usually $V = [n]$), E is a set of (nonoriented) edges, and h_G is a function $h_G : V \to \mathbb{N}$, called a *grading*. We will drop the subscript when the underlying graded graph is clear, and we will refer to $h(v)$ as the *height* of v. The *height* of G, denoted $h(G)$, is the largest height of a vertex of G. The vertices v such that $h(v) = r$ form the r-th *level* of G, for each $r \geq 0$. If $u < v$ are connected by edge e, the *slope* of e is $s(e) = h(u) - h(v)$. A graded graph is an *A-graph* if the slopes of all edges of G are in A. A graded graph is *planted* if each connected component has a vertex on the 0-th level.

Recall that, for a graph G, we let $e(G)$ be the number of edges and $c(G)$ be the number of connected components of G. We also let $v(G)$ be the number of vertices of G.

Proposition 5.6. *Let $\mathcal{E}_n$ be the arrangement (5.1). Then, for $n \geq 1$,*

$$q\,\overline{\chi}_{\mathcal{E}_n}(q,t) = \sum_G q^{c(G)}(t-1)^{e(G)},$$

where the sum is over all planted graded A-graphs on $[n]$.

Proof. We associate to each planted graded A-graph $G = (V, E, h)$ on $[n]$ an arrangement $\mathcal{A}_G$ in $\mathbb{R}^n$. It consists of the hyperplanes $x_i - x_j = h(i) - h(j)$, for each $i < j$ such that ij is an edge in G. This is a subarrangement of $\mathcal{E}_n$ because $h(i) - h(j)$, the slope of edge ij, is in A. It is central because the point $(h(1), \ldots, h(n)) \in \mathbb{R}^n$ belongs to all these hyperplanes.

This is in fact a bijection between planted graded A-graphs on $[n]$ and central subarrangements of $\mathcal{E}_n$. To see this, take a central subarrangement $\mathcal{A}$. We will recover the planted graded A-graph G that it came from. For each pair (i, j) with $1 \leq i < j \leq n$, $\mathcal{A}$ can have at most one hyperplane of the form $x_i - x_j = a_t$. If this hyperplane is in $\mathcal{A}$, we must put edge ij in G, and demand that the heights $h(i)$ and $h(j)$ satisfy $h(i) - h(j) = a_t$. When we do this for all the hyperplanes in $\mathcal{A}$, the height requirements that we introduce are consistent, because $\mathcal{A}$ is central. However, these requirements do not fully determine the heights of the vertices; they only determine the relative heights within each connected component of G. Since we want G to be planted, we demand that the vertices with the lowest height in each connected component of G should have height 0. This does determine G completely, and clearly $\mathcal{A} = \mathcal{A}_G$.

With this bijection in hand, and keeping equation (4.1) in mind, we just need that $r(\mathcal{A}_G) = n - c(G)$ and $|\mathcal{A}_G| = e(G)$. Both of these claims are easy. $\square$

Theorem 5.7. *Let $\mathcal{E}_n$ be the arrangement (5.1), and let*

$$A_r(t,x) = \sum_{n \geq 0} \left(\sum_{f:[n]\to[r]} t^{a(f)} \right) \frac{x^n}{n!}, \tag{5.2}$$

where $a(f)$ denotes the number of pairs (i,j) with $1 \le i < j \le n$ such that $f(i) - f(j) \in A$. Then

$$1 + q\sum_{n\ge 1} \overline{\chi}_{\mathcal{E}_n}(q,t)\frac{x^n}{n!} = \left(\lim_{r\to\infty} \frac{A_r(t,x)}{A_{r-1}(t,x)}\right)^q. \tag{5.3}$$

Remark. The limit in (5.3) is a limit in the sense of convergence of formal power series. For more information on this notion of convergence, see [17, Section 1.1] or [9].

Proof of Theorem 5.7. First we prove that

$$A_r(t,x) = \sum_G (t-1)^{e(G)} \frac{x^{v(G)}}{v(G)!} \tag{5.4}$$

where the sum is over all graded A-graphs G of height less than r. The coefficient of $\frac{x^n}{n!}$ in the right-hand side of (5.4) is $\sum_G (t-1)^{e(G)}$, summing over all graded A-graphs G on $[n]$ with height less than r. We have

$$\begin{aligned}\sum_G (t-1)^{e(G)} &= \sum_{h:[n]\to[0,r-1]} \sum_{\substack{G \text{ such that}\\ h_G = h}} (t-1)^{e(G)}\\ &= \sum_{h:[n]\to[0,r-1]} (1+(t-1))^{a(h)}\\ &= \sum_{f:[n]\to[r]} t^{a(f)}\end{aligned}$$

The only tricky step here is the second: if we want all graded A-graphs G on $[n]$ with a specified grading h, we need to consider the possible choices of edges of the graph. Any edge ij can belong to the graph, as long as $h(i) - h(j) \in A$, so there are $a(h)$ possible edges.

Equation (5.4) suggests the following definitions. Let

$$B_r(t,x) = \sum_G t^{e(G)} \frac{x^{v(G)}}{v(G)!}$$

where the sum is over all *planted* graded A-graphs G of height less than r, and let

$$B(t,x) = \sum_G t^{e(G)} \frac{x^{v(G)}}{v(G)!}$$

where the sum is over all *planted* graded A-graphs G.

The equation

$$1 + q\sum_{n\ge 1} \overline{\chi}_{\mathcal{E}_n}(q,t)\frac{x^n}{n!} = B(t-1,x)^q, \tag{5.5}$$

follows from Proposition 5.6, using the compositional formula for exponential generating functions.

It is not difficult to see that $B(t-1,x) = \lim_{r\to\infty} B_r(t-1,x)$, so it suffices to show that

$$B_r(t-1,x) = A_r(t,x)/A_{r-1}(t,x) \tag{5.6}$$

or, equivalently, that $A_r(t,x) = B_r(t-1,x)A_{r-1}(t,x)$. We need to show that the ways of putting the structure of a graded A-graph G with $h(G) < r$ on $[n]$ can be put in correspondence with the ways of doing the following: first splitting $[n]$ into two disjoint sets S_1 and S_2, then putting the structure of a *planted* graded A-graph G_1 with $h(G_1) < r$ on

S_1, and then putting the structure of a graded A-graph G_2 with $h(G_2) < r-1$ on S_2. We also need that, in that correspondence, $(t-1)^{e(G)} = (t-1)^{e(G_1)}(t-1)^{e(G_2)}$.

We do this as follows. Let G be a graded A-graph G with $h(G) < r$. Let G_1 be the union of the connected components of G which contain a vertex on the 0-th level. Put a grading on G_1 by defining $h_{G_1}(v) = h_G(v)$ for $v \in G_1$. Let $G_2 = G - G_1$. Since $h_G(v) \geq 1$ for all $v \in G_2$, we can put a grading on G_2 by defining $h_{G_2}(v) = h_G(v) - 1$ for $v \in G_2$. G_1 is a planted graded A-graph with $h(G_1) < r$, and G_2 is a graded A-graph with $h(G_2) < r-1$.

It is clear that our map from G to a pair (G_1, G_2) is a one-to-one correspondence and that $(t-1)^{e(G)} = (t-1)^{e(G_1)}(t-1)^{e(G_2)}$. This completes the proof of (5.6), and Theorem 5.7 follows. □

The *Catalan arrangement* C_n in $\mathbb{R}^n$ consists of the hyperplanes

$$x_i - x_j = -1, 0, 1, \qquad 1 \leq i < j \leq n. \tag{5.7}$$

When the arrangement in Theorem 5.7 is a subarrangement of the Catalan arrangement, we can say more about the power series A_r of equation (5.2). Let

$$A(t,x,y) = \sum_r A_r(t,x)y^r = \sum_{n\geq 0}\sum_{r\geq 0}\left(\sum_{f:[n]\to[r]} t^{a(f)}\right)\frac{x^n}{n!}y^r \tag{5.8}$$

be the generating function for the power series in Theorem 5.7, and let

$$S(t,x,y) = \sum_{n\geq 0}\sum_{r\geq 0}\left(\sum_{f:[n]\to[r]} t^{a(f)}\right)\frac{x^n}{n!}y^r \tag{5.9}$$

where the inner sum is over all *surjective* functions $f : [n] \to [r]$. The following proposition reduces the computation of $A(t,x,y)$ to the computation of $S(t,x,y)$, which is easier in practice.

Proposition 5.8. *If $A \subseteq \{-1,0,1\}$ in the notation of Theorem 5.7, we have*

$$A(t,x,y) = \frac{S(t,x,y)}{1 - yS(t,x,y)}.$$

Considering the different subsets of $\{-1,0,1\}$, we get six nonisomorphic subarrangements of the Catalan arrangement. They come from the subsets $\emptyset$, $\{0\}$, $\{1\}$, $\{0,1\}$, $\{-1,1\}$ and $\{-1,0,1\}$. The corresponding subarrangements are the empty arrangement, the braid arrangement, the *Linial arrangement*, the *Shi arrangement*, the *interval arrangement* and the Catalan arrangement, respectively. The empty arrangement is trivial, and the braid arrangement was already treated in detail in Section 5.1. We now have a technique that lets us talk about the remaining four arrangements under the same framework. We will do this in the remainder of this section.

5.3.1. *The Linial arrangement.* The Linial arrangement $\mathcal{L}_n$ consists of the hyperplanes $x_i - x_j = 1$ for $1 \leq i < j \leq n$. This arrangement was first considered by Linial and Ravid. It was later studied by Athanasiadis [1] and Postnikov and Stanley [13], who independently computed its characteristic polynomial:

$$\chi_{\mathcal{L}_n}(q) = \frac{q}{2^n}\sum_{k=0}^{n}\binom{n}{k}(q-k)^{n-1}.$$

They also put the regions of $\mathcal{L}_n$ in bijection with several different sets of combinatorial objects. Perhaps the simplest such set is the set of *alternating trees* on $[n+1]$: the set of trees such that every vertex is either larger or smaller than all its neighbors.

Now we present the consequences of Proposition 5.6, Theorem 5.7 and Proposition 5.8 for the Linial arrangement. Recall that a poset P on $[n]$ is *naturally labeled* if $i < j$ in P implies $i < j$ in $\mathbb{Z}^+$.

Proposition 5.9. *For all $n \geq 1$ we have*

$$q\,\overline{\chi}_{\mathcal{L}_n}(q,t) = \sum_P q^{c(P)}(t-1)^{e(P)}$$

where the sum is over all naturally labeled, graded posets P on $[n]$. Here $c(P)$ and $e(P)$ denote the number of components and edges of the Hasse diagram of P, respectively.

Theorem 5.10. *Let*

$$\frac{1+ye^{x(1+y)}}{1-y^2e^{x(1+y)}} = \sum_{r\geq 0} A_r(x)y^r. \tag{5.10}$$

Then we have

$$\sum_{n\geq 0} \chi_{\mathcal{L}_n}(q)\frac{x^n}{n!} = \left(\lim_{r\to\infty} \frac{A_r(x)}{A_{r-1}(x)}\right)^q.$$

In particular, if f_n is the number of alternating trees on $[n+1]$,

$$\sum_{n\geq 0}(-1)^n f_n \frac{x^n}{n!} = \lim_{r\to\infty}\frac{A_{r-1}(x)}{A_r(x)}.$$

Proof. In view of Theorem 5.7 and Proposition 5.8, the exponential generating functions that we are after are determined by $S(0,x,y)$. From equation (5.9), the coefficient of $\frac{x^n}{n!}y^r$ in $S(0,x,y)$ is equal to the number of surjective functions $f : [n] \to [r]$ which never have $f(i) - f(j) = 1$ for $i < j$. These are just the nondecreasing surjective functions $f : [n] \to [r]$. For $n \geq 1$ there are $\binom{n-1}{r-1}$ such functions. For $n = 0$ and $r \geq 1$ there are no such functions, and for $n = r = 0$ there is one such function. Therefore

$$\begin{aligned} S(0,x,y) &= 1 + \sum_{n\geq 1}\sum_{r\geq 1}\binom{n-1}{r-1}\frac{x^n}{n!}y^r \\ &= 1 + \sum_{n\geq 1}\frac{x^n}{n!}y(1+y)^{n-1} \\ &= \frac{1+ye^{x(1+y)}}{1+y}. \end{aligned}$$

Proposition 5.8 then implies that

$$A(0,x,y) = \frac{1+ye^{x(1+y)}}{1-y^2e^{x(1+y)}},$$

in agreement with equation (5.10). The theorem then follows since $\chi_{\mathcal{L}_n}(q) = q\overline{\chi}_{\mathcal{L}_n}(q,0)$, and the number of regions of $\mathcal{L}_n$ is $f_n = (-1)^n\chi_{\mathcal{L}_n}(-1)$ by Theorem 2.1. □

Now we apply our machinery to the Shi arrangement, the interval arrangement and the Catalan arrangement.

The Shi arrangement $\mathcal{S}_n$ consists of the hyperplanes $x_i - x_j = 0, 1$ for $1 \leq i < j \leq n$. Its number of regions is $(n+1)^{n-1}$, the number of labeled trees on $n+1$ vertices; its characteristic polynomial is $\chi_{\mathcal{S}_n}(q) = q(q-n)^{n-1}$ [15], [16].

Theorem 5.11. *Let*

$$A_r(x) = \sum_{n=0}^{r} (r-n)^n \frac{x^n}{n!}.$$

Then we have

$$\sum_{n\geq 0} \chi_{\mathcal{S}_n}(q) \frac{x^n}{n!} = \left(\lim_{r\to\infty} \frac{A_r(x)}{A_{r-1}(x)} \right)^q.$$

In particular,

$$\sum_{n\geq 0} (-1)^n (n+1)^{n-1} \frac{x^n}{n!} = \lim_{r\to\infty} \frac{A_{r-1}(x)}{A_r(x)}.$$

A *semiorder on* $[n]$ is a poset P on $[n]$ for which there exist n unit intervals $I_1, \ldots, I_n$ of $\mathbb{R}$, such that $i < j$ in P if and only if I_i is disjoint from I_j and to the left of it. Let i_n be the number of semiorders on $[n]$.

The interval arrangement $\mathcal{I}_n$ consists of the hyperplanes $x_i - x_j = -1, 1$ for $1 \leq i < j \leq n$. Its number of regions is i_n [18],[13].

Theorem 5.12. *Let*

$$\frac{1-y+ye^x}{1-y+y^2-y^2e^x} = \sum_{r\geq 0} A_r(x) y^r.$$

Then we have

$$\sum_{n\geq 0} \chi_{\mathcal{I}_n}(q) \frac{x^n}{n!} = \left(\lim_{r\to\infty} \frac{A_r(x)}{A_{r-1}(x)} \right)^q.$$

In particular,

$$\sum_{n\geq 0} (-1)^n i_n \frac{x^n}{n!} = \lim_{r\to\infty} \frac{A_{r-1}(x)}{A_r(x)}.$$

The Catalan arrangement $\mathcal{C}_n$ consists of the hyperplanes $x_i - x_j = -1, 0, 1$ for $1 \leq i < j \leq n$. Its number of regions is $n!\, C_n$, where $C_n = \frac{1}{n+1}\binom{2n}{n}$ is the n-th Catalan number [18].

Theorem 5.13. *Let*

$$A_r(x) = \sum_{n=0}^{\lfloor \frac{r+1}{2} \rfloor} \binom{r-n+1}{n} x^n.$$

Then we have

$$\sum_{n\geq 0} \chi_{\mathcal{C}_n}(q) \frac{x^n}{n!} = \left(\lim_{r\to\infty} \frac{A_r(x)}{A_{r-1}(x)} \right)^q.$$

In particular,

$$\frac{\sqrt{1+4x}-1}{2x} = \sum_{n\geq 0} (-1)^n C_n x^n = \lim_{r\to\infty} \frac{A_{r-1}(x)}{A_r(x)}. \tag{5.11}$$

References

[1] C. A. Athanasiadis. Characteristic polynomials of subspace arrangements and finite fields, *Adv. Math.* **122** (1996), 193-233.

[2] C. A. Athanasiadis. Deformations of Coxeter hyperplane arrangements and their characteristic polynomials, in M. Falk and H. Terao (eds.), *Arrangements - Tokyo 1998*, Advanced Studies in Pure Mathematics, in press.

[3] A. Björner, M. Las Vergnas, B. Sturmfels, N. White and G. M. Ziegler. *Oriented matroids*, Encyclopedia of Mathematics and Its Applications 46, Cambridge University Press, Cambridge, 1993.

[4] T. Brylawski. A decomposition for combinatorial geometries, *Trans. Amer. Math. Soc.* **171** (1972), 235-82.

[5] T. Brylawski and J. Oxley. The Tutte polynomial and its applications, in N. White (ed.), *Matroid Applications*, Encyclopedia of Mathematics and Its Applications, 40, Cambridge Univ. Press, Cambridge, 1992, 123-225.

[6] H. H. Crapo. The Tutte polynomial, *Aequationes Math.* **3** (1969), 211-229.

[7] C. Greene. Weight enumeration and the geometry of linear codes, *Stud. Appl. Math.* **55** (1976), 119-128.

[8] J. Humphreys. *Reflection groups and Coxeter groups*, Cambridge Studies in Advanced Mathematics 29, Cambridge University Press, Cambridge, 1990.

[9] I. Niven. Formal power series, *Amer. Math. Monthly*, **76** (1969), 871-889.

[10] P. Orlik and L. Solomon. Combinatorics and topology of complements of hyperplanes, *Invent. Math.* **56** (1980), 167-189.

[11] P. Orlik and H. Terao. *Arrangements of hyperplanes*, Springer-Verlag, Berlin/Heidelberg/New York, 1992.

[12] J. G. Oxley and D. J. A. Welsh. The Tutte polynomial and percolation, in J.A. Bondy and U.S.R. Murty (eds.), *Graph theory and related topics*, Academic Press, New York, 1979, 329-339.

[13] A. Postnikov, R. Stanley. Deformations of Coxeter hyperplane arrangements, *J. Combin. Theory, Ser. A.* **91** (2000), 544-597.

[14] V. Reiner. An interpretation for the Tutte polynomial, *European J. Combin.* **20** (1999), 149-161.

[15] J.-Y. Shi. *The Kazhdan-Lusztig cells in certain affine Weyl groups*, Lecture Notes in Mathematics, no. 1179, Springer, Berlin/Heidelberg/New York, 1986.

[16] J.-Y. Shi. Sign types corresponding to an affine Weyl group, *J. London Math. Soc.* **35** (1987), 56-74.

[17] R. P. Stanley. *Enumerative Combinatorics, vol. 1*, Wadsworth and Brooks - Cole, Belmont, CA, 1986; reprinted by Cambridge University Press, Cambridge, 1997.

[18] R. P. Stanley. Hyperplane arrangements, interval orders, and trees. *Proc. Nat. Acad. Sci. U.S.A.* **93** (1996), 2620-2625.

[19] R. P. Stanley. Graph colorings and related symmetric functions: ideas and applications, *Discrete Mathematics* **193** (1998), 267-286.

[20] W. T. Tutte. A contribution to the theory of chromatic polynomials, *Canad. J. of Math.* **6** (1953), 80-91.

[21] M. L. Wachs and J. W. Walker. On geometric semilattices, *Order.* **2** (1986), 367-385.

[22] D. J. A. Welsh. *Matroid theory*, Academic Press, New York, 1976.

[23] D.J.A. Welsh and G.P. Whittle. Arrangements, channel assignments and associated polynomials, *Advances in Applied Mathematics* **23** (1999), 375-406.

[24] H. Whitney. A logical expansion in mathematics, *Bull. Amer. Math. Soc.* **38** (1932), 572-579.

[25] T. Zaslavsky. Facing up to arrangements: face-count formulas for partitions of space by hyperplanes, *Mem. Amer. Math. Soc.*, vol. 1, no. 154, 1975.

[26] T. Zaslavsky. Biased graphs III. Chromatic and dichromatic invariants, *J. Combin. Theory, Ser. B.* **64** (1995), 17-88.

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