EXPLICIT FORMULAE FOR SOME KAZHDAN-LUSZTIG POLYNOMIALS FOR S_n

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ABSTRACT. We consider Kazhdan-Lusztig polynomials $P_{u,v}(q)$ indexed by permutations u, v having particular forms with regard to their monotonicity patterns. First we present a solution to the problem of computing $P_{u,v}(q)$ when $u, v \in S_n$ satisfy $u^{-1}(n) - v^{-1}(n) \leq 3$ by reducing it to a problem for Kazhdan-Lusztig polynomials for S_{n-1}. As a corollary we obtain an "almost" explicit expression for $P_{e,4\ldots n123}$. We also show explicit formulae for $P_{e,v}(q)$ when v is obtained by inserting n in a permutation of S_{n-1} that is allowed to rise only in the first and in the last position generalizing a theorem due to B. Shapiro, M. Shapiro and A. Vainshtein. As an application of this result we write down explicit formulae for $P_{e,\sigma(n-2)\sigma(n-1)\sigma(n)\,n-3\ldots 4\,\tau(1)\tau(2)\tau(3)}$ for $(\sigma, \tau) \in S_3 \times S_3 \setminus (e, e)$, where σ act on the set $\{n-2, n-1, n\}$ in the natural way, establishing, in particular, two conjectures due to F. Brenti and R. Simion. Our proofs are based on the fact that the polynomials under consideration (together with some others) satisfy some nice recurrence relations. Moreover they are purely combinatorial and make no use of the geometry of the corresponding Schubert varieties.

RÉSUMÉ. Nous considérons polynômes de Kazhdan-Lusztig $P_{u,v}(q)$ indexés par permutations u, v avec des formes particulières par rapport à leurs andaments de monotonicité. D'abord nous proposons une solution au problème du calcul de $P_{u,v}(q)$ quand $u, v \in S_n$ vérifient $u^{-1}(n) - v^{-1}(n) \leq 3$ en le réduisant à un problème pour les polynômes de Kazhdan-Lusztig pour S_{n-1}. Comme première application de ce résultat nous obtiendrons une expression "presque" explicite pour $P_{e,4\ldots n123}$. Nous montrerons aussi des formules explicites pour $P_{e,v}(q)$ quand v est obtenu en insérant n dans une permutation de S_{n-1} qui a au plus 2 montées dans la première et la dernière position. Ce résultat généralise un théorème de B. Shapiro, M. Shapiro et A. Vainshtein. Comme application de ce résultat nous montrerons des formules explicites pour $P_{e,\sigma(n-2)\sigma(n-1)\sigma(n)\,n-3\ldots 4\,\tau(1)\tau(2)\tau(3)}$ avec $(\sigma, \tau) \in S_3 \times S_3 \setminus (e, e)$, oùσ agit sur l'ensemble $\{n-2, n-1, n\}$ de la façon naturelle, en prouvant deux conjectures de F. Brenti et R. Simion. Nos démonstrations sont fondeès sur le fait que les polynômes que nous considérons vérifient de bonnes relations de recurrence. De plus, elles sont purement combinatoires et n'utilisent pas la geometrie des variétés de Schubert correspondentes.

1. INTRODUCTION

In [KL1] Kazhdan and Lusztig defined, for every Coxeter system W, a family of polynomials, parametrized by pairs of elements of W, which have become known as the Kazhdan-Lusztig polynomials of W. These polynomials are intimately related to the Bruhat order of W and have proven to be of fundamental importance in representation theory and in the geometry of the Schubert varieties. We focus our attention to the case of the symmetric group, where these polynomials can be computed with some particular purely combinatorial rules (see, for example, [Br2, Corollary 4.6]. Despite the rather elementary recursion relations they satisfy, these polynomials are in general quite difficult to compute explicitly. In fact the only families of Kazhdan-Lusztig polynomials that are known correspond to situations where the geometry of the corresponding Schubert varieties is easier (see, for example, [LSc],[Bo], [P] and [SSV, Theorems 1 and 2]) , where the interval $[u, v]$ has some special shape (see, for example, [Br1, Corollaries 6.8 and 6.9] or when the shape of the indexing permutation lead in some natural way to the use of induction (see [BS, Corollary 3.2 and Theorem 3.3]). This work gives results in the direction of explicit formulae for the Kazhdan-Lusztig polynomials of the symmetric group when the indexing permutations are of particular forms.

The main results are the following. First we reduce the calculation of $P_{u,v}(q)$ when $u, v \in S_n$ satisfy $u^{-1}(n) - v^{-1}(n) \leq 3$ to an (easier) problem in S_{n-1}. Next we will show some applications of this result. In particular we obtain a formula for $P_{1\ldots n-3\,\sigma(n-2)\,\sigma(n-1)\,\sigma(n)\,,4\ldots n123}$, where σ is any permutation of the set $\{n-2, n-1, n\}$ that shows how these families of polynomials are intimately related among each other. Then we will focus our attention to permutations in S_n that are obtained from an element of S_{n-1} allowed to rise only in the first and in the last position by inserting n (or 1) anywhere in its complete notation. We write down some recurrence relations they satisfy and we obtain explicit formulae from these relations. Finally, as an application of this result, we find explicit formulae for $P_{e,\sigma(n-2)\sigma(n-1)\sigma(n)\,n-3\ldots4\,\tau(1)\tau(2)\tau(3)}$ where $(\sigma, \tau) \in S_3 \times S_3 \setminus (e, e)$ act on the set $\{n-2, n-1, n, 1, 2, 3\}$ in the most natural way, establishing, in particular, two conjectures due to F. Brenti and R. Simion (see [BS, Conjecture 4.1 and 4.2]). The proofs rely on the special shape of the permutation under consideration that will allow us to deduce some easy recursions satisfied by these polynomials (together with some other Kazhdan-Lusztig polynomials) with no use of geometry.

In §2 we fix the notation and we provide the necessary preliminaries on the Bruhat order of S_n and some known results about the Kazhdan-Lusztig polynomials which are used in the proofs of our results.

2. Notation and preliminaries

In this section we collect some definitions and results that have been used in the proofs of this work.

We let $\mathbb{N} := \{0, 1, 2, 3, \ldots\}$ be the set of non-negative integers and for $a \in \mathbb{N}$ we let $[a] := \{1, 2, \ldots, a\}$ (where $[0] = \emptyset$) and for $a \in \mathbb{R}$, $\lfloor a \rfloor$ is the largest integer $\leq a$. Given $n, m \in \mathbb{N}$, $n \leq m$, we let $[n, m] := \{n, n+1, \ldots, m\}$. We write $S = \{a_1, \ldots, a_r\}_<$ to mean that $S = \{a_1, \ldots, a_r\}$ and $a_1 < \cdots < a_r$

For $i \in \mathbb{Z}$ we denote by

$$[i]_q := \sum_{j=0}^{i-1} q^j$$

so that $[n]_q = 0$ if $n \leq 0$. Given a polynomial $P(q)$ and $i \in \mathbb{N}$ we denote by $[q^i](P(q))$ the coefficient of q^i in $P(q)$.

Given a set T we let $S(T)$ be the set of all bijections of T. In particular, $S_n = S([n])$ is the symmetric group on n elements and we denote by e the identity of S_n. If $u \in S([n, n+k])$ for some $n, k \in \mathbb{N}$, then we write $u = u_1 u_2 \ldots u_{k+1}$ to mean that $u(n+i) = u_{i+1}$ for $i = 0, \ldots, k$, while we denote by s_i the transposition $(i, i+1)$. Given $\sigma, \tau \in S(T)$, we let $\sigma\tau := \sigma \circ \tau$, i.e. we compose permutations as functions, from right to left. Given $\sigma \in S_n$, the right descent set of σ is

$$D_R(\sigma) := \{i \in [n-1] : \sigma(i) > \sigma(i+1)\}$$

and the left descent set is

$$D_L(\sigma) := \left\{i \in [n-1] : \sigma^{-1}(i) > \sigma^{-1}(i+1)\right\}$$

and the length of σ is defined by the number of inversions:

$$\ell(\sigma) := \mathrm{inv}(\sigma) := \#\left\{(a, b) \in [n] \times [n] : a < b, \sigma(a) > \sigma(b)\right\}$$

For example, if $\sigma = 6\,3\,5\,2\,4\,1$ then $D_R(\sigma) = \{1,3,5\}$, $D_L(\sigma) = \{1,2,4,5\}$ and $\ell(\sigma) = 11$. Throughout this work we view S_n as a poset ordered by the strong Bruhat order. We are not going to define this order in the usual way (see [H, Section 5.9] for its definition), but we shall use the following characterization of it due to Ehresmann [E]. For $\sigma \in S_n$ and $j \in [n]$, let

$$\{\sigma^{j,1}, \dots, \sigma^{j,j}\}_< := \{\sigma(1), \dots, \sigma(j)\}$$

Theorem 2.1. *Let $\sigma, \tau \in S_n$. Then $\sigma \leq \tau$ if and only if $\sigma^{j,i} \leq \tau^{j,i}$ for all $1 \leq i \leq j \leq n-1$.*

We take the following fundamental result (see [H, Section 7.11] for a proof) as the definition of the Kazhdan-Lusztig polynomials:

Theorem 2.2. *There exists a unique family of polynomials $\{P_{u,v}(q),\ u,v \in S_n\} \subset \mathbb{Z}[q]$ such that:*
(1) $P_{u,v}(q) = 0$ if $u \not\leq v$
(2) $P_{u,v}(q) = 1$ if $u = v$
(3) If $u \leq v$ and $i \in D_R(v)$ then

$$P_{u,v}(q) = q^{1-c} P_{us_i,vs_i}(q) + q^c P_{u,vs_i}(q) - \sum_{\{z : i \in D(z)\}} q^{\frac{\ell(v)-\ell(z)}{2}} \mu(z, vs_i) P_{u,z}(q)$$

where, for $u, v \in S_n$,

$$\mu(u,w) := \begin{cases} \left[q^{\frac{1}{2}(\ell(w)-\ell(u)-1)}\right](P_{u,w}(q)) & \text{if } u < w \text{ and } \ell(w)-\ell(u) \text{ is odd} \\ 0 & \text{otherwise} \end{cases}$$

and $c = 1$ if $i \in D(u)$, and $c = 0$ otherwise.

The following result, whose proof can be found in [KL2], is a consequence of the role played by the Kazhdan-Lusztig polynomials in the geometry of Schubert varieties and no combinatorial proof of it is known so far :

Theorem 2.3. *Let $u, v \in S_n$. Then $P_{u,v}(q) \in \mathbb{N}[q]$.*

Two important consequences of Theorem 2.2 are the following:

Proposition 2.4. *Let $u, v \in S_n$ such that $u < v$. Then*

$$\deg(P_{u,v}(q)) \leq \frac{1}{2}(\ell(v) - \ell(u) - 1)$$

and, if $i \in D_R(v)$

$$P_{u,v}(q) = P_{us_i,v}(q)$$

It should be remarked that Theorem 2.2 and the second part of Proposition 2.4 can be reformulated in a similar way using left descents instead of right descents. An immediate consequence of Proposition 2.4 is the following

Corollary 2.5. *Let $z, w \in S_n$, $z \leq w$, be such that $\mu(z,w) \neq 0$ and $\ell(w) - \ell(z) > 1$. Then $D_R(z) \supseteq D_R(w)$ and $D_L(z) \supseteq D_L(w)$.*

Two other properties that we often use are the following (see [H, Theorem 7.9 and Corollary 7.14] for proofs:

Proposition 2.6. *Let $u, v \in S_n$. Then*

$$\begin{aligned} P_{u,v}(q) &= P_{u^{-1},v^{-1}}(q) \\ &= P_{w_0uw_0,w_0vw_0}(q) \end{aligned}$$

where $w_0 = n \dots 2\,1$ is the longest element of S_n.

Let $w \in S_n$. We denote by $\overline{w}$ the permutation of S_{n-1} obtained from w by suppressing the value n from its notation. Then the following result can be obtained as a corollary of Theorem 2.2 and will be of fundamental importance in the rest of our work.

Proposition 2.7. *Let $u, v \in S_n$ such that n occurs in the same position in both u and v. Then*

$$P_{u,v}(q) = P_{\overline{u},\overline{v}}(q)$$

It should be mentioned that Proposition 2.7 can be stated in a "dual" version when 1 occur in the same position in both u and v.

The next result gives an explicit formula for $P_{u,v}$ when v is allowed to rise only in the first and in the last place (see [SSV]) and will be generalized in §4:

Theorem 2.8. *Let $u, v \in S_n$, $u \le v$, be such that $[2, n-2] \subseteq D_R(v)$. Then*

$$P_{u,v}(q) = \begin{cases} 1 & \text{if } v(1) < v(n) \text{ or } v(n) \le u(1) \text{ or } v(1) \ge u(n) \\ 1 + q^{v(1)-v(n)} & \text{otherwise} \end{cases}$$

We conclude this section with an easy characterization of the permutations that gives rise to Kazhdan-Lusztig polynomials equal to 1 (see [LSa] for a proof). Let $\tau \in S_m$ and $\sigma \in S_n$ with $n \ge m$. We say that σ avoids τ if there is no subsequence $1 \le i_1 < \cdots < i_m \le n$ such that

$$\sigma\left(i_{\tau(1)}\right) < \cdots < \sigma\left(i_{\tau(m)}\right)$$

Theorem 2.9. *Let $v \in S_n$. Then*

$$P_{u,v}(q) = 1 \;\forall u \le v \iff v \text{ avoids both } 3412 \text{ and } 4231$$

3. A reduction theorem

Definition. Let $u, v \in S_n$. Then we set

$$d(u, v) := u^{-1}(n) - v^{-1}(n)$$

Note that by Theorem 2.1, if $u \le v$ we have $d(u, v) \ge 0$.

We are going to reduce the calculation of $P_{u,v}(q)$ to a problem for Kazhdan-Lusztig polynomials for S_{n-1} when $d(u, v) \le 3$. We have already seen that if $d(u, v) = 0$ then $P_{u,v}(q) = P_{\overline{u},\overline{v}}(q)$, so we may focus our attention to the case $d(u, v) > 0$.

The next results, for $d(u, v) = 1$ or 2, represent a reformulation and a generalization of a theorem due to F. Brenti and R. Simion (see [BS, Theorem 3.1]).

Theorem 3.1. *Let $u, v \in S_n$ such that $u \le v$ and $i = v^{-1}(n)$. Then*
(1) If $d(u, v) = 1$

$$P_{u,v}(q) = P_{\overline{u},\overline{v}}(q)$$

(2) If $d(u, v) = 2$

$$P_{u,v}(q) = \begin{cases} q^{1-c} P_{us_i,vs_i}(q) + q^c P_{u,vs_i}(q) & \text{if } i+1 \notin D_R(v) \\ P_{\overline{u},\overline{v}}(q) & \text{if } i+1 \in D_R(v) \end{cases}$$

where $c = 1$ if $i \in D(u)$ and $c = 0$ otherwise.

Note that the first part of Theorem 3.1 follows easily from Proposition 2.4 and Proposition 2.7.

Suppose now that $d(u, v) = 3$ and again set $i = v^{-1}(n)$. To fix the ideas we write

$$u := \ldots u_i u_{i+1} u_{i+2} n \ldots$$

and

$$v := \ldots n v_{i+1} v_{i+2} v_{i+3} \ldots$$

If $v_{i+2} > v_{i+3}$ then, by Proposition 2.4, we may swap u_{i+2} and n in u and hence we go back to the case $d(u,v) = 2$. So, with no lack of generality, we may suppose $v_{i+2} < v_{i+3}$, i.e. $i+2 \notin D_R(v)$. We would like to use Theorem 2.2 taking i as a right descent for v. The next result will allow us to simplify the sum in that formula in this case.

Proposition 3.2. *Let $u, v \in S_n$ such that $u \leq v$, $d(u,v) = 3$, $i = v^{-1}(n)$ and $i+2 \notin D_R(v)$. Then the application $F : z \mapsto \overline{z}$ establishes a bijection between*

$$\left\{ \begin{array}{c} z \in S_n \text{ such that } z \geq u \\ i \in D_R(z),\ \mu(z, vs_i) \neq 0 \\ \text{and } \ell(vs_i) - \ell(z) > 1 \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} z \in S_{n-1} \text{such that } z \geq \overline{u} \\ i, i+1 \in D_R(z) \\ \text{and } \mu(z, \overline{v}) \neq 0 \end{array} \right\}$$

Moreover, for z in the set of the left-hand side, we have $\mu(z, vs_i) = \mu(\overline{z}, \overline{v})$, $\ell(v) - \ell(z) = \ell(\overline{v}) - \ell(\overline{z}) + 3$ and $P_{u,z}(q) = P_{\overline{u},\overline{z}}(q)$.

We are now ready to state the main result of this section.

Theorem 3.3. *Let $u, v \in S_n$ such that $u \leq v$, $d(u,v) = 3$, $i = v^{-1}(n)$ and $i+2 \notin D_R(v)$. Then*

$$\begin{aligned} P_{u,v}(q) &= q^{1-c} P_{us_i, vs_i}(q) + q^c P_{u, vs_i}(q) - \sum_{\{z \in S_{n-1} : i, i+1 \in D_R(z)\}} q^{\frac{\ell(\overline{v}) - \ell(z) + 3}{2}} \mu(z, \overline{v}) P_{\overline{u}, z}(q) \\ &- \varepsilon_0 q P_{\overline{u}, \overline{v}}(q) - \varepsilon_1 q P_{\overline{u}, \overline{v} s_{i+1}} \end{aligned}$$

where

$$\varepsilon_j := \left\{ \begin{array}{ll} 0 & v_{i+1} < v_{i+j+2} \\ 1 & \text{otherwise} \end{array} \right.$$

for $j = 0, 1$ and, as usual, $c = 1$ if $i \in D_R(u)$ and $c = 0$ otherwise.

Note that the polynomials in the first two summands in the right-hand side of the previous formula are indexed by permutations x, y verifying $d(x,y) = 2$.

It should be mentioned that both Theorems 3.1 and 3.3 can also be stated in a "dual" version when $u, v \in S_n$ satisfy $\widetilde{d}(u,v) := v^{-1}(1) - u^{-1}(1) \leq 3$.

The next example shows us that, unfortunately, there could be many summands different from 0 in the previous sum.

Example 3.4. Let $n \geq 5$ and $v := 3 \ldots (n-2)\, n\, (n-1)\, 1\, 2$ and hence $\overline{v} = 3 \ldots (n-2)\, (n-1)\, 1\, 2$ and $u = e$. Then it is easy to check that for every $i \in [3, n-2]$, $(1,i)\overline{v}$ gives rise to a non-zero summand in Theorem 3.3.

Before showing the first application of Theorem 3.3, let us state the following:

Problem 3.5. Let $i, n \in \mathbb{N}$, $1 \leq i \leq n$. Find an explicit formula for

$$P_{e, i \ldots n\, 1 \ldots i-1}(q)$$

Problem 3.5 is trivial for $i = 1$ and it is an easy consequence of Theorem 2.9 for $i = 2$. In 1992 M. Haiman conjectured a formula to solve this problem for $i = 3$ which has been proved by F. Brenti and R. Simion in 2000 (see [BS], Corollary 3.2):

Theorem 3.6. *Let*

$$F_n(q) := \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n-i}{i} q^i$$

be the q-analogue of the Fibonacci number. Then $\forall n \geq 3$

$$P_{e, 3 \ldots n\, 1\, 2}(q) = F_{n-2}(q)$$

If we use Theorems 2.2 and 3.1 then we can deduce

$$P_{s_{n-1},3\dots n\,1\,2}(q) = P_{e,3\dots n-1\,1\,2}$$

$$P_{e,3\dots n\,1\,2}(q) = P_{e,3\dots n-1\,1\,2}(q) + qP_{s_{n-2},3\dots n-1\,1\,2}(q)$$

This can be restated in the following way:

Proposition 3.7. *Let $n \geq 3$. Then*

$$\begin{pmatrix} P_{e,3\dots n\,1\,2}(q) \\ P_{s_{n-1},3\dots n\,1\,2}(q) \end{pmatrix} = \begin{pmatrix} 1 & q \\ 1 & 0 \end{pmatrix}^{n-3} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

The next result is a direct consequence of Theorem 3.3 and is the starting point to the solution of Problem 3.5 for $i = 4$.

Lemma 3.8. *Let $n \in \mathbb{N}$, $n \geq 4$, $v_n \in S_n$, $v_n := 4 \dots n\,1\,2\,3$ and $u \leq v$. Then*

$$P_{u,v_n} = q^{1-c}P_{us_{n-3},v_ns_{n-3}}(q) + q^cP_{u,v_ns_{n-3}}(q)$$

where $c = 1$ if $n-3 \in D_R(u)$ and $c = 0$ otherwise.

Now let S_3 act on the set $\{n-2, n-1, n\}$ by identifying $1, 2$ and 3 with $n-2, n-1$ and n respectively. Then, $\forall \sigma \in S_3$ we set

$$\Phi_\sigma(q) := P_{1\dots(n-3)\sigma(n-2)\sigma(n-1)\sigma(n),v_n}$$

In 2001 Billey and Warrington (see [BW, Theorem 4]) gave a generating function description of $\Phi_{123}(q)$.

The following theorem shows an explicit formula for all $\Phi_\sigma(q)$, $\sigma \in S_3$ at the same time and in particular it solves Problem 3.5 for $i = 4$.

Theorem 3.9. *Let $n \in \mathbb{N}$ $n \geq 4$.*

$$\begin{pmatrix} \Phi_{123}(q) \\ \Phi_{132}(q) \\ \Phi_{213}(q) \\ \Phi_{231}(q) \\ \Phi_{312}(q) \\ \Phi_{321}(q) \end{pmatrix} = \begin{pmatrix} 1 & q & q & q^2 & 0 & 0 \\ 1 & 0 & q & 0 & 0 & 0 \\ 1 & q & 0 & 0 & q & q^2 \\ 0 & 1 & 0 & 0 & q & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}^{n-4} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

4. Explicit formulae

The main target of this section is to find an explicit formula for all polynomials $P_{u,v}$, $u, v \in S_n$, when $D_R(\overline{v}) \supseteq [2, n-2]$, $v^{-1}(n) \neq 1$ and u have some particular shape depending on that of v. This result will turn out to be a generalization of Theorem 2.8 and will allow us to prove two conjectures due to F. Brenti and R. Simion.

With this purpose we fix $x, y, n \in \mathbb{N}$ such that $x, y \in [2, n-1]$ and $x \neq y$. We denote by σ_0 the unique element v of S_{n-1} such that $v(1) = x$, $v(n-1) = y$ and $[2, n-2] \subseteq D_R(v)$.

For any $i \in [2, n]$ we denote by v_i the unique permutation of S_n satisfying the following two conditions:

$$\overline{v_i} = \sigma_0$$

$$v_i^{-1}(n) = i$$

We also set, $\forall a, i \in \mathbb{N}$, $2 \leq a \leq i \leq n$,

$$u_{i,a} := s_as_{a+1}\cdots s_{i-1}$$

so that, in particular, $u_{i,i} = e$ $\forall i$.

For example, for $n = 6$, $x = 4$ and $y = 2$ we have $v_4 = 4\,5\,3\,6\,1\,2$, $v_5 = 4\,5\,3\,1\,6\,2$ and $u_{4,2} = 1\,3\,4\,2\,5\,6$.

We denote by $R_{i,a}(q) := P_{u_{i,a},v_i}(q)$. We also set, for the notation convenience, $R_{n+1,a} = 0$.

The fundamental result about this family of Kazhdan-Lusztig polynomials is the following:

Theorem 4.1. *Let $2 \leq a \leq i \leq n-1$. Then the polynomials $R_{i,a}(q)$ verify the following relations:*

If $x > y$

$$\begin{aligned} R_{i,a}(q) &= qR_{i+1,a}(q) + R_{i+1,i+1}(q) - qR_{i+2,i+2}(q) \\ &\quad -\delta_{i,n-y-1}q - \delta_{i,n-x}q^{x-y+1} \end{aligned}$$

and if $x < y$

$$\begin{aligned} R_{i,a}(q) &= qR_{i+1,a}(q) + R_{i+1,i+1}(q) \\ &\quad -qR_{i+2,i+2}(q) - \delta_{i,n-y}q \end{aligned}$$

where $\delta_{i,j}$ is the usual Kronecker symbol.

Theorem 4.1 can be used to find explicit formulae for the polynomials $R_{i,a}(q)$ using a double induction on n and $n-i$.

Corollary 4.2. *Suppose $x > y$. Then if we set:*

$$H_i(q) = q\left[n-y-i\right]_q + q^{n-y-1}\left[x-i\right]_q + q^{x-y+1}\left[n-x+1-i\right]_q + q^{n-y}\left[y-1-i\right]_q$$

we have:

$$R_{i,a}(q) = \begin{cases} (1+q^{x-y})\left[n-1-i\right]_q - H_i(q) & \text{if } a \in [2, y-1] \\ \left[n-i\right]_q + q^{x-y}\left[n-1-i\right]_q - H_i(q) & \text{if } a \in [y, x] \\ (1+q^{x-y})\left[n-i\right]_q - H_i(q) & \text{if } a \in [x+1, n-1] \end{cases}$$

Corollary 4.3. *Suppose $x < y$. Then if we set*

$$K_i(q) := q\left[n-y+1-i\right]_q + q^{n-y}\left[y-1-i\right]_q$$

we have

$$R_{i,a}(q) = \begin{cases} \left[n-1-i\right]_q - K_i(q) & \textit{if } a \in [2, y-1] \\ \left[n-i\right]_q - K_i(q) & \textit{if } a \in [y, n-1] \end{cases}$$

Note that in this case the polynomials $R_{i,a}$ don't depend on x.

Remark 4.4. We could have defined $R_{i,1}(q)$, in a similar way, but we have chosen not to do it because they satisfy a slight different recursion (and the general discussion would have been more complicated) and hence we have preferred to suppose $a \geq 2$.

Note that by an easy argument on induction on n based on Theorems 2.4 and 2.9, in order to prove Theorem 2.8 it's enough to show it when $u = e$ and $v = \sigma_0$. Hence our result provides also a combinatorial proof of this theorem which was originally proved by B. Shapiro, M. Shapiro and A. Vainshtein in a geometric way.

We are now going to show further explicit formulae for some other families of Kazhdan-Lusztig polynomials whose proofs still use Theorems 3.1 and 3.3 as well as other recursive relations similar to that of Theorem 4.1.

With this purpose we let $n \in \mathbb{N}$, $n \geq 6$ and S_3 act at the same time on $\{1, 2, 3\}$ in the usual way and on $\{n-2, n-1, n\}$ as it did in the §3,

i.e. in the natural way identifying $n-2, n-1$ and n with $1, 2$ and 3 respectively.

Definition. $\forall(\sigma,\tau)\in S_3\times S_3$ we denote by $D_{\sigma,\tau}(q)$ the following Kazhdan-Lusztig polynomial:

$$D_{\sigma,\tau}(q) := P_{e,\sigma(n-2)\,\sigma(n-1)\,\sigma(n)\,n-3\ldots 4\,\tau(1)\,\tau(2)\,\tau(3)}(q)$$

Theorem 4.5. *$\forall n\geq 6$ the following formulae hold:*

(1) $D_{123,321}(q)=D_{321,123}(q)=1$
(2) $D_{132,321}(q)=D_{321,132}(q)=D_{321,213}(q)=D_{213,321}(q)=1$
(3) $D_{231,321}(q)=D_{321,312}(q)=1$
(4) $D_{321,321}(q)=1$
(5) $D_{312,321}(q)=D_{321,231}(q)=1+q$
(6) $D_{231,312}(q)=1+q^{n-3}$
(7) $D_{213,312}(q)=D_{231,132}(q)=D_{231,132}(q)=D_{132,312}(q)=1+q^{n-4}$
(8) $D_{132,213}(q)=D_{213,132}(q)=1+q^{n-5}$
(9) $D_{132,132}(q)=D_{213,213}(q)=1+q^{n-5}(1+q)$
(10) $D_{123,312}(q)=D_{231,123}(q)=1+2q^{n-4}$
(11) $D_{123,132}(q)=D_{132,123}(q)=D_{213,123}(q)=D_{123,213}(q)=1+q^{n-5}(2+q)$
(12) $D_{123,231}(q)=D_{312,123}(q)=(1+2q^{n-5})(1+q)$
(13) $D_{231,231}(q)=D_{312,312}(q)=1+q+q^{n-4}$
(14) $D_{132,231}(q)=D_{312,132}(q)=D_{312,213}(q)=D_{213,231}(q)=(1+q)(1+q^{n-5})$
(15) $D_{312,231}(q)=(1+q)^2(1+q^{n-5})$

All the equalities among the $D_{\sigma,\tau}(q)$'s in each row of Theorem 4.5 are due to Theorem 2.6 while equations 1,2,3 and 4 follows directly from Theorem 2.9. Equations 6,7,8,9,10 and 11 are particular cases of the explicit formulae arising from Theorem 4.1, while all the others need a further proof. Equations 10 and 11 were conjectured by F. Brenti and R. Simion (see [BS, Conjectures 4.1 and 4.2]) while equation 6 had been conjectured by M. Haiman and already proved by B. Shapiro, M. Shapiro and A. Vainshtein as a particular case of Theorem 2.8.

The only missing case from Theorem 4.5 is $D_{123,123}$. This has turned out to be much more difficult than the others and will be treated apart in a joint work of the author and M. Marietti.

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