$\mathbb{Z}$-TILINGS OF POLYOMINOES AND STANDARD BASIS

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ABSTRACT. In this paper, we prove that for every set F of *polyominoes* (for us, a polyomino is a finite union of unit squares of a square lattice), we can find a $\mathbb{Z}$-tiling (*signed tile*) of polyominoes by copies of elements of F in polynomial time. We use for this the theory of generalized standard basis. So, we can algorithmically extend results of Conway and Lagarias on $\mathbb{Z}$-tiling problems.

RÉSUMÉ. Nous montrons que, pour toute famille F de *polyominos généraux* (union finie de cases d'une grille), le problème du $\mathbb{Z}$-pavage (*pavage signé*) des polyominos par des copies d'éléments de F peut être résolu en temps polynomial par l'usage de la théorie des bases de Grobner. De plus, nous pouvons ainsi retrouver et étendre de manière algorithmique des résultats obtenus par Conway et Lagarias sur les $\mathbb{Z}$-pavages.

1. INTRODUCTION

A *cell* $c(i,j)$ in the square lattice denotes the set :

$$c(i,j) := \{(x,y)\,; i \leq x < i+1, j \leq y < j+1\}.$$

So, cells are labelled by their lower left corner. For us, a *polyomino* is a finite -not necessary connected- union of cells. In this paper, we are interested in the study of a variant of the problem of tiling, called the $\mathbb{Z}$*-tiling problem.* Precisely, let P a polyomino and F a set of polyominoes (*the tiles*), a $\mathbb{Z}$*-tiling of* P *by* F consists of a finite number of translated tiles placed in the lattice (possibly with overlaps), with each tile assigned a sign of +1 or -1, such that for each cell $c(i,j)$ in $\mathbb{Z}^2$ the sum of the signs of the tiles covering $c(i,j)$ is +1 if $c(i,j) \in P$ and 0 if $c(i,j) \notin P$ (fig.1). Obviously, a polyomino which is tilable by a set of tiles is also $\mathbb{Z}$-tilable by this set. So, the study of $\mathbb{Z}$-tiling creates important necessary conditions of tilability. J.H. Conway and J.C. Lagarias [3] have previously studied this notion. They particularly obtained the following necessary and sufficient condition for a simply connected polyomino P :

P has a $\mathbb{Z}$-tiling of P by F if and only if the combinatorial boundary $[\partial P]$ is included in the tile boundary group $\mathbf{B}(F)$. For these definitions, we can refer to the paper of J.H. Conway and J.C. Lagarias [3].

Nevertheless, this group theoretic theorem presents some drawbacks : Firstly, it only applies to simply connected polyominoes. Secondly, the new criterion seems in general no easier to verify than to solve the original problem. Thirdly, it seems to be impossible to extend theoretic group arguments in higher dimension. In this paper, we propose another way of solving the problem. We associate for each polyomino P a polynomial in $\mathbb{Z}[X_1, X_2, Y_1, Y_2]$, called P*-polynomial.* We denote it by Q_P. We prove that, given a set F of polyominoes, a polyomino P is $\mathbb{Z}$-tilable by F if and only if $Q_P \in \langle Q_{P'}\ with\ P' \in E, X_1Y_1 - 1, X_2Y_2 - 1\rangle_{\mathbb{Z}}$ (i.e. the ideal of $\mathbb{Z}[X_1, ..., X_n]$ generated by the polynomials $Q_{P'}\ where\ P' \in E$ and by $X_1Y_1 - 1, X_2Y_2 - 1$). This new formulation allows us to use commutative algebraic tools like

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the standard basis algorithm to solve the problem. The reader can find a good introduction to standard basis (for ideals in $\mathbb{K}[X_1, ..., X_n]$ where $\mathbb{K}$ is a field) in [2] [4].

This leads us to reconsider coloring arguments. These tools are frequently found in the literature [5],[6],[7]). This notion gives in general important necessary conditions of tilability. We define in this paper the *generalized coloring* associated to a set of tiles F. This coloring groups all the generalized coloring arguments defined by Conway-Lagarias [3]. Moreover, the generalized coloring of a polyomino P is null if and only if P is $\mathbb{Z}$-tilable by the set F. Finally, we prove that it is possible to determine the generalized coloring of a classical polyomino when we only know the colors of the squares which are adjacent to the boundary of P.

So, if a polyomino P is in a sense "big" then we have a better algorithm to determine the tilability of P. Now, we are going to introduce the abstract notions that constitute the general framework of this paper. Given a subdivision S of $\mathbb{R}^d$ in cells and $\mathbb{A}$ a unitary ring, a $\mathbb{A}$-*weighted polycell* or simply a $\mathbb{A}$-polycell is a map P of S in $\mathbb{A}$ with a finite support. For each cell c, we call *weight of* P in c the number $P(c)$. The space $\mathbb{P}_{\mathbb{A}}$ of $\mathbb{A}$-weighted polycells has a natural structure of free $\mathbb{A}$-module. Clearly, the cells of weight 1 constitute a base of $\mathbb{P}_{\mathbb{A}}$. We can canonically embed the set of polycells in the $\mathbb{A}$-module of the $\mathbb{A}$-weighted polycells (in assigning 1 to cells covered by the polycell and 0 to the other cells). We say that a $\mathbb{A}$-weighted polycell P is $\mathbb{A}$-*tilable* by a set of $\mathbb{A}$-weighted tiles if and only if P is a $\mathbb{A}$-linear combination of translated elements of this set.

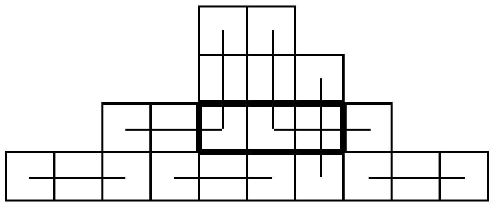

Fig.1. A (classical) polyomino P which is $\mathbb{Z}$-tilable by bars of length 3. In bold, a negative copy of a bar. The segments indicate positive copies of bars. These positive and negative bars constitute a $\mathbb{Z}$-tiling of P.

2. P-polynomials and $\mathbb{Z}$-tilings

In this section, in order to simplify, we only deal with $\mathbb{Z}$-polyominoes and $\mathbb{Z}$-polyhexes (sum of weighted hexagons of the regular hexagonal lattice). But all the theorems can be made in a more general framework. For instance, we have similar results for $\mathbb{Z}$-polycubes (in this case, the cells are the unit cubes of the regular cubical lattice of $\mathbb{R}^d$) or for $\mathbb{Z}$-polyamands (the cells are the triangles of the regular triangular lattice). The reader can find in [1] a more general presentation.

Firstly, for $a \in \mathbb{Z}^2$, we put by convention that

$$X^a = X_1^{\frac{a_1+|a_1|}{2}} X_2^{\frac{a_2+|a_2|}{2}} Y_1^{\frac{|a_1|-a_1}{2}} Y_2^{\frac{|a_2|-a_2}{2}}.$$

We encode the plan with 4 parameters to avoid to work with Laurent polynomials. Let us recall that we denote by $\langle P_1, ..., P_k \rangle_{\mathbb{Z}}$ the ideal of $\mathbb{Z}[X_1, ..., X_n]$ generated by the polynomials $P_1, ..., P_k$. For each $\mathbb{Z}$-polyomino P, we can define its *P-polynomial*

$$Q_P = \sum_{(a_1,a_2)\in\mathbb{Z}^2} P(c(a_1, a_2)) X^{(a_1,a_2)}$$

Lemma 2.1. *The space* $\mathbb{P}_{\mathbb{Z}}$ *is isomorphic to* $\mathbb{Z}[X_1, X_2, Y_1, Y_2] / \langle (X_1Y_1 - 1), (X_2Y_2 - 1) \rangle_{\mathbb{Z}}$.

Proof. There exists a unique linear map f from $\mathbb{Z}[X_1,X_2,Y_1,Y_2]$ to $\mathbb{P}_{\mathbb{Z}}$ such that : $f\left(X_1^{a_1}Y_1^{b_1}X_2^{a_2}Y_2^{b_2}\right)$ is the cell (a_1-b_1,a_2-b_2) with weight 1. Now, we must prove that $\ker(f)=\langle (X_1Y_1-1),(X_2Y_2-1)\rangle_{\mathbb{Z}}$. We proceed by successive divisions by (X_1Y_1-1) and (X_2Y_2-1) in the successive rings

$$\mathbb{Z}[X_2,Y_1,Y_2][X_1] \text{ and } \mathbb{Z}[X_1,Y_1,Y_2][X_2].$$

So, we can write all polynomial Q as follows $Q=R+\sum_{i=1}^{2}Q_i(X_iY_i-1)$ with R containing only monomials of the form X^a where $a\in\mathbb{Z}^2$ (i.e. without simultaneously X_i and Y_i). We have the following equivalence : $f(Q)$ is the empty polyomino, denoted by 0 (i.e. the polyomino P with 0 weight on all the squares), if and only if $f(R)=0$ (because of $f(Q_i(X_iY_i-1))=0$). Moreover, it is clear that $f(R)=0\Leftrightarrow R=0$ and that $R=0\Leftrightarrow Q\in\langle (X_1Y_1-1),(X_2Y_2-1)\rangle_{\mathbb{Z}}$. So, $\ker(f)=\langle (X_1Y_1-1),(X_2Y_2-1)\rangle_{\mathbb{Z}}$. □

Theorem 2.2. *Let E be a set of $\mathbb{Z}$-polyominoes. A $\mathbb{Z}$-polyomino P is $\mathbb{Z}$-tilable by E if and only if*

$$Q_P\in\left\langle Q_{P'} \text{ with } P'\in E, X_1Y_1-1, X_2Y_2-1\right\rangle_{\mathbb{Z}}.$$

Proof. By definition, a $\mathbb{Z}$-polyomino P is $\mathbb{Z}$-tilable by E if and only if there exists an integer t and for all i, $1\le i\le t$, $\lambda_i\in\mathbb{Z}$, $P^i\in E$ and $a^i=\left(a_1^i,a_2^i\right)\in\mathbb{Z}^2$ such that $P=\sum_{i=1}^{t}\lambda_i P^i_{\left(a_1^i,a_2^i\right)}$ where $P^i_{\left(a_1^i,a_2^i\right)}$ denotes the translation of P^i by the vector $\left(a_1^i,a_2^i\right)$. So, in

$$\mathbb{Z}[X_1,X_2,Y_1,Y_2]/\langle (X_1Y_1-1),(X_2Y_2-1)\rangle_{\mathbb{Z}}$$

we have $Q_{P^i_{a^i}}=X^{a^i}Q_{P^i}$ and consequently, $Q_P=\sum_{i=1}^{t}\lambda_i X^{a_i}Q_{P^i}$. Finally, P is $\mathbb{Z}$-tilable by E if and only if

$$Q_P\in\left\langle Q_{P'} \text{ with } P'\in E, X_1Y_1-1, X_2Y_2-1\right\rangle_{\mathbb{Z}}.$$

□

In fact, when we have a periodic tiling, we can always define the notion of polycell (finite union of cells of this tiling) and so, it is possible to translate, as we have done it, the problem of the $\mathbb{Z}$-tiling of a polycell by a set of polycells to algebraic problem of membership in an ideal of a polynomial ring on $\mathbb{Z}$. For instance, for the hexagonal lattice built in gluing copies of the hexagonal convex hull of the points $(0,0),(0,1),\left(\frac{-1}{2},\frac{\sqrt{3}}{2}\right),\left(\frac{3}{2},\frac{\sqrt{3}}{2}\right),(0,\sqrt{3}),(1,\sqrt{3})$, a polycell is generally called a *polyhexe*. We denote by $[a_1,a_2]$ the hexagonal cell which the lower left corner is the point $\left(\frac{3}{2}(a_1+a_2),\frac{\sqrt{3}}{2}(-a_1+a_2)\right)$ where $(a_1,a_2)\in\mathbb{Z}^2$ (fig.2). For each $\mathbb{Z}$-polyhexe P, we can define the *P-polynomial*

$$Q_P=\sum_{(a_1,a_2)\in\mathbb{Z}^2}P([a_1,a_2])\,X^{(a_1,a_2)}$$

Then, we have the following theorem :

Theorem 2.3. *Let E be a set of $\mathbb{Z}$-polyhexes. A $\mathbb{Z}$-polyhexe P is $\mathbb{Z}$-tilable by E if and only if*

$$Q_P\in\left\langle Q_{P'} \text{ avec } P'\in E, X_1Y_1-1, X_2Y_2-1\right\rangle_{\mathbb{Z}}.$$

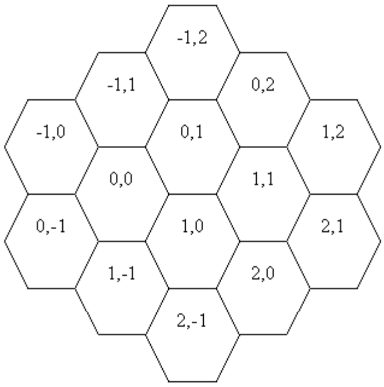

Fig.2. The hexagonal lattice with its coordinates.

3. STANDARD BASIS ON $\mathbb{Z}[X_1,...,X_n]$.

In this section, we indicate briefly how to solve the problem of membership in an ideal of $\mathbb{Z}[X_1,...,X_n]$. In fact, we use a non trivial extended version to $\mathbb{Z}[X_1,...,X_n]$ of the Buchberger algorithm [1]. The original one only works for an ideal of $\mathbb{K}[X_1,...,X_n]$ where $\mathbb{K}$ is a field and can be found in [4], [2]. First of all, we have to define a total order on the monomials of $\mathbb{Z}[X_1,...,X_n]$. Let $\leq^*$ be the lexicographic order on the n-tuples and let $\alpha=(\alpha_1,...,\alpha_n)$ be in $\mathbb{N}^n$, we denote by X^α the monomial $X_1^{\alpha_1}...X_n^{\alpha_n}$. Then, we put by definition that $X^\alpha \leq^* X^\beta$ if and only if $\alpha \leq^* \beta$. It is easy to verify that $\leq^*$ is a total order on the monomials of $\mathbb{Z}[X_1,...,X_n]$ and that we have the following property :
For all $\gamma \in \mathbb{N}^n$, if $X^\alpha \leq^* X^\beta$, then $X^{\alpha+\gamma} \leq^* X^{\beta+\gamma}$. This is the *lexicographic order* induced by $X_1 > ... > X_n$. Now, we recall useful terminologies for multivariable polynomials. Let $P = \sum_{\alpha\in\mathbb{N}^n} a_\alpha X^\alpha$ be a non empty polynomial of $\mathbb{Z}[X_1,...,X_n]$:
The *support* of P is $S(P) = \{\alpha \in \mathbb{N}^n \textit{ such that } a_\alpha \neq 0\}$. In particular $S(P)$ is always finite.
The *multidegree* of P is $m(P) = \max^*(\alpha \in S(P))$.
The *leading coefficient* of P is $LC(P) = a_{m(P)}$.
The *leading monomial* of P is $LM(P) = X^{m(P)}$.
The *leading term* of P is $LT(P) = LC(P)LM(P)$.

Theorem 3.1. *Let $F = (P_1,...,P_s)$ be a s-tuple of polynomials of $\mathbb{Z}[X_1,...,X_n]$. Then every polynomial P of $\mathbb{Z}[X_1,...,X_n]$ can be written in the following non unique form $P = R + \sum_{k=1}^{s} Q_k P_k$ where :*
i) $Q_1,...,Q_s, R \in \mathbb{Z}[X_1,...,X_n]$.
ii) $R = \sum_{\alpha\in\mathbb{N}^n} c_\alpha X^\alpha$ and $\forall\alpha \in S(R), c_\alpha X^\alpha$ is not divisible by any of $LT(P_1),...,LT(P_s)$.

Proof. This proof is an easy consequence of the following generalized division algorithm. $\square$

Algorithm 3.2. *Generalized Division Algorithm.*
We denote by trunc(s) the integer part of s.
Input : $(P_1,...,P_s), P$
Output : $(a_1,...,a_s), R$
$a_1 := 0,...,a_s := 0,\ R := 0$
$Q := P$

While $Q \neq 0$ *Do*
  $i := 1$
  $division := false$
  While ($i \leq s$ *and division=false*) *Do*
    If $LM(P_i)$ *divides* $LM(Q)$ *and* $|LC(P_i)| \leq |LC(Q)|$ *Then*
      $a_i := a_i + trunc(LC(Q)/LC(P_i))LM(Q)/LM(P_i)$
      $Q := Q - (trunc(LC(Q)/LC(P_i))LM(Q)/LM(P_i))P_i$
      $division := true$
    Else
    $i := i + 1$
    EndIf
  EndWhile
  If $division = false$ *Then*
    $R := R + LT(Q)$
    $Q := Q - LT(Q)$
  EndIf
EndWhile
Return $(a_1, ..., a_s), R$

R is the *remainder* of P by $(P_1, ..., P_s)$. We denote it by $\bar{P}^{(P_1,...,P_s)}$.

Example 3.3. *If we have* $P = X_1X_2^2 + X_1X_2 + X_2^2$ *and* $(P_1 = X_2^2 - 1, P_2 = X_1X_2 - 1)$, *then we obtain* $P = P_1 \times (X_1 + 1) + P_2 + X_1 + 2$. *The remainder is* $X_1 + 2$.

Example 3.4. *The remainder of the division of* $P = X_1X_2^2 - X_2^2$ *by* $(P_1 = X_2^2 - 1, P_2 = X_1X_2 - 1)$ *is null. Nevertheless, the division of* $P = X_1X_2^2 - X_2^2$ *by* $(P_1 = X_1X_2 - 1, P_2 = X_2^2 - 1)$ *gives* $\bar{P}^{(P_1,P_2)} = -X_2^2 + 1$. *So, we point out that the division depends on the ordering in the s-tuple of the polynomials. Actually, the division does not allow us to determine if a polynomial belongs or not to an ideal* I *of* $\mathbb{Z}[X_1, ..., X_n]$.

We recall that in $\mathbb{R}[X]$ a polynomial $P \in I$ if and only if Q divides P where Q is the minimal polynomial of I. We have the following analogous version in $\mathbb{Z}[X_1, ..., X_n]$:

Theorem 3.5. *For every ideal* I *of* $\mathbb{Z}[X_1, ..., X_n]$ *other than* $\{0\}$, *there exists a s-tuple of polynomials* $(P_1, ..., P_s)$ *such that* P *belongs to* I *if and only if the remainder of* P *by* $(P_1, ..., P_s)$ *is null.*

Such a s-tuple is called a *standard basis* of I. We do not proof here this important theorem. The reader who wants to take this theorem further can find a constructive proof of this in the following report [1]. We continue this section by an application to a classical problem solved by Conway and Lagarias [3] by using group theoretic arguments. Let T_N denote the triangular array of cells in the hexagonal lattice having $N(N+1)/2$ cells (fig.3).

Theorem 3.6. *(Conway-Lagarias, Theo 1.3 and 1.4)*
a) The triangular region T_N *in the hexagonal lattice has a* $\mathbb{Z}$*-tiling by congruent copies of* T_2 *polyhexes if and only if* $N = 0$ *or* $2 \mod 3$.
b) The triangular region T_N *in the hexagonal lattice has a* $\mathbb{Z}$*-tiling by congruent copies of three-in-line polyhexes if and only if* $N = 0$ *or* $8 \mod 9$.

Proof.
a) Firstly, we put $Y_1 > Y_2 > X_1 > X_2$ and we compute a standard basis of

$$\langle X_1 + X_2 + 1, X_1X_2 + X_1 + X_2, Y_1X_1 - 1, Y_2X_2 - 1\rangle_{\mathbb{Z}}.$$

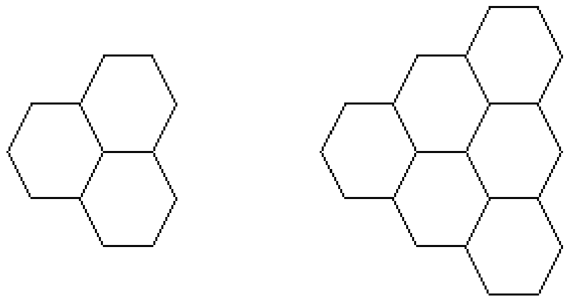

Fig.3. T_2 and T_3

We obtain $(X_1+X_2+1, X_2^2+X_2+1, Y_2+X_2+1, Y_1-X_2)$. Now, as $Q_{T_N} = \sum_{i=0}^{N} X_2^i \left(\sum_{j=0}^{N-i-1} X_1^j \right)$, we can easily compute that the remainder of Q_{T_N} by $(X_1+X_2+1, X_2^2+X_2+1, Y_2+X_2+1, Y_1-X_2)$ is equal to $\begin{cases} 0 & \text{if } N = 0 \text{ or } 2 \mod 3. \\ 1 & \text{if } N = 1 \mod 3. \end{cases}$

b) we compute a standard basis of

$$\langle X_1^2+X_1+1, X_2^2+X_2+1, X_1^2Y_2^2+X_1Y_2+1, Y_1X_1-1, Y_2X_2-1\rangle_{\mathbb{Z}}.$$

with $Y_1 > Y_2 > X_1 > X_2$. We obtain $B = (X_1^2+X_1+1, X_2^2+X_2+1, X_1+Y_1+1, X_2+Y_2+1, 3X_2+3X_1+3, X_2X_1-X_1-X_2-2)$.

The remainder of Q_{T_N} by B is equal to $\begin{cases} 0 & \text{if } N = 0 \text{ or } 8 \mod 9. \\ 1 & \text{if } N = 1 \mod 9. \\ X_1+X_2+1 & \text{if } N = 2 \text{ or } 3 \mod 9. \\ -2X_1-2X_2-1 & \text{if } N = 4 \mod 9. \\ 2X_1+2X_2+2 & \text{if } N = 5 \text{ or } 6 \mod 9. \\ -X_1-X_2 & \text{if } N = 7 \mod 9. \end{cases}$

□

4. General Coloring

Definition 4.1. *Let E be a set of $\mathbb{Z}$-polyominoes, the ideal*

$$I(E) = \langle Q_P; P \in E \text{ and } X_1Y_1-1, X_2Y_2-1\rangle_{\mathbb{Z}}$$

is the ideal associated to E. *If B is a standard basis of I then B is said to* be associated *to E.*

Definition 4.2. *A* general coloring χ_E *is the map from* $\mathbb{Z}[X_1, X_2, Y_1, Y_2]$ *into* $\mathbb{Z}[X_1, X_2, Y_1, Y_2]$ *such that* $\chi_E(Q) = \bar{Q}^B$.

Theorem 4.3. *A $\mathbb{Z}$-polyomino P is $\mathbb{Z}$-tilable by a set of $\mathbb{Z}$-polyominoes $E = \{P_1, ..., P_s\}$ if and only if $\chi_E(P) = 0$.*

Proof. By theorem 2.2, P is $\mathbb{Z}$-tilable by $E = \{P_1, ..., P_s\}$ if and only if Q_P belongs to $I(E)$, and by definition, Q_P belongs $I(E)$ if and only if $\chi_E(Q_P) = 0$. □

Remark 4.4. *This definition seems to be tautological. Indeed, this is very useful to have a geometric visualization. Let us observe the following explicit example.*

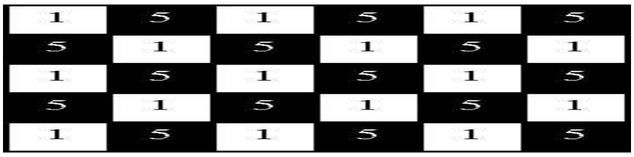

Fig.4. The T-tetraminoes

Example 4.5. *Consider that we want to have a chromatic characterization of the $\mathbb{Z}$-tilability by the set E of classical polyominoes described below (fig.4) called* T-tetraminoes.
We have

$$I(E) = \langle X_1^2 + X_1X_2 + X_1 + 1, X_1^2X_2 + X_1X_2 + X_1 + X_2, X_1X_2 + X_2^2 + X_2 + 1, \\ X_1X_2^2 + X_1X_2 + X_1 + X_2, X_1Y_1 - 1, X_2Y_2 - 1\rangle.$$

We compute a standard basis for the order $Y_1 > Y_2 > X_1 > X_2$: $(X_1+3, X_2+3, 8, Y_1+3, Y_2+3)$. So, we have $\chi_E(Q_{c(i,j)}) = \chi_E(X^{(i,j)}) = \begin{cases} 1 & \text{if } i+j=0 \mod 2 \\ 5 & \text{if } i+j=1 \mod 2 \end{cases}$. Moreover, we always have the following classical remainder property $\chi_E(Q_P) = \chi_E(\sum_{c(i,j)\in P} Q_{c(i,j)}) = \chi_E\left(\sum_{c(i,j)\in P} \chi_E\left(Q_{c(i,j)}\right)\right)$. Now, as $A = \chi_E\left(\sum_{c(i,j)\in P} Q_{c(i,j)}\right)$ is an integer, $\chi_E(A) = A \mod 8$. So, suppose that the squares of the plan have a chessboard-like coloration (fig.5), a polyomino P is $\mathbb{Z}$-tilable by E if and only if when assigning 5 on the white squares and 1 on the black ones in P, the sum of values on the squares is a multiple of 8.

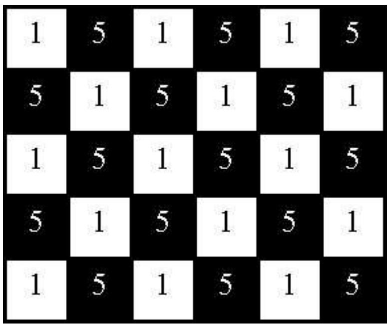

Fig.5. A general coloring for the T-tetraminoes. The lower left corner square is the square (0,0).

5. $\mathbb{Z}$-TILABILITY AND BOUNDARY CONDITIONS

In this section, we only deal with classical polyominoes and not with $\mathbb{Z}$-polyominoes. In the paper of Conway and Lagarias, it is possible to know if a polyomino P has a $\mathbb{Z}$-tiling in travelling the boundary of P. We prove that we have a similar situation with our characterization. We do not need to compute the remainder of Q_P, but only the remainder of a shorter polynomial associated to the boundary of P.

Theorem 5.1. *Let P be a polyomino and χ_E a general coloring associated to E. We suppose that*

$$\chi_E\left(\sum_{i=1}^{2}(X_i+Y_i)\right) - 4$$

is not a zero divisor of $\mathbb{Z}[X_1, X_2, Y_1, Y_2]/I(E)$. *In this case,*

$$\chi_E(Q_P) = 0 \text{ if and only if } \sum_{(c_1,c_2)\in S} (\chi_E(Q_{c_1}) - \chi_E(Q_{c_2})) = 0$$

where (c_1, c_2) *belongs to* S *if* c_1 *is a square in* P, c_2 *does not belong to* P *and* c_2 *has a common side with* c_1.

Proof. Let χ_E the general coloring, we denote by

$$v_{\chi_E}(X^a) = \sum_{i=1}^{2} (\chi_E(X^a) - \chi_E(X^a X_i)) + \sum_{i=1}^{2} (\chi_E(X^a) - \chi_E(X^a Y_i)).$$

We have $\sum_{(c_1,c_2)\in S} (\chi_E(Q_{c_1}) - \chi_E(Q_{c_2})) = \sum_{c\in P} v_{\chi_E}(Q_c)$ because, if the squares c and c' belong to P, the contributions of the couples (c, c') et (c', c) vanish themselves. We denote by T the set of (c, c') where c is a square in P, and c' has a common side with c_1. Moreover, we have $\sum_{(c_1,c_2)\in S} (\chi_E(Q_{c_1}) - \chi_E(Q_{c_2})) = \sum_{c\in P} v_{\chi_E}(Q_c) = \sum_{c\in P} \sum_{(c,c')\in T} (\chi_E(Q_c) - \chi_E(Q_{c'}))$. Now, if we consider that the image of χ_E is in $\mathbb{Z}[X_1, X_2, Y_1, Y_2]/I(E)$, it is obvious that χ_E is a morphism of algebra. So,

$$\sum_{c\in P} \sum_{(c,c')\in T} (\chi_E(Q_c) - \chi_E(Q_{c'})) = \chi_E(Q_P) \left(4 - \chi_E\left(\sum_{i=1}^{2} (X_i + Y_i)\right)\right) \sum_{c\in P} \chi_E(Q_c)$$

in $\mathbb{Z}[X_1, X_2, Y_1, Y_2]/I(E)$. As $\chi_E\left(\sum_{i=1}^{2} (X_i + Y_i)\right) - 4$ is not a zero divisor, $\chi_E(Q_P) = 0$ if and only if $\sum_{(c_1,c_2)\in S} (\chi_E(Q_{c_1}) - \chi_E(Q_{c_2})) = 0$. □

To conclude this section, we give an example related to the paper of Thurston [8]. Let us consider that we want to $\mathbb{Z}$-tile a polyomino with dominoes (union of two adjacent squares). The associated ideal is $I(E) = \langle X_1 + 1, X_2 + 1, X_1Y_1 - 1, X_2Y_2 - 1\rangle$. We obtain that $(1+X_1, 1+X_2, Y_1+1, Y_2+1)$ is a standard basis of $I(E)$ for the order $Y_1 > Y_2 > X_1 > X_2$. So, $\chi_E(X^{(i,j)}) = \begin{cases} 1 & \text{if } i+j = 0 \mod 2 \\ -1 & \text{if } i+j = 1 \mod 2 \end{cases}$. Hence, a polyomino is $\mathbb{Z}$-tilable by dominoes if and only if it has the same number of black (when $i + j = 1 \mod 2$) and white (when $i+j = 0 \mod 2$) squares $c_{(i,j)}$. In this case, the polyomino is said *balanced.* Independently, we have $\chi_E(\sum_{i=1}^{2} (X_i + Y_i)) - 4 = -8$ which is not a zero divisor. So, we can apply the theorem 5.1. The values of $\chi_E(Q_{c_1}) - \chi_E(Q_{c_2}) = \begin{cases} 2 & \text{if } c_1 = c_{(i,j)} \text{ and } i+j = 0 \mod 2 \\ -2 & \text{if } c_1 = c_{(i,j)} \text{ and } i+j = 1 \mod 2 \end{cases}$ where $(c_1, c_2) \in S$. Thus, P is balanced if and only if we have the same number of black and white edges on its boundary (a *black* (resp. *white*) *edge* is an edge which borders a black (resp. white) square of P).

References

[1] O. Bodini, Pavage des polyominos et Bases de Grobner, Rapport de recherche N^oRR2001-51, LIP, 2001.
[2] B. Buchberger, Introduction to Grobner basis, Logic of computation (Marktoberdorf 95) 35-66, NATO Adv. Sci. Inst. Ser. F Comput. Systems Sci.,157, Springer Berlin (1997).
[3] J.H Conway, J.C. Lagarias, Tiling with polyominoes and combinatorial group theory, J.C.T. Series A 53 (1990) 183-208.
[4] D. Cox, J. Little, D. O'Shea, Ideals,varieties and algorithms, 2nde edition, Undergraduate Text in Mathematics, Springer Verlag, New York (1997) XV 536 pp.
[5] S. W. Golomb, Tiling with polyominoes, J.C.T. Series A 1 (1966) 280-296.
[6] S. W. Golomb, Polyominoes which tile rectangles, J.C.T. Series A 51 (1989) 117-124.
[7] D.A. Klarner, Packing a rectangle with congruent n-ominoes, J.C.T. Series A 7 (1969) 107-115.
[8] W.P. Thurston, Conway's tiling groups, Amer. Math. Monthly, 97, num. 8 (Oct. 1990) 757-773.

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