DESCENT NUMBER AND MAJOR INDICES FOR THE EVEN-SIGNED PERMUTATION GROUP

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ABSTRACT. We introduce and study three new statistics on the even-signed permutation group D_n. We show that two of these are Mahonian, i.e. are equidistributed with length, and that a pair of them gives a generalization of Carlitz's identity on the Euler-Mahonian distribution of the descent number and major index over S_n.

RÉSUMÉ. Nous presentons et étudions trois nouvelles statistiques sur le groupe de Coxeter de type D_n. Nous demontrons que parmi ces trois, deux sont "Mahonian", c'est-à-dire equidistribuées avec la longueur, et que deux autres portent à une généralisation de l'identité de Carlitz sur la distribution de Euler-Mahonian du nombre de descentes et du major index sur S_n.

1. INTRODUCTION

A well known classical result due to MacMahon (see [15]) asserts that the inversion number and the major index are equidistributed on the symmetric group. The joint distribution of major index and descent number was studied by Carlitz [7] and others. Several results of this nature have been generalized to the hyperoctahedral group B_n (see, e.g., [6],[13]) and many candidates for a major index for B_n have been suggested (see, e.g., [8],[9],[10],[12],[18]), but no generalizations of MacMahon's result have been found until the discovery of the flag major index in the recent paper [1]. After that, Foata posed the problem of finding a "descent statistic" that, together with the flag major index, allows the generalization to B_n of the well known Carlitz's identity on the Euler-Mahonian distribution of descent number and major index over S_n. In [2] Adin, Brenti and Roichman give two answers to Foata's question. Now it's natural to wonder if some of these statistics and results can be generalized to the even-signed permutation group D_n.

The goal of this paper is to show that this is the case. More precisely, we introduce and study three new statistics on D_n; the D-negative descent number ($ddes$), the D-negative major index ($dmaj$) and the D-flag major index ($fmaj_D$). When restricted to S_n, $ddes$ reduces to descent number and $dmaj$ to the major index. The two major indices on D_n are equidistributed with length, and the pair $(ddes, dmaj)$ gives a generalization of Carlitz's identity to D_n.

The organization of this extended abstract is as follows. In the next section we collect some definitions, notation and results that are needed in the rest of the work. In §3 we introduce a new "descent set" and hence in a very natural way new definitions of "descent number" and "major index" on D_n. It's shown that $dmaj$ is equidistributed with length and that $(ddes, dmaj)$ gives a generalization of Carlitz's identity. In §4 we define, in terms of Coxeter elements, the D-flag major index for D_n and we show that it's equidistributed with length. Furthermore, we describe a combinatorial algorithm to compute it.

2. Notation, Definitions and Preliminaries

In this section we give some definitions, notation and results that will be used in the rest of this work. We let $\mathbf{P} := \{1,2,3, \ldots\}$, $\mathbf{N} := \mathbf{P} \cup\{0\}$, and $\mathbf{Z}$ be the set of integers; for $a \in \mathbf{N}$ we let $[a] := \{1, 2, \ldots, a\}$ (where $[0] := \emptyset$). Given $n, m \in \mathbf{Z}$, $n \leq m$, we let $[n, m] := \{n, n+1, \ldots, m\}$. The cardinality of a set A will be denoted by $|A|$ and we let $\binom{[n]}{2} := \{S \subseteq [n] : |S| = 2\}$. More generally, given a multiset $M = \{1^{a_1}, 2^{a_2}, \ldots, r^{a_r}\}$ we denote by $|M|$ its cardinality, so $|M| = \sum_{i=1}^{r} a_i$. Given a variable q and a commutative ring R we denote by $R[q]$ (respectively, $R[[q]]$) the ring of polynomials (respectively, formal power series) in q with coefficient in R. For $i \in \mathbf{N}$ we let, as customary, $[i]_q := 1+q+q^2+\ldots+q^{i-1}$ (so $[0]_q = 0$).

Given a sequence $\sigma = (a_1, \ldots, a_n) \in \mathbf{Z}^n$ we say that $(i, j) \in [n] \times [n]$ is an *inversion* of σ if $i < j$ and $a_i > a_j$. We say that $i \in [n-1]$ is a *descent* of σ if $a_i > a_{i+1}$. We denote by $Inv(\sigma)$ and $Des(\sigma)$ the set of inversions and the set of descents of σ and with $inv(\sigma)$ and $des(\sigma)$ their cardinality, respectively. We also let

$$maj(\sigma) := \sum_{i \in Des(\sigma)} i \tag{1}$$

and call it the *major index* of σ.

Let S_n be the set of all bijections $\sigma : [n] \to [n]$. If $\sigma \in S_n$ then we write $\sigma = \sigma_1 \ldots \sigma_n$ to mean that $\sigma(i) = \sigma_i$, for $i = 1, \ldots, n$. If $\sigma \in S_n$ then we may also write σ in *disjoint cycle form* (see, e.g., [16, p.17]) and we will usually omit to write the 1-cycles of σ. For example, if $\sigma = 64175823$ then we also write $\sigma = (2, 4, 7)(1, 6, 8, 3)$. Given $\sigma, \tau \in S_n$ we let $\sigma\tau := \sigma \circ \tau$ (composition of functions) so that, for example, $(1, 2)(2, 3) = (1, 2, 3)$.

We denote by B_n the group of all bijections π of the set $[-n, n] \setminus \{0\}$ onto itself such that

$$\pi(-a) = -\pi(a)$$

for all $a \in [-n, n] \setminus \{0\}$, with composition as the group operation. This group is usually known as the group of *signed permutations* on $[n]$, or as the *hyperoctahedral group* of rank n. We identify S_n as a subgroup of B_n, and B_n as a subgroup of S_{2n}, in the natural ways.

If $\pi \in B_n$ then we write $\pi = [a_1, \ldots, a_n]$ to mean that $\pi(i) = a_i$ for $i = 1, \ldots, n$, we call this the *window* notation of w, and we let

$$\begin{array}{rcl} inv(\pi) & := & inv(a_1, \ldots, a_n), \\ des(\pi) & := & des(a_1, \ldots, a_n), \\ maj(\pi) & := & maj(a_1, \ldots, a_n), \\ Neg(\pi) & := & \{i \in [n] : a_i < 0\}, \\ N_1(\pi) & := & |Neg(\pi)|, \end{array} \tag{2}$$

and

$$N_2(\pi) := |\{\{i, j\} \in \binom{[n]}{2} : a_i + a_j < 0\}|. \tag{3}$$

We denote by D_n the subgroup of B_n consisting of all the signed permutations having an even number of negative entries in their window notation, more precisely

$$D_n := \{\pi \in B_n : N_1(\pi) \equiv 0 \ (\ mod\ 2 \ )\}.$$

Obviously, the definitions in (2) and (3) are still valid for $\pi \in D_n$.

It is well known (see, e.g., [5, §8.2]) that D_n is a Coxeter group with respect to the generating set $S := \{s_0, s_1, \ldots, s_{n-1}\}$ where

$$s_0 := [-2, -1, 3, \ldots, n]$$

and

$$s_i := [1, 2, \dots, i-1, i+1, i, i+2, \dots, n]$$

for $i = 1, \dots, n-1$. This gives rise to another natural statistic on D_n the *length* (similarly definable for any Coxeter group), namely

$$l(\pi) := min\{r \in \mathbf{N} : \pi = s_{i_1} \dots s_{i_r} \text{ for some } i_1, \dots, i_r \in [0, n-1]\}.$$

There is a well known direct combinatorial way to compute this statistic for $\pi \in D_n$ (see, e.g., [5, §8.2]), namely

$$l(\pi) = inv(\pi) - \sum_{i \in Neg(\pi)} \pi(i) - N_1(\pi). \tag{4}$$

It's not hard to prove that for all $\pi \in B_n$ (and so also for $\pi \in D_n$),

$$N_1(\pi) + N_2(\pi) = - \sum_{i \in Neg(\pi)} \pi(i), \tag{5}$$

so equivalently (4) becomes

$$l(\pi) = inv(\pi) + N_2(\pi). \tag{6}$$

For example, if $\pi := [-4, 1, 3, -5, -2, -6] \in D_6$ then $inv(\pi) = 10$, $des(\pi) = 2$, $maj(\pi) = 8$, $N_1(\pi) = 4$, $N_2(\pi) = 13$ and $l(\pi) = 23$.

We follow [5] for general Coxeter group notation and terminology. In particular, let (W, S) be a Coxeter system, for $J \subseteq S$ we let W_J be the subgroup of W generated by J, and

$$W^J := \{w \in W : l(ws) > l(w) \text{ for all } s \in J\}.$$

We call W_J the *parabolic subgroup* generated by J and W^J the *set of minimal left coset representatives* of W_J or the *quotient*. The quotient W^J is a poset according to the Bruhat order (see, e.g., [5] or [14]).
The following is well known, (see, e.g., [5] or [14]).

Proposition 1. *Let $J \subseteq S$. Then:*

- i) *Every $w \in W$ has a unique factorization $w = w^J w_J$ such that $w^J \in W^J$ and $w_J \in W_J$.*
- ii) *For this factorization $l(w) = l(w^J) + l(w_J)$.*

Now we let

$$T := \{\pi \in D_n : des(\pi) = 0\}. \tag{7}$$

It is well known, and easy to see, that

$$D_n = \biguplus_{u \in S_n} \{\pi u : \pi \in T\}, \tag{8}$$

where $\biguplus$ denotes disjoint union. We will often use this decomposition in this paper. Note that (8) is one case of the multiplicative decomposition of a Coxeter group into a parabolic subgroup and its minimal coset representatives (see Proposition 1), more precisely T is the quotient corresponding to the maximal parabolic subgroup generated by $J := S \setminus \{s_0\}$. In the next section we will analyze this issue in more detail.
For $n \in \mathbf{P}$ we let

$$A_n(t, q) := \sum_{\sigma \in S_n} t^{des(\sigma)} q^{maj(\sigma)},$$

and $A_0(t,q) := 1$. For example, $A_3(t,q) = 1 + 2tq^2 + 2tq + t^2q^3$. The following result is due to Carlitz, and we refer the reader to [7] for its proof.

Theorem 2. *Let* $n \in \mathbf{P}$. *Then*

$$\sum_{r\geq 0} [r+1]_q^n t^r = \frac{A_n(t,q)}{\prod_{i=0}^{n}(1-tq^i)} \tag{9}$$

in $\mathbf{Z}[q][[t]]$.

3. THE “NEGATIVE” STATISTICS

In this section we define a new “descent set” for elements of D_n. This gives rise, in a very natural way, to the definitions of “major index” and “descent number” for D_n. We then show that these two statistics give a generalization of Carlitz's identity to D_n, and that the former is equidistributed with length.

3.1. **The D-Negative Descent Multiset.**

For $\pi \in D_n$ let

$$Des(\pi) := \{i \in [n-1] \ : \ \pi(i) > \pi(i+1)\},$$

we define the *D-negative descent multiset*

$$DDes(\pi) := Des(\pi) \biguplus \{-\pi(i) - 1 \ : \ i \in Neg(\pi)\} \setminus \{0\}. \tag{10}$$

For example, if $\pi = [-4, 1, 3, -5, -2, -6] \in D_6$ then $Des(\pi) = \{3, 5\}$ and $DDes(\pi) = \{1, 3^2, 4, 5^2\}$.

Note that if $\pi \in S_n$ then $DDes(\pi)$ is a set and coincides with the usual descent set of π. Also, note that $DDes(\pi)$ can be defined rather naturally also in purely Coxeter group theoretic terms. In fact, for $i \in [n-1]$ let $\xi_i \in D_n$ be defined by

$$\xi_i := [-1, 2, \dots, i, -i-1, i+2, \dots, n].$$

Then $\xi_1, \dots, \xi_{n-1}$ are reflections (in the Coxeter group sense, see e.g., [5] or [14]) of D_n and it is clear from (4) that

$$DDes(\pi) := \{i \in [n-1] : l(\pi s_i) < l(\pi)\} \biguplus \{i \in [n-1] : l(\pi^{-1}\xi_i) < l(\pi^{-1})\}.$$

These considerations explain why it is natural to think of $DDes(\pi)$ as a “descent set”, so the following definitions are natural.
For $\pi \in D_n$ we let

$$ddes(\pi) := |DDes(\pi)|$$

and

$$dmaj(\pi) := \sum_{i \in DDes(\pi)} i.$$

For example if $\pi = [-4, 1, 3, -5, -2, -6] \in D_6$ then $ddes(\pi) = 6$, and $dmaj(\pi) = 21$.
Note that from (10) there follows that

$$dmaj(\pi) = maj(\pi) - \sum_{i \in Neg(\pi)} \pi(i) - N_1(\pi) = maj(\pi) + N_2(\pi). \tag{11}$$

This formula is also one the motivations behind our definition of $dmaj(\pi)$, because of the corresponding formulas (4) and (6), (see also [2]) .
Also note that

$$ddes(\pi) = des(\pi) + N_1(\pi) + \epsilon(\pi), \tag{12}$$

where

$$\epsilon(\pi) := \begin{cases} -1 & \text{if } 1 \notin \pi([n]) \\ 0 & \text{if } 1 \in \pi([n]). \end{cases}$$

3.2. **Equidistribution.** Our first result shows that $dmaj$ and l are equidistribuited in D_n.

Proposition 3. [4, Proposition 3.1] *Let* $n \in \mathbf{P}$. *Then*

$$\sum_{\pi \in D_n} q^{dmaj(\pi)} = \sum_{\pi \in D_n} q^{l(\pi)}.$$

3.3. **Generalization of Carlitz's Identity.** We start with some notation and terminology concerning partitions (see [17, §7.2]). By an (integer) *strict partition* we mean a sequence of positive integers $\lambda = (\lambda_1, \ldots, \lambda_k)$ such that $\lambda_1 > \lambda_2 > \ldots > \lambda_k$. We denote by $|\lambda| := \sum_i \lambda_i$. We denote by $\tilde{P}_S$ the set of all (integer) strict partitions. Given $n \in \mathbf{P}$ we let

$$\tilde{P}_S(n) := \{\lambda \in \tilde{P}_S : \lambda \subseteq (n, n-1, \ldots, 2, 1)\}.$$

As before, let $T = \{\pi \in D_n : des(\pi) = 0\}$ so

$$T = \{\pi \in D_n : \pi(1) < \pi(2) < \ldots < \pi(n)\}.$$

Therefore, given $\pi \in T, \pi \neq e$, there is a unique $k \in [n]$ such that

$$\pi(k) < 0 < \pi(k+1).$$

Given $\pi \in T$ we associate to π the strict partition

$$\Lambda(\pi) := (-\pi(1) - 1, -\pi(2) - 1, \ldots, -\pi(k) - 1). \tag{13}$$

The following is known, (see, e.g., [3]).

Proposition 4. *The map* Λ *defined by (13) is a bijection between* T *and* $\tilde{P}_S(n-1)$. *Furthermore* $\pi \leq \sigma$ *in* T *if and only if* $\Lambda(\pi) \subseteq \Lambda(\sigma)$ *and* $l(\pi) = |\Lambda(\pi)|$ *for all* $\pi, \sigma \in T$.

We begin with the following lemma.

Lemma 1. [4, Lemma 3.3] *Let* $n \in \mathbf{P}$. *Then*

$$\sum_{\sigma \in T} t^{N_1(\sigma)+\epsilon(\sigma)} q^{N_2(\sigma)} = \sum_{S \subseteq [n-1]} t^{|S|} q^{\sum_{i \in S} i} = \prod_{i=1}^{n-1} (1 + tq^i).$$

We are now ready to state the main result of this work, namely that the pair of statistics $(ddes, dmaj)$ solves Foata's problem for the group of even-signed permutations D_n.

Theorem 5. [4, Theorem 3.5] *Let* $n \in \mathbf{P}$. *Then*

$$\sum_{r \geq 0} [r+1]_q^n t^r = \frac{\sum_{\pi \in D_n} t^{ddes(\pi)} q^{dmaj(\pi)}}{(1-t)(1-tq^n) \prod_{i=1}^{n-1} (1 - t^2 q^{2i})} \tag{14}$$

in $\mathbf{Z}[q][[t]]$.

Note that, as in (9) for S_n and in ([2, Theorem 3.2]) for B_n, the powers of q in the denominator of formula (14) are the Coxeter degrees of D_n (see [14, p.59]).

4. The Flag Major Index for D_n

In this section we introduce another new "major index" statistic for D_n. This is an analogue of the flag major index introduced in [1]. We show that this statistic is equidistributed with length and we give a combinatorial algorithm to compute it.

4.1. **The D-Flag Major Index.** For $i = 0, \dots, n-1$ we define

$$t_i := s_i s_{i-1} \cdots s_0, \tag{15}$$

explicitly for all $i \in [n-1]$

$$t_i = [-1, -i-1, 2, 3, \dots, i, i+2, \dots, n], \tag{16}$$

and for $i = 0$

$$t_0 = [-2, -1, 3, \dots, n] = s_0. \tag{17}$$

These are Coxeter elements (see e.g., [14, §3.16]), in a distinguished flag of parabolic subgroups

$$1 < G_1 < G_2 < \ldots < G_n = D_n$$

where $G_i \simeq D_i$ $(i \geq 2)$ is the parabolic subgroup of D_n generated by $s_0, s_1, \dots, s_{i-1}$. The family $\{t_i\}_i$ is a new set of generators for D_n, and we have the following proposition.

Proposition 6. [4, Proposition 4.1] *For every $\pi \in D_n$ there exists a unique representation*

$$\pi = t_0^{h_{n-1}} t_{n-1}^{k_{n-1}} t_0^{h_{n-2}} t_{n-2}^{k_{n-2}} \cdots t_0^{h_1} t_1^{k_1} \tag{18}$$

with $0 \leq h_r \leq 1$, $0 \leq k_r \leq 2r-1$ and

$$k_r \in \{2r-1, r-1\} \text{ if } h_r = 1 \tag{19}$$

for all $r = 1, \dots, n-1$.

Note that the representation (18) is not unique if we drop the condition (19). For example consider $\pi = [3, -2, 1, -4] \in D_4$, then π has two different representations of type (18), namely, $\pi = t_3^3 t_0 t_2^3 t_0$ and $\pi = t_0 t_3^3 t_2 t_0 t_1$. The representation of Proposition 6 is the first one.

Let $\pi \in D_n$, then we define the *D-flag major index* of π by

$$fmaj_D := \sum_{i=1}^{n-1} k_i + \sum_{i=1}^{n-1} h_i. \tag{20}$$

4.2. **Equidistribution.** For $0 \leq m \leq 2n-1$, $n \geq 2$, we define $r_{n,m} \in D_n$ as follows: for $n = 2$,

$$r_{2,m} := \begin{cases} e & \text{if } m = 0 \\ s_1 & \text{if } m = 1 \\ s_1 s_0 & \text{if } m = 2 \\ s_0 & \text{if } m = 3, \end{cases}$$

and for $n > 2$,

$$r_{n,m} := \begin{cases} e & \text{if } m = 0 \\ s_{n-m} s_{n-m+1} \cdots s_{n-1} & \text{if } 0 < m < n \\ s_{m-n+1} s_{m-n} \cdots s_0 s_2 s_3 \cdots s_{n-1} & \text{if } n \leq m < 2n-1 \\ s_0 s_2 s_3 \cdots s_{n-1} & \text{if } m = 2n-1. \end{cases}$$

The set $\{r_{n,m} \ : \ 0 \leq m < 2n\}$ forms a complete set of representatives of minimal length for the left cosets of D_{n-1} in D_n. Moreover this is still valid for every $i \in [2, n]$, namely, $r_{i,m} \in D_i^{J_i}$ for all $m \in [0, 2i-1]$, where $J_i := S \setminus \{s_{n-1}, \dots, s_{i-1}\}$. Hence we have the following decomposition

$$D_n = D_n^{J_n} D_{n-1}^{J_{n-1}} \cdots D_2.$$

Note that the length of $r_{i,m}$ is $\bar{m}$, where

$$\bar{m} := \begin{cases} m & \text{if } 0 \le m \le 2i-2 \\ i-1 & \text{if } m = 2i-1. \end{cases}$$

From *i*) of Proposition 1 we know that each element $\pi \in D_n$ has a unique representation as a product

$$\pi = \prod_{k=1}^{n-1} r_{n+1-k, m_{n+1-k}} \tag{21}$$

where $0 \le m_j < 2j$ for all j. From *ii*) of Proposition 1 there follows that

$$l(\pi) = \sum_{j=2}^{n} \bar{m}_j. \tag{22}$$

Thanks to the unique representation (21) we define a map $\phi : D_n \to D_n$ in the following way:

$$\phi(\prod_{k=1}^{n-1} r_{n+1-k, m_{n+1-k}}) := \prod_{k=1}^{n-1} \phi(r_{n+1-k, m_{n+1-k}}),$$

where for $i \neq 2$,

$$\phi(r_{i,m}) := \begin{cases} t_{i-1}^{m} & \text{if } m < 2i-2 \\ t_0 t_{i-1}^{\bar{m}-1} & \text{if } 2i-2 \le m \le 2i-1, \end{cases}$$

and for $i = 2$,

$$\phi(r_{2,m}) := \begin{cases} e & \text{if } m = 0 \\ t_1 & \text{if } m = 1 \\ t_0 t_1 & \text{if } m = 2 \\ t_0 & \text{if } m = 3. \end{cases}$$

The definition of ϕ, together with Proposition 6 and (21), imply the following result.

Proposition 7. [4, Proposition 4.2] *The map $\phi : D_n \to D_n$ is a bijection.*

This implies the main result of this section, namely that the D-flag major index is equidistributed with the length in D_n.

Theorem 8. [4, Theorem 4.3] *Let $n \in \mathbf{P}$. Then*

$$\sum_{\pi \in D_n} q^{fmaj_D(\pi)} = \sum_{\pi \in D_n} q^{l(\pi)}.$$

Note that the B-flag major index ($fmaj$) defined on B_n (see [1]) does not work on D_n. Namely if we consider $\pi \in D_n$ as an element of B_n, then $fmaj(\pi)$ is not equidistributed with length on D_n. For example let $\pi = [-2, -1]$ then $fmaj_D(\pi) = 1$ while $fmaj(\pi) = 4$, and in D_2 there is no element of length 4.

Note also that $fmaj_D$ restricted to S_n is not the major index and it's not equidistributed with length. It seems to be a new statistic on S_n. It's easy to see that for each $\pi \in S_n$, $fmaj_D(\pi)$ is always even and that $fmaj_D(\pi) \ge maj(\pi)$. If we let $E_n(q) := \sum_{\pi \in S_n} q^{fmaj_D(\pi)}$, for $n \le 4$ we have $E_1(q) = 1$, $E_2(q) = 1 + q^2$, $E_3(q) = 1 + 3q^2 + q^4 + q^6$ and $E_4(q) = 1 + 5q^2 + 6q^4 + 7q^6 + 3q^8 + q^{10} + q^{12}$.

4.3. **A combinatorial algorithm.** Let $\sigma = (a_1, \dots, a_n) \in \mathbf{Z}^n$, we use this *split-notation*

$$\sigma = [a_1][a_2, \dots, a_{i+1}][a_{i+2}, \dots, a_n].$$

Sometimes it will be useful to denote the first part with A and the second with C_i where i represents the number of its elements.
We define the following operations on $\sigma \in \mathbf{Z}^n$:

$$\overset{\rightarrow_i^0}{\sigma} := [-a_2][-a_1, a_3, \dots, a_{i+1}][a_{i+2}, \dots, a_n],$$

and

$$\overset{\rightarrow_i^1}{\sigma} := [-a_1][-a_{i+1}, a_2, \dots, a_i][a_{i+2}, \dots, a_n].$$

In these cases we will write $\overset{\rightarrow_i^0}{\sigma} = (A^0, C_i^0, [a_{i+2}, \dots, a_n])$ and $\overset{\rightarrow_i^1}{\sigma} = (\overset{\rightarrow^1}{A}, \overset{\rightarrow^1}{C_i}, [a_{i+2}, \dots, a_n])$.
Moreover for all $n \in \mathbf{P}$ we define

$$\rightarrow_i^n := \rightarrow_i^1 \circ \dots \circ \rightarrow_i^1 \quad n\text{-times}. \tag{23}$$

Note that for every $\sigma \in \mathbf{Z}^n$, $\overset{\rightarrow_i^{2i}}{\sigma} = \sigma$.
For example let $\pi \in D_4$, $\pi = [-2][1, 3, -4, 5] = (A, C_4)$, then

$$\overset{\rightarrow_4^0}{\pi} = [-1][2, 3, -4, 5] = (A^0, C_4^0),$$

$$\overset{\rightarrow_4^5}{\pi} = [2][5, -1, -3, 4] = (\overset{\rightarrow^5}{A}, \overset{\rightarrow^5}{C_4}),$$

and

$$\overset{\rightarrow_3^2}{\pi} = [-2][-3, 4, 1][5] = (\overset{\rightarrow^2}{A}, \overset{\rightarrow^2}{C_3}, [5]).$$

These are the two technical properties that we will use in the algorithm. Fix $i \in [n-1]$, let t_i be as in (16),

$$t_i = [-1][-i-1, 2, 3, \dots, i][i+2, \dots n].$$

It's easy to see that for all $i \in [n-1]$ we have

$$t_i^2 = t_i t_i = \overset{\rightarrow_i^1}{t_i}, \tag{24}$$

and by (23) that for $k \in \mathbf{P}$

$$t_i^k = \overset{\rightarrow_i^{k-1}}{t_i}. \tag{25}$$

Now consider $t_{i-1} = [-1][-i, 2, \dots, i-1][i+1, \dots, n]$. As before it is not hard to see that

$$t_i t_{i-1} = \overset{\rightarrow_{i-1}^1}{t_i}. \tag{26}$$

Now we are able to state the algorithm to compute the unique representation of π as in Proposition 6, namely

$$\pi = f_{n-1} \cdots f_1$$

where for all $r \in [n-1]$, $f_r = t_0^{h_r} t_r^{k_r}$ with $h_r \in [0, 1]$ and $k_r \in [0, 2r-1]$.
We construct a sequence $e_0, \dots, e_{n-1}$ of elements of D_n such that

i) $e_0 = e$, $e_{n-1} = \pi$;
ii) $e_i = f_{n-1} \cdots f_{n-i}$, for all $i \in [1, n-1]$;
iii) $\pi(j) = e_i(j)$, for all $j > n - i$.

From *iii*) there immediately follows that $e_{n-1} = \pi$.

We need to do $n-1$ steps. From now to avoid confusion we put on A an index corresponding to the number of steps. We begin with $e_0 = [1][2,\ldots,n]$. We describe the $(n-i+1)$-th step. Assume that e_{n-i} has been constructed. Then by *iii*),

$$e_{n-i} = (A_{n-i}, C_{i-1}, [\pi(i+1), \ldots \pi(n)]).$$

For simplicity, we define $p(i)$ and $p(-i)$ to be the positions of $\pi(i)$ and $-\pi(i)$ in C_{i-1} or C^0_{i-1} respectively. There are four cases to consider.

1) $\pi(i) \in C_{i-1}$

Then we let $k_{i-1} = i-1-p(i)$ and $h_{i-1} = 0$. Hence $f_{i-1} = t_{i-1}^{i-1-p(i)}$.

2) $-\pi(i) \in C_{i-1}$

Then we let $k_{i-1} = 2i-2-p(-i)$ and $h_{i-1} = 0$. Hence $f_{i-1} = t_{i-1}^{2i-2-p(-i)}$.

3) $\pi(i) \in A_{n-i}$

Then $-\pi(i) \in C^0_{i-1}$ and in particular $p(-i) = 1$. We let $k_{i-1} = 2i-3$ and $h_{i-1} = 1$. Hence $f_{i-1} = t_0 t_{i-1}^{2i-3}$.

4) $-\pi(i) \in A_{n-i}$

Then $\pi(i) \in C^0_{i-1}$ and $p(i) = 1$. We let $k_{i-1} = i-2$ and $h_{i-1} = 1$. Hence $f_{i-1} = t_0 t_{i-1}^{i-2}$.

We have determined the factor f_{i-1}. From (23) and (25) there follows that $e_{n-i+1}(i) = \pi(i)$ and by (26) *iii*) again holds.

Therefore

$$e_{n-i+1} = (A_{n-i+1}, C_{i-2}, [\pi(i), \ldots, \pi(n)]),$$

where in cases 1) and 2),

$$A_{n-i+1} := \overset{\to k_{i-1}}{A_{n-i}}\ , \quad C_{i-2} := \overset{\to k_{i-1}}{C_{i-1}} \setminus [\pi(i)],$$

while in cases 3) and 4),

$$A_{n-i+1} := \overset{\to k_{i-1}}{A^0_{n-i}}\ , \quad C_{i-2} := \overset{\to k_{i-1}}{C^0_{i-1}} \setminus [\pi(i)].$$

Observe that in the first step $p(n) = \pi(n) - 1$ and $p(-n) = -\pi(n) + 1$. These can be used for the computation of e_1.

We finish this section by illustrating the procedure with an example.

Let $\pi = [5, 3, -4, 1, -2] \in D_5$. We start from

$$e = e_0 = [1][2,3,4,5] = (A_0, C_4).$$

$1^{st} - \mathit{step}$) $\quad -\pi(5) = -2 \in C_4$

We are in case 2) and $p(-5) = 1$, so $k_4 = 7$, $h_4 = 0$ and $f_4 = t_4^7$.

It follows that $A_1 = \overset{\to 7}{A_0} = [-1]$ and $C_3 = \overset{\to 7}{C_4} \setminus [-2] = [3,4,5]$. Hence,

$$e_1 = [-1][3,4,5][-2].$$

$2^{nd} - \mathit{step}$) $\quad -\pi(4) = -1 \in A_1$

We are in case 4) so $k_3 = 2$, $h_3 = 1$ and $f_3 = t_0 t_3^2$.

It follows that $A_2 = \overset{\to 2}{A^0_1} = [-3]$ and $C_2 = \overset{\to 2}{C^0_3} \setminus [1] = [-4, -5]$. Hence,

$$e_2 = [-3][-4,-5][1,-2].$$

$3^{rd} - \mathit{step}$) $\quad \pi(3) = -4 \in C_2$

We are in case 1) and $p(3) = 1$, so $k_2 = 1$, $h_2 = 0$ and $f_2 = t_2$.

It follows that $A_3 = \overset{\rightarrow 1}{A_2} = [3]$ and $C_1 = \overset{\rightarrow 1}{C_2} \setminus [-4] = [5]$. Hence,

$$e_3 = [5][-4, 1, -2].$$

$4^{th} - step)$ $\pi(2) = 3 \in A_3$

We are in case 3) so $k_1 = 1$, $h_1 = 1$ and $f_1 = t_0 t_1$.

It follows that $A_4 = \overset{\rightarrow 1}{A_3^0} = [5]$ and $C_0 = \emptyset$. Hence,

$$e_4 = [5][3, -4, 1, -2] = \pi,$$

and we are done. Finally $\pi = t_4^7 t_0 t_3^2 t_2 t_0 t_1$ and $fmaj_D(\pi) = 12$.

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