# SKEW OSCILLATING SEMI-STANDARD TABLEAUX (EXTENDED ABSTRACT)

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ABSTRACT. We introduce an analogue of the Robinson-Schensted correspondence for skew oscillating semi-standard tableaux which generalize the correspondence for skew oscillating tableaux. We give the geometric construction for skew oscillating semi-standard tableaux and examine its combinatorial properties.

RÉSUMÉ. Nous introduisons une construction analogue de la correspondance de Robinson-Schensted pour les tableaux semi-standards oscillants gauches qui généralise la correspondance pour les tableaux oscillants gauches. Nous donnons la construction géométrique pour les tableaux semi-standards oscillants gauches et exeminons ses propriétés combinatoires.

## 1. INTRODUCTION

The Robinson-Schensted correspondence between permutations and pairs of standards tableaux of the same shape is introduced by Robinson ([6]) and it is given by Schensted ([10]) a little different form. After generalization by Knuth ([5]) to generalized permutations and pairs of semi-standards tableaux, various analogues of the Robinson-Schensted correspondence have been produced on different kinds of tableaux ([8],[13],[3],[9]).

More recently, Dulucq and Sagan ([4]) have given the Robinson-Schensted correspondence for oscillating tableaux and skew oscillating tableaux.

In this article, we extended the properties and constructions of analogue of Robinson-Schensted correspondence in [4] to skew oscillating semi-standard tableaux. In sections 2, we give basic definitions of generalised biwords and skew oscillating semi-standard tableaux. An algorithm of Robinson-Schensted for skew oscillating semi-standard tableaux is given in section 3, which is an extension of the algorithm of Robinson-Schensted correspondence for skew oscillating standard tableaux given in [4]. Then we give a geometric construction of a generalized biword due to Viennot, Chauve and Dulucq ([1],[8],[13]).

## 2. DEFINITION AND NOTATIONS

Let $\lambda = (\lambda_1, ..., \lambda_k)$, $\lambda_1 \leq ... \leq \lambda_k$, be a partition of $n$ such that $\sum_{i=1}^{k} \lambda_i = n$. The partition $\lambda$ can be displayed a Ferrers diagram with the part $\lambda_i$ in the row $i$. If $\mu \subseteq \lambda$ then the corresponding skew shape $\lambda/\mu$ is the set $\{c|c \in \lambda, c \notin \mu\}$. If $|\lambda/\mu| = n$ then we write $\lambda/\mu \vdash n$ and say that $\lambda/\mu$ is a skew partition of $n$. A skew semi-standard tableau $S$ of shape $\lambda/\mu$ is a labeling of the cells of $\lambda/\mu$ with positive integers so that the rows are strictly increasing and the columns are weakly increasing. $\emptyset_\alpha$ denotes the empty tableaux of the shape $\alpha$ (or a skew tableau of the shape $\alpha/\alpha$). $T(\lambda/\mu)$ denotes the set of skew semi-standard tableaux of shape $\lambda/\mu$.

$S(i, j)$ denotes the label of the cell in the $i^{th}$ row and $j^{th}$ column of a skew semi-standard tableau $S$ so that $k \in S$ means $k = S(i, j)$ for some $i, j$. $\overline{T}(\lambda/\mu)$ denotes the set of tableaux of shape $\lambda/\mu$ with rows strictly decreasing and columns weakly decreasing. For example, when $\lambda = (5, 4, 3, 1)$ $\mu = (2, 2)$, the two following tableaux belong to $T(\lambda/\mu)$ and $\overline{T}(\lambda/\mu)$ respectively.

$$\begin{array}{lllll} 1 & & & & \\ 1 & 3 & 6 & & \\ \square & \square & 2 & 5 & \\ \square & \square & 2 & 3 & 5 \end{array} \qquad\qquad \begin{array}{lllll} 1 & & & & \\ 5 & 4 & 3 & & \\ \square & \square & 6 & 3 & \\ \square & \square & 7 & 3 & 2 \end{array}$$

Four kinds of insertions and deletions in a skew semi-stanaard tableau ([1],[4]) are defined below. Let $S$ be a skew semi-standard tableau of shape $\lambda/\mu$.

1. The *external insertion* inserts an integer $x$ in $S$ by using the Knuth-Robinson-Schensted algorithm([2],[5]). We denote the new tableau obtained after this insertion by $ExtI(S, x)$. The inverse process is called *external deletion*, denoted by $ExtD(S, x)$, which ends with the expulsion of an integer out of $S$.

2. The *internal insertion* occurs only in a cell $(u, v)$ of $S$ such that $(u, v) \notin \mu$ and it belongs to one of three cases: (i) $(u-1, v) \in \mu$ and $(u, v-1) \in \mu$, (ii) $v = 1$ and $(u-1, v) \in \mu$, (iii) $u = 1$ and $(u, v-1) \in \mu$.

The *internal insertion* of the cell $(u, v)$ inserts the integer $x$ contained in $S(u, v)$ from the row $u+1$ using the external insertion algorithm. We denote the new tableau by $IntI(S, (u, v))$. The external deletion is called *internal deletion* if it ends in filling a cell of $\mu$. $IntD(S, (u, v))$ denotes the internal deletion.

3. The *empty insertion* adds an empty cell $(u, v)$ in $S$ such that $(u, v) \notin \lambda$, satisfying (i)$(u-1, v) \in \mu$ and $(u, v-1) \in \mu$, (ii)$u = 1$, $(u, v-1) \in \mu$ or (iii)$v = 1$, $(u-1, v) \in \mu$. $EmpI(S, (u, v))$ denotes the new tableau obtained after this insertion and the inverse process is called *empty deletion*, denoted by $EmpD(S, (u, v))$.

4. A cell can simply be *attached* or *erased* using neither the insertion algorithms nor the deletion algorithms.

Example 2.1.

$$\mathrm{P} = \begin{array}{llll} \square & & & \\ \square & 2 & & \\ \square & 2 & 4 & \\ \square & \square & 3 & 7 \end{array}$$

$$\mathrm{ExtI(P,6)} = \begin{array}{llll} \square & & & \\ \square & 2 & & \\ \square & 2 & 4 & 7 \\ \square & \square & 3 & 6 \end{array} \qquad \mathrm{ExtD(P,3)} = \begin{array}{llll} \square & & & \\ \square & 2 & & \\ \square & 2 & & \\ \square & \square & 4 & 7 \end{array}$$

$$\mathrm{IntI(P,(2,2))} = \begin{array}{llll} 2 & & & \\ 2 & & & \\ \square & \square & 4 & \\ \square & \square & 3 & 7 \end{array} \qquad \mathrm{IntD(P,(3,2))} = \begin{array}{llll} \square & & & \\ \square & & & \\ \square & 2 & 4 & \\ \square & 2 & 3 & 7 \end{array}$$

$$\mathrm{EmpI(P,(5,1))} = \begin{array}{llll} \square & & & \\ \square & & & \\ \square & 2 & & \\ \square & 2 & 4 & \\ \square & \square & 3 & 7 \end{array} \qquad \mathrm{EmpD(P,(4,1))} = \begin{array}{llll} \square & 2 & & \\ \square & 2 & 4 & \\ \square & \square & 3 & 7 \end{array}$$

A skew oscillating semi-standard tableau of length $n$ is a sequence of semi-standard tableaux $P = (P_0, P_1, ..., P_n)$ where $P_k$ is obtained from $P_{k-1}$ by an insertion or a deletion of a cell.

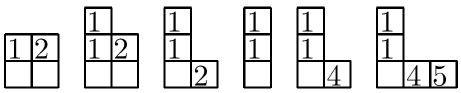
$\Theta_n(\alpha/\gamma \to \beta/\mu)$ denotes the set of skew oscillating tableaux $P = (P_0, P_1, ..., P_n)$ of length $n$ satisfying the following conditions:

(1) the shape of $P_0$ is $\alpha/\gamma$, and the shape of $P_n$ is $\beta/\mu$,

(2) $P_k$ is obtained from $P_{k-1}$ by attaching a cell with a label (this is not by the insertion algorithms) or a deletion of a cell by external deletion, internal deletion or empty deletion.

(3) if $x_i, x_j, ..., x_m$ are inserted respectively in $P_i, P_j, ..., P_m$, $i < j < ... < m$, then $x_i \leq x_j \leq ... \leq x_m$.

For example, if $\alpha = (2, 2)$, $\gamma = (2)$, $\beta = (3, 1, 1)$ and $\mu = (1)$ then the following tableau belongs to $\Theta_5(\alpha/\gamma \to \beta/\mu)$. 1 is inserted in $P_1$, 4 in $P_4$ and 5 in $P_5$.




For a $P \in \Theta_n(\alpha/\gamma \to \beta/\mu)$, we define a set of nondecreasing sequences of positive integers $I(P) = \cup_{j\in N} I_j$, where $I_j = \{j_0, j_1, j_2, ..., j_n\}$, $j_0 = 0 \leq j_1 \leq ... \leq j_n$ and $j_k = x$ if $P_k = P_{k-1} + (u, v)$ with $P_k(u, v) = x$ for $1 \leq k \leq n$. An $I_j$ of the example above is $\{0, 1, j_2, j_3, 4, 5\}$, where $j_2, j_3$ are positive integers.

A skew oscillating semi-standard tableau of $\Theta(\emptyset\mu \to \lambda/\mu)$ having only insertion steps, is a skew semi-standard tableau of shape $\lambda/\mu$, the label of a cell being given by its creation.

A generalized biword $\pi$ of size $2n$ is a sequence of vertical pairs of positive integers $\pi = \begin{pmatrix} u_1 & u_2 & ... & u_n \\ v_1 & v_2 & ... & v_n \end{pmatrix}$ where $u_1 \geq u_2 \geq ... \geq u_k$, $u_i \geq v_i$ for $i = 1, ..., k$, and $v_i \geq v_2$ if $u_i = u_j$. $\hat{\pi}$ denotes the

top row of $\pi$ and $\check{\pi}$ its bottom row.

$GB$ denotes the set of generalized biwords. The size of $\pi$ is $2\times$ the number of pairs of $\binom{u_i}{v_i}$, or $|\pi| = 2n$. $GB_{2n}$ denotes the set of generalized biwords of size $2n$.

## 3. Generalized biwords and skew oscillating semi-standard tableaux

We give a description of an algorithm to examine the relation between skew oscillating semi-standard tableaux and the triples $(S, U, \pi) \in \cup_{\mu\subseteq\alpha\cap\beta}[T(\beta/\mu) \times \overline{T}(\alpha/\mu)] \times GB$.

Algorithm **OSCIL**

(a)The input is $(S, U, \pi) \in \cup_{\mu\subseteq\alpha\cap\beta}[T(\beta/\mu) \times \overline{T}(\alpha/\mu)] \times GB$,

(b) The output is $(P, I)$ where $P \in \Theta_n(\emptyset_\alpha \to \beta/\mu)$, and $I = \{i_0 = 0, i_1, i_2, ..., i_n\} \in I(P)$, i.e., $I$ satisfies the following conditions :

(1) $i_1, ..., i_n$ being a nondecreasing sequence of positive integers,

(2) if we obtain $P_k$ from $P_{k-1}$ by attaching a cell $(u, v)$ with label $a$, that is, $P_k = P_{k-1} + (u, v)$, with $P_k(u, v) = a$, then $i_k = a$.

We construct a sequence of nonnegative integers $I = \{i_0, i_1, i_2, ..., i_n\}$ as follows : let $i_0 = 0$ and $i_1, ..., i_n$ be the rearranged elements of $S$, $U$, $\hat{\pi}$ and $\check{\pi}$ in nondecreasing order. We have $n = |S| + |U| + |\pi|$.

Let $P_n = S$.
For $k$ from $n$ to 1 :
(a) if there is a cell $P_k(u, v) = i_k$, then erase this cell to obtain $P_{k-1}$,
(b) else if the pair $(i_k, x)$ belongs to $\pi$, then $P_{k-1} = ExtI(P_k, x)$,
(c)else if $U(u, v) = i_k$ and $P_k(u, v)$ exists (with label $x$), then $P_{k-1} = IntI(P_k, (u, v))$,
(d) else $P_{k-1} = EmpI(P_k, (u, v))$.
The tableaux $P_k$ have respective shapes $\lambda_k/\mu_k$. $P = (P_0, ..., P_n)$ and $I = \{i_0, i_1, ..., i_n\}$ satisfy that $i_k = a$ when $P_k = P_{k-1} + (u, v)$, with $P_k(u, v) = a$, so $I \in I(P)$.

Algorithm $\mathbf{OSCIL^{-1}}$.
(a) The input is $(P, I)$ where $P \in \Theta_n(\emptyset_\alpha \to \beta/\mu)$ with $\mu \subseteq \alpha \cap \beta$ and $I \in I(P)$.
(b) The output is a triple $(S, U, \pi) \in \cup_{\mu \subseteq \alpha \cap \beta}[T(\beta/\mu) \times \overline{T}(\alpha/\mu)] \times GB$.

Let $\pi = \emptyset$, and $U_0 = P_0$.
For $k$ from 1 to $n$:
(a)if $P_k = P_{k-1} + (u, v)$, then $U_i = U_{i-1}$,
(b) else $(P_k = P_{k-1} - (u, v))$, we have three cases :
$(b_1)$ if the deletion is external ($x$ ejected out of $P_{k-1}$), then add the pair $(i_k, x)$ to $\pi$, $U_k = U_{k-1}$,

$(b_2)$ else if it is internal (the cell $P_{k-1}(u, v)$ with label $x$ is erased), then label the cell $U_{i-1}(u, v)$ with $i_k$ to obtain $U_k$,

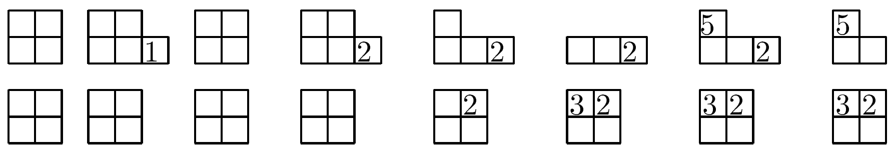
$(b_3)$ else label the cell $U_{k-1}(u, v)$ with $i_k$ to obtain $U_k$.
Finally, we obtain $S = P_n$, $U = U_n$ and $\pi \in GB$

Example 3.1.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| k = | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| $i_k =$ | 0 | 1 | 1 | 2 | 2 | 3 | 5 | 6 |

$P_k =$

$U_k =$



| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $\pi_k =$ | $\emptyset$ | $\emptyset$ | $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ | $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ | $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ | $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ | $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ | $\begin{pmatrix} 6 & 1 \\ 2 & 1 \end{pmatrix}$ |

**Theorem 1.** *Let $\alpha$, $\beta$ be fixed partitions. There is a bijection $\Phi$ from triples $(S, U, \pi)$ of*

*$\cup_{\mu\subseteq\alpha\cap\beta}[T(\beta/\mu)\times\overline{T}(\alpha/\mu)]\times GB$ to $(P,I)$ with a skew oscillating semi-standard tableau $P$ of $\Theta_n(\emptyset_\alpha\to\beta/\mu)$, $n=|S|+|U|+|\pi|$ and $I=\{i_0,i_1,i_2,...,i_n\}\in I(P)$*

**Proof:** For a triple $(S,U,\pi)\in\cup_{\mu\subseteq\alpha\cap\beta}[T(\beta/\mu)\overline{T}(\alpha/\mu)\times GB]$, we obtain directly a nondecreasing sequence $I=\{i_0,i_1,i_2,...,i_n\}$ with $i_0=0$ and $\{i_1,i_2,...,i_n\}$ rearranging the elements of $S$, $U$, $\hat{\pi}$ and $\check{\pi}$. A skew oscillating semi-standard tableaux $P\in\Theta_n$ results by applying the algorithm $OSCIL$.

To give the inverse operation, we construct a nondecreasing sequence $I=\{i_0,i_1,i_2,...,$ $i_n\}$ from $P\in\Theta_n(\emptyset_\alpha\to\beta/\mu)$ as follows : (1) $i_0=0$ (2) if $P_k=P_{k-1}+(u,v)$, with $P_k(u,v)=x$ then $i_k=x$, else $i_k$ is a positive integer satisfying $i_{k-1}\le i_k\le i_{k+1}$, so $I\in I(P)$. Next, we construct a sequence $(S_0,U_0,\pi_0)=(P_0,P_0,\emptyset)$, $(S_1,U_1,\pi_1),...,(S_n,U_n,\pi_n)=(S,U,\pi)$ from $P$ and $I(P)$ by applying the algorithm $OSCIL^{-1}$. The algorithm $OSCIL^{-1}$ corresponds exactly to the inverse construction of the cell produced by the algorithm $OSCIL$. So $(S,U,\pi)$ is in bijection with $(P,I)$. Example 3.2 shows the application of the algorithm $OSCIL$ and $OSCIL^{-1}$. $\diamondsuit$

*Remark 3.2* If the skew oscillating semi-standard tableaux $P\in\Theta_n(\emptyset_\alpha\to\beta/\alpha)$ has only insertion steps, the bijection $\Phi$ is $\Phi^{-1}(P,I)=(P_n,\emptyset_\alpha,\emptyset)$

$\overline{\Theta}_n(\beta/\mu\to\alpha/\gamma)$ denotes the set of skew oscillating tableaux of length $n$, $Q=(Q_0,Q_1,...,$ $Q_n)$, satisfying the following conditions:

(1) the shape of $Q_0$ is $\beta/\mu$, and the shape of $Q_n$ is $\alpha/\gamma$,

(2) $Q_k$ is obtained from $Q_{k-1}$ by erasing of a labelled cell ( not by deletion algorithms) or an insertion of a cell by external insertion, internal insertion or empty insertion.

(3) if $x_i,x_j,...,x_m$ are deleted respectively from $Q_i,Q_j,...Q_m$, $i<j<...<m$, then $x_i\ge x_j\ge...\ge x_m$.

We know that $P=(P_0,P_1,...,P_n)\in\Theta_n(\alpha/\gamma\to\beta/\mu)$ if and only if $\overline{P}=(P_n,P_{n-1},...,P_0)\in\overline{\Theta}_n(\beta/\mu\to\alpha/\gamma)$.

We define a set of nonincreasing sequences of positive integers $\overline{I}(Q)=\cup_{j\in N}\overline{I}_j$,
$\overline{I}_j=\{\overline{j}_1,\overline{j}_2,...,\overline{j}_n\}$ for $Q\in\overline{\Theta}_n(\beta/\mu\to\alpha/\gamma)$ as follows:
(1) $\overline{j}_1\ge\overline{j}_2\ge...\ge\overline{j}_n$
(2) if $Q_{k+1}=Q_k-(u,v)$ with $Q_k(u,v)=x$, then $\overline{j}_k=x$.

**Theorem 2.** *Let $\pi$ be a generalized biword of size $2n$ and $\alpha$ be an empty partition (of shape $\alpha$). There is a bijection $\Phi_{RS}$ from pairs $(\emptyset_\alpha,\pi)$ to $\{(P,I_1),(Q,\overline{I}_2))\}$ of $\cup_\beta[\{\Theta_n(\emptyset_\alpha\to\beta/\alpha)\times I(P)\},\times\{\overline{\Theta}_n(\emptyset_\alpha\to\beta/\alpha)\times\overline{I}(Q)\}]$.*

**Proof:** According to the previous theorem, we obtain $(P_0=\emptyset_\alpha,...,P_n...,P_{2n}=\emptyset_\alpha)$
with $I=\{i_0,i_1,...i_n,..,i_{2n}\}$, and the result $(P,I_1)=(P_0,P_1,...,P_n(\text{of shape}\beta/\alpha),\{i_0,i_1,...,i_n\})$ with $I_1\in I(P)$, and $(Q,\overline{I}_2)=(P_{2n},P_{2n-1},...,P_n(\text{of shape}\beta/\alpha),\{i_{2n},i_{2(n-1)},...,i_n\})$. Therefore $Q\in\overline{\Theta}_n(\emptyset_\alpha\to\beta/\alpha)$ and $\overline{I}_2\in\overline{I}(Q)$ $\diamondsuit$

Taking an empty initial and final semi-standard tableaux in the theorem 1 and 2, we have an analog of Robinson-Schensted correspondence for oscillating semi-standard tableaux, as stated in the following results.

**Corollaire 1.** *Let $\beta$ be fixed partitions and $n$ a fixed integer. There is a bijection $\Phi_{\emptyset}$ from pairs $(S, \pi)$ of $T(\beta) \times GB$ such that $n = |S| + |\pi|$ to pairs $(P, I)$ with $P$ of $\Theta_n(\emptyset \to \beta)$ and $I \in I(P)$.*

**Corollaire 2.** *Let $n$ be a fixed integer. There is a bijection $\Phi_{RS_{\emptyset}}$ from generalized biwords $\pi$ of $GB_{2n}$ to pairs $\{(P, I_1), (Q, \overline{I}_2)\}$ of $\cup_{\beta}[(\Theta_n(\emptyset \to \beta) \times I(P)) \times (\overline{\Theta}_n(\emptyset \to \beta)] \times \overline{I}(Q)]$.*

## 4. Geometric description of a generalized biword

In this section, we represent a geometric description of a generalized biword in the the first quadrant of the Cartesian plane by applying the geometric construction of Viennot in [1] and [13] for standard tableaux. We obtain an oscillating semi-standard tableau from the geometric description of
a generalized biword.

First, we present a method to standardize a generalized biword. Let $\mathbb{N}$ be a set of positive integers. For a given generalized biword $\pi = \begin{pmatrix} u_1 & u_2 & ... & u_n \\ v_1 & v_2 & ... & v_n \end{pmatrix}$, we propose a new alphabet $\mathbb{N} \cup \{j^{(h)} : j, h \in \mathbb{N}\}$ such that

$$... < j < j^{(1)} < j^{(2)} < ... < j+1 < j+1^{(1)} < j+1^{(2)} < ...$$

If $u_j = u_{j+1} = ... = u_{j+m} = v_{i_1} = v_{i_2} = ... = v_{i_k}$, $i_1 < i_2 < ... < i_k$, in $\pi$ then we translate $v_{i_k} \to u_j$, $v_{i_{k-1}} \to u_j^{(1)}$, ..., $u_j \to u_j^{(m+k-1)}$.

For example,

$$\pi = \begin{pmatrix} 5 & 5 & 4 & 3 & 3 & 3 \\ 4 & 2 & 1 & 2 & 2 & 1 \end{pmatrix} \leftrightarrow \tau = \begin{pmatrix} 5^{(1)} & 5 & 4^{(1)} & 3^{(2)} & 3^{(1)} & 3 \\ 4 & 2^{(2)} & 1^{(1)} & 2^{(1)} & 2 & 1 \end{pmatrix}$$

The translation from $\pi$ to $\tau$ is bijective and is called **standardization of** $\pi$. It is denoted by $\tau = \varphi(\pi)$. We know that $\tau$ has the properties of the generalized biword on the new alphabet $N \cup \{j^{(h)} : j, h \in N\}$.

Then, we represent $\tau = \varphi(\pi)$ in the place of $\pi$ in the part $\{0, 1, 2, ..., n\} \times \{0, 1, ..., n\}$ of the Cartesian plane as follows:

- Define a map $\Psi :$ abscissas $x$ $(x = 0, 1, 2, ..., n) \to \{$ the greatest element of $\hat{\pi}+1\} \cup \hat{\tau}$ with

$$\Psi(x) = \begin{cases} \text{the greatest element of } \hat{\pi}+1 & \text{if } x = 0 \\ x^{th} \text{ greatest element of } \hat{\tau} & \text{else} \end{cases}$$

- Define a map $\Gamma :$ abscissas $y$ $(y = 0, 1, 2, ..., n) \to \{0\} \cup \check{\tau}$ with

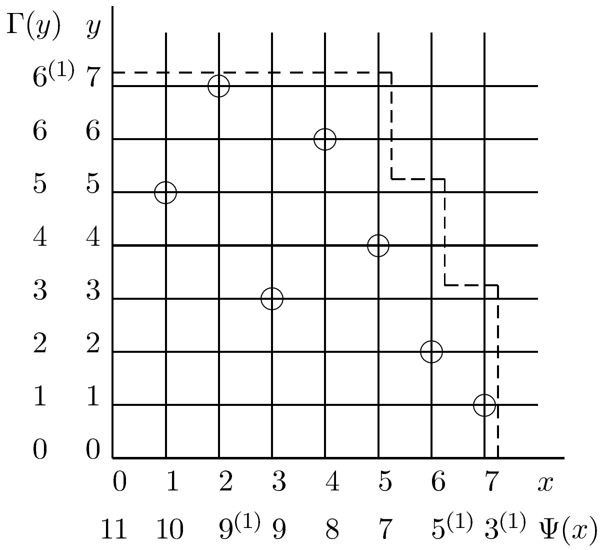
$$\Gamma(x) = \begin{cases} 0 & \text{if } y = 0 \\ y^{th} \text{ lowest element of } \check{\tau} & \text{else} \end{cases}$$

- We define valid domain as the set of points $(x, y)$ such that $\Psi(x) \geq \Gamma(y)$.
- For each pair $(u_k, v_k)$ of $\tau$, we set up the point $(\Psi^{-1}(u_k), \Gamma^{-1}(v_k))$ which is in the valid domain.

Example 4.1. For a generalized biword $\pi$ of $GB_{14}$ we obtain $\tau = \varphi(\pi)$ by the standardization of $\pi$ :

$$\pi = \begin{pmatrix} 10 & 9 & 9 & 8 & 7 & 5 & 3 \\ 5 & 6 & 3 & 6 & 4 & 2 & 1 \end{pmatrix} \to \tau = \begin{pmatrix} 10 & 9^{(1)} & 9 & 8 & 7 & 5^{(1)} & 3^{(1)} \\ 5 & 6^{(1)} & 3 & 6 & 4 & 2 & 1 \end{pmatrix}.$$

Here we give the representation of $\tau$. The dashed line describes the limit of valid domain, which is slightly extended on the figure for readability.




Description of $\tau = \varphi(\pi)$

Figure 4.1

From the Figure 4.1, we construct an oscillating semi-standard tableaux $T = \{T_0 = \emptyset, T_1, T_2, ..., T_{2n} = \emptyset\}$ as follows:

(a) let $A = \{\Psi(x_i)\}_{i=1..n}$ and $B = \{\Gamma(y_i)\}_{i=0..n}$. $I = \{i_0, ...., i_{2n}\}$ describes $A$ and $B$ lined up in nondecreasing order.

(b) For $k$ from $2n$ to 1:
if $i_k \in A$ and $(\Psi^{-1}(i_k), y)$ is SW-corner of a Shadow line, then $T_{k-1} = ExtI(T_k, \Gamma(y))$.
else if $i_k \in B$ and $(x, \Gamma^{-1}(i_k))$ is SW-corner of a Shadow line, then $T_{k-1} = T_k - (u, v)$ with $T_k(u, v) = i_k$.

Figure 4.2 is the skew oscillating semi-standard tableaux corresponding to $\tau$ in Example 4.1. If $\Phi(\emptyset, \emptyset, \pi) = (P, J)$ in theorem 1, then we know that $(P, J)$ is exactly equal to $(T, I)$ if the exponents of contents of $T$ and $I$ are removed, resulting in the following theorem.

**Theorem 3.** *Let $\pi$ be a generalized biword and $\tau = \varphi(\tau)$ and $D(\tau)$ be a description of $\tau$ with shadow lines in a Cartesian plane. Then there is a bijection from (T,I) of $\Theta_{2n}(\emptyset \to \emptyset) \times I(T)$*

*to $D(\tau)$ of $\{D(\tau)|\tau = \varphi(\pi), \pi \in GB_{2n}\}$. If $\Phi(\emptyset, \emptyset, \pi) = (P, J)$, then $(P, J)$ is equal to $(T, I)$ in removing the exponents of contents of $T$ and $I$.*

**Definition 1.** *The shadow $S(\tau)$ of a generalized biword $\tau$ is the set of points $(x, y)$ such that there is a point $(x', y')$ of the representation of $\tau$ with $x' \le x$, $y' \le y$.*

Shadow lines of $\tau$ are defined recursively. The first shadow line $L_1$ of $\tau$ is the boundary of $S(\tau)$. To construct the shadow line $L_{i+1}$ of $\tau$ remove the points of the representation of $\tau$ lying on $L_i$ and construct the shadrow line of the remaining points. This procedure ends when there is no remaining point on the plane. The $SW$-coners of a shadow line are the points of the representation of $\tau$ located on this line ([8],[13]). The $NE$-coners of a shadow line are the points $(x, y)$ of the shadow line such that $(x+1, y)$ and $(x, y+1)$ are not a part of this shadow line.

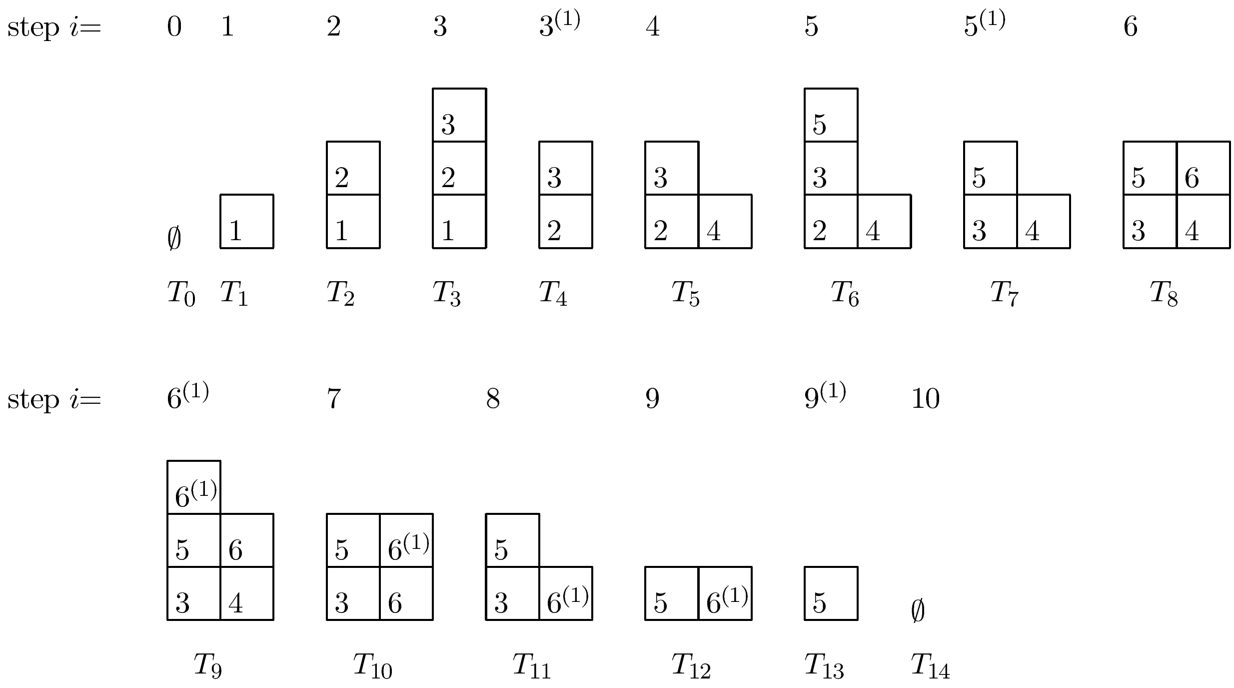


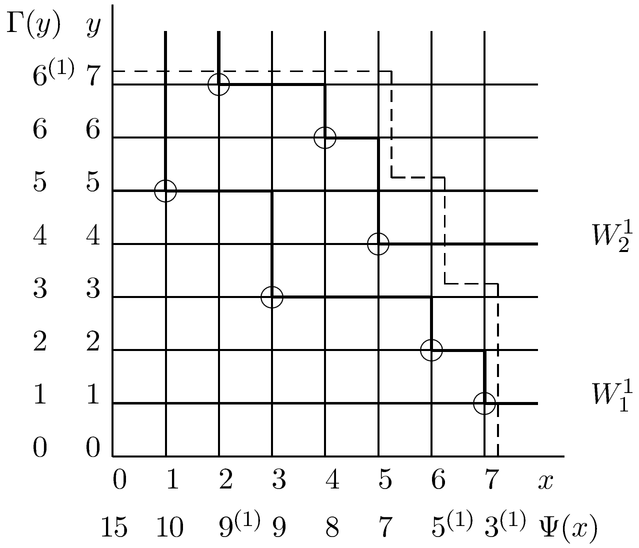

Figure 4.2

**Definition 2.** *The $k^{th}$ skeleton of a generalized biword defined recursively by*

*1. $\tau^{(1)} = \tau$*

*2. $\tau^{(k+1)} = \begin{pmatrix} \Psi(a_1) & \Psi(a_2) & \dots & \Psi(a_m) \\ \Gamma(b_1) & \Gamma(b_2) & \dots & \Gamma(b_n) \end{pmatrix}$ where $(a_1, b_1), \dots, (a_m, b_m)$ are the NE-corners of $\tau^{(k)}$. The shadow diagram of $\tau$ is the set of shadow lines of all the skeletons $\tau^{(k)}$ of $\tau$. The shadow lines of $\tau^{(k)}$ are denoted by $W_j^{(k)}$.*

Example 4.2. Let $\pi$ be the generalized biword of size $2n$ defined in example 4.1. Here we have the description of shadow lines $W_j^{(1)}$, $j = 1, 2$, of $\tau = \varphi(\pi)$.




Description of generalized biword $\tau = \varphi(\pi)$ with shadow lines

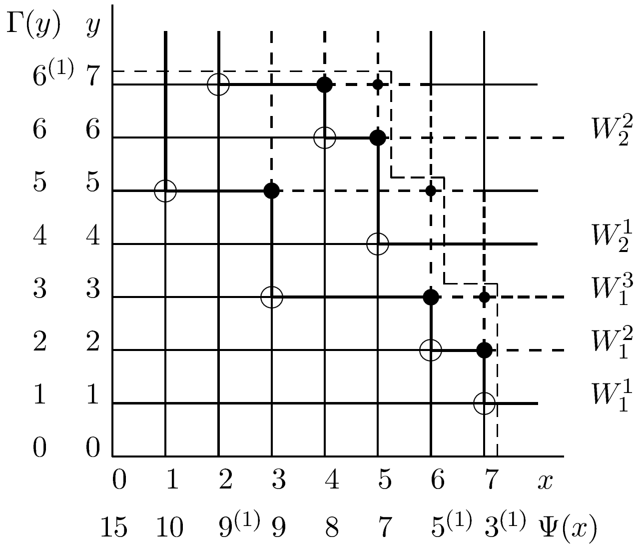
Figure 4.3

We can see that the shadow line $W_1^1$ in Figure 4.3 describes the behaviour of the first cell of the first row during the construction of $T_{14}, T_{13}, ..., T_0$. The shadow line $W_1^1$ has four SW-corners at (1,5), (3,3), (6,2) and (7,1). For the SW-corner (1,5), with $\Psi(1) = 10$ and $\Gamma(5) = 5$, followed by (3,3) with $\Psi(3) = 9$ and $\Gamma(3) = 3$. During the construction of the tableaux $T_{14}$ to $T_0$, the first cell of first row is created during step 10 with label 5, this label is replaced during step 9 by the label 3. The label 3 is replaced during step $5^{(1)}$ by the label 2 and during step $3^{(1)}$ by the label 1, because $\Psi(6) = 5^{(1)}$ and $\Gamma(2) = 2$, $\Psi(7) = 3^{(1)}$ and $\Gamma(1) = 1$. The cell is deleted during step 1.

In the same way the shadow line $W_j^i$ describes the behaviour of the $j^{th}$ cell of the $i^{th}$ row. So the theorem 4.1 in [1] is satisfied for a generalized biword and an oscillating semi-standard tableau as follows.

**Theorem 4.** *Let $\pi$ be a generalized biword of size $2n$ and $\tau$ the standardization of $\pi$, i.e. $\tau = \varphi(\pi)$. If $\Phi(\emptyset, \emptyset, \tau) = (T, I)$ then the shadow line $W_j^{(i)}$ of $\tau$ describes the behavior of the $j^{th}$ cell of the $i^th$ row of the tableaux $T_{2n}, ..., T_0$ in the following way :*

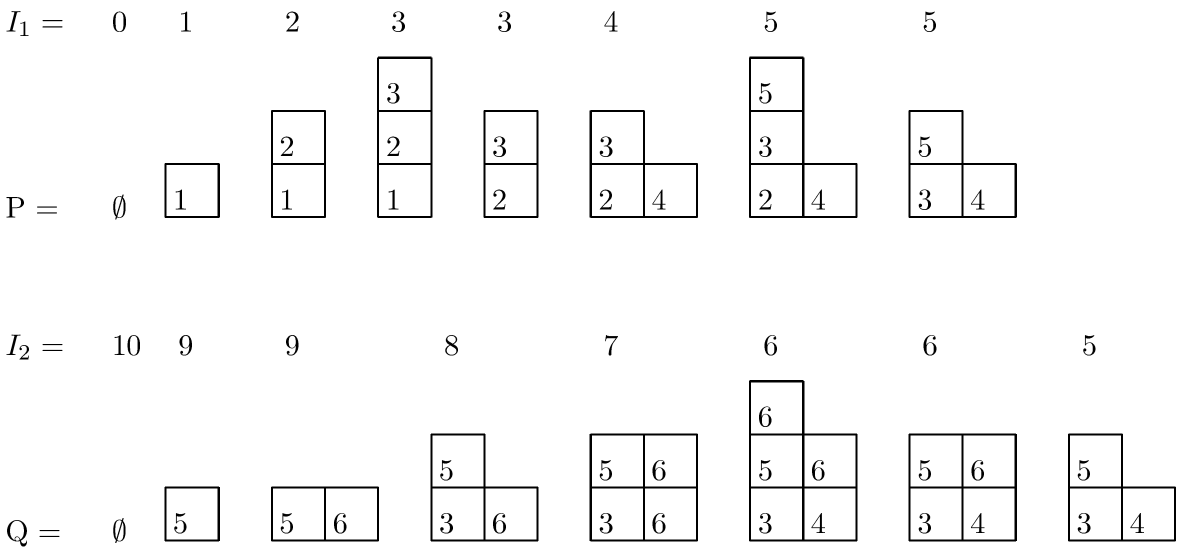
1. *a SW-corner $(x, y)$ indicates that during step $\Psi(x)$, the label $\Gamma(y)$ fills in this cell,*
2. *when the line leaves the valid domain at $(x, y)$, this cell is deleted during step $\Gamma(y)$,*
3. *otherwise, the cell remains unchanged.*




Description of a biword $\pi$

Figure 4.4

From Figure 4.2 the generalized biword $\pi$ in Example 4.1 is in bijection with the following pairs $((P, I_1), (Q, I_2))$, where $(P, I_1) \in (\Theta_n(\emptyset \to \beta), I(P))$ and $(Q, I_2) \in (\overline{\Theta}_n(\emptyset \to \beta), \overline{I}(Q))$:

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