PARTITIONS AND COMPOSITIONS DEFINED BY (IN)EQUALITIES (EXTENDED ABSTRACT)

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ABSTRACT. We consider sequences of integers $(\lambda_1, \ldots, \lambda_k)$ defined by a system of linear inequalities with integer coefficients. We show that when the constraints are strong enough to guarantee that all λ_i are nonnegative, the weight generating function for the integer solutions has a finite product form $\prod_i (1 - q^{b_i})^{-1}$, where the b_i are positive integers that can be computed from the coefficients of the inequalities. The results are proved bijectively and are used to give several examples of interesting identities for integer partitions and compositions. The method can be adapted to accomodate equalities along with inequalities and can be used to obtain multivariate forms of the generating function. Our initial results were conjectured thanks to the Omega package [6].

We generalize the method to handle special cases with rational coefficients (including lecture hall partitions) and obtain new identities involving partitions and compositions defined by the ratio of consecutive parts. In particular, we obtain a surprising result about "anti-lecture hall" compositions.

RÉSUMÉ. Nous considérons des suites d'entiers $(\lambda_1, \ldots, \lambda_k)$ définies par un système d'inégalités linéaires à coefficients entiers. Nous montrons que si les solutions du système sont toujours des suites d'entiers positifs ou nuls alors la série génératrice des solutions selon leur poids est le produit $\prod_{i=0}^{k-1}(1-q^{b_i})^{-1}$. Les b_i sont des entiers positifs que l'on calcule à partir du système. Les résultats sont démontrés bijectivement et donnent des identités intéressantes pour les partitions et les compositions. La méthode peut être adaptée aux systèmes d'(in)égalités et permet aussi d'obtenir des séries génératrices multivariées. Les résultats initiaux ont été conjecturés grâce au package Omega [6].

Nous généralisons la méthode pour manipuler des cas avec des coefficients rationnels (comme les lecture hall partitions) et obtenons des nouvelles identités pour les partitions et compositions définies par le rapport entre parts successives. En particulier nous obtenons un résultat surprenant pour les "anti-lecture hall" compositions.

1. INTRODUCTION

For a sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ of integers, define the *weight* of λ to be $\lambda_1 + \cdots + \lambda_k$ and call each λ_i a *part* of λ. If a sequence λ of weight n has all parts nonnegative, we call it a *composition* of n into k nonnegative parts and if, in addition, λ is a nonincreasing sequence, we call it a *partition* of n into at most k parts. In the remainder of the paper we will consider that $\lambda_i = 0$ if $i < 0$ or $i > k$.

In this paper we want to study partitions and compositions into k nonnegative parts defined by equalities and inequalities. This work was motivated by results of the form :

- Given a positive integer r, the partitions $\lambda = (\lambda_1, \ldots, \lambda_k)$ of n which satisfy $\lambda_i \geq r\lambda_{i+1}$ for $1 \leq i \leq k$ have weight generating function $\prod_{i=0}^{k-1}\left(1 - q^{1+r+\cdots+r^i}\right)^{-1}$ [14].
- Given a positive integer r the weight generating function of the partitions λ with at most k parts and $\lambda_i \geq \sum_{j=1}^{r}(-1)^{j+1}\binom{r}{j}\lambda_{j+i}$, $1 \leq i < k$ is : $\prod_{i=0}^{k-1}\left(1 - q^{\binom{i+r}{r}}\right)^{-1}$. See [2, 13, 18].

- The weight generating function of the partitions λ with at most k parts and $\lambda_i/(k-i+1) \geq \lambda_{i+1}/(k-i)$, $1 \leq i < k$ is : $\prod_{i=0}^{k-1}\left(1-q^{2i+1}\right)^{-1}$. See the *Lecture Hall Theorem* in [10].

More generally, we consider integer sequences λ of length k satisfying $\lambda_i \geq \sum_{j=1}^{k-i} a_{ij}\lambda_{i+j}$ where the a_{ij} guarantee that all $\lambda_i \geq 0$. We show in Section 2 that when the a_{ij} are all integers, the weight generating function for these compositions is $\prod_{i=0}^{k-1}(1-q^{b_i})^{-1}$, where the $b = (b_0, \ldots, b_{k-1})$ is a sequence of positive integers that may be readily derived from the a_{ij}. Several generalizations and a linear algebra proof are included.

In Section 3, we consider rational coefficients a_{ij}. We show how to use the results of Section 2 to give an explicit form for the generating function for *any* set of compositions defined by the ratio of consecutive parts:

$$\lambda_1 \geq \frac{n_1}{d_1}\lambda_2; \quad \lambda_2 \geq \frac{n_2}{d_2}\lambda_3; \quad \lambda_3 \geq \frac{n_3}{d_3}\lambda_4; \quad \ldots; \quad \lambda_{k-1} \geq \frac{n_{k-1}}{d_{k-1}}\lambda_k; \quad \lambda_k \geq 0,$$

This result has been implemented in Maple and our experiments have led to several interesting results. We focus here on some related to the Lecture Hall Theorem.

In [10], Bousquet-Mélou and Eriksson considered the set L_k of partitions λ into at most k parts satisfying $\lambda_i/(k-i+1) \geq \lambda_{i+1}/(k-i)$, for $1 \leq i < k$, and proved the following *Lecture Hall Theorem*:

$$\sum_{\lambda \in L_k} q^{|\lambda|} = \prod_{i=0}^{k-1} \frac{1}{1-q^{2i+1}}. \tag{1}$$

This result was generalized in [11] to an (m,l)-*Lecture Hall Theorem* $(m, l \geq 2)$ for partitions into at most k parts satisfying $\lambda_i/a_{k-i+1} \geq \lambda_{i+1}/a_{k-i}$, for $1 \leq i < k$, where $\{a_i\}$ is the (m,l)-sequence defined by:

$$a_{2i} = la_{2i-1} - a_{2i-2}; \qquad a_{2i-1} = ma_{2i-2} - a_{2i-3}, \quad i \geq 2,$$

with the initial conditions $a_1 = 1$ and $a_2 = l$, $m, l \geq 2$. A different approach in [12] led to the *Refined Lecture Hall Theorem* (setting $u = v = 1$ gives (1)) :

$$\sum_{\lambda \in L_k} q^{|\lambda|} u^{|\lceil\lambda\rceil|} v^{o(\lceil\lambda\rceil)} = \prod_{i=1}^{k} \frac{1+uvq^i}{1-u^2q^{k+i}},$$

with $\lceil\lambda\rceil = (\lceil\lambda_1/(k-i+1)\rceil, \lceil\lambda_2/(k-i)\rceil, \ldots, \lceil\lambda_k/k\rceil)$ and $o(\lambda)$ is the number of odd parts in λ.

In Section 4, we show a slight generalization of (1) in which the constraints on λ_1 can be modified. In a footnote in [11], Bousquet-Mélou and Erikson note that their proof of the (m,l)-Lecture Hall Theorem can be simplified. In Section 5, we shall describe the resulting short and elegant proof of the (m,l)-Lecture Hall Theorem and show that it can be generalized to compositions when $m = 1$ and $l > 3$ or $m > 3$ and $l = 1$.

So, for one example, compositions satisfying,

$$\frac{\lambda_1}{12} \geq \frac{\lambda_2}{5} \geq \frac{\lambda_3}{8} \geq \frac{\lambda_4}{3} \geq \frac{\lambda_5}{4} \geq \frac{\lambda_6}{1}$$

have generating function

$$[(1-q)(1-q^4)(1-q^5)(1-q^7)(1-q^{11})(1-q^{17})]^{-1}.$$

Finally, in Section 6, we prove another new result, the *Anti-Lecture Hall Theorem* for the set A_k of compositions into at most k parts satisfying $\lambda_i/i \geq \lambda_{i+1}/(i+1)$, for $1 \leq i < k$:

$$\sum_{\lambda \in A_k} q^{|\lambda|} = \prod_{i=1}^{k} \frac{1+q^i}{1-q^{i+1}}. \tag{2}$$

In fact, we prove the following refinement of (2):

$$\sum_{\lambda \in A_k} q^{|\lambda|} u^{|\lfloor\lambda\rfloor|} v^{o(\lfloor\lambda\rfloor)} = \prod_{i=1}^{k} \frac{1+uvq^i}{1-u^2q^{i+1}}.$$

where $\lfloor\lambda\rfloor = (\lfloor\lambda_1/1\rfloor, \lfloor\lambda_2/2\rfloor, \ldots, \lfloor\lambda_k/k\rfloor)$. The bijective proof we give follows the idea of Yee's beautiful proof [17] of the Refined Lecture Hall Theorem.

Because of space constraints we give only sketches of the proofs in this extended abstract.

Our initial results were conjectured thanks to experiments with the Omega package [6, 4], a Mathematica implementation of the Omega operator defined by MacMahon [15] as:

$$\underset{\geq}{\Omega} \sum_{s_1=-\infty}^{\infty} \cdots \sum_{s_r=-\infty}^{\infty} A_{s_1,\ldots,s_r} x_1^{s_1} \ldots x_r^{s_r} = \sum_{s_1=0}^{\infty} \cdots \sum_{s_r=0}^{\infty} A_{s_1,\ldots,s_r}.$$

This operator was then not used for 85 years except by Stanley in 1973 [16]. A few years ago Andrews revived this operator [1, 2] and used it in [1] to give a second proof of the Lecture Hall Theorem. In conjunction with Paule and Riese, he implemented the operator in the Omega package and together they have continued to identify the power of the Omega operator for such combinatorial problems as magic squares [9], hypergeometric multisums [3], constrained compositions [8], plane partitions diamonds [5], and k-gons partitions [7].

2. Integer coefficients

Let $A[1...k-1, 1..k-1]$ be an upper diagonal matrix of integers and let P_A be the set of sequences $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ satisfying

$$\begin{aligned} \lambda_1 &= \sum_{j=1}^{k-1} A[1,j]\lambda_{j+1} \\ \lambda_i &\geq \sum_{j=i}^{k-1} A[i,j]\lambda_{j+1} \quad \text{for } 2 \leq i \leq k-1 \\ \lambda_k &\geq 0. \end{aligned} \tag{3}$$

Define a sequence of matrices $A^{(t)}, 1 \leq t \leq k-1$, as follows: $A^{(1)} = A$ and $A^{(t)}$ is a matrix of dimension $(k-t) \times (k-t)$, whose entries are defined by the recurrence :

$$\begin{aligned} A^{(t)}[1,j] &= (A^{(t-1)}[1,1]+1)A^{(t-1)}[2,j+1] + A^{(t-1)}[1,j+1] \\ A^{(t)}[i,j] &= A^{(t-1)}[i+1,j+1] \ \text{ if } 2 \leq i \leq k-t\ . \end{aligned}$$

Then each matrix $A^{(t)}$ is an upper diagonal matrix of integers and we can show the following.

Lemma 1. *If every element of P_A is a composition, then the same is true of $P_{A^{(t)}}$ for $1 \leq t \leq k-1$.*

The main result of this section is the following.

Theorem 1. *Let $P_A(k)$ be the set of sequences $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ satisfying (3). Let $P_A(n,k)$ be the set of sequences $P_A(k)$ of weight n. If every $\lambda \in P_A$ is a composition, then the weight generating function for $P_A(k)$ is*

$$\sum_{n=0}^{\infty} |P_A(n,k)| q^n = \prod_{i=1}^{k-1} \frac{1}{1-q^{b_i}}, \tag{4}$$

where $b_i = A^{(i)}[1,1] + 1 > 0$.

Sketch of proof. The mapping

$$\Theta(\lambda) = (b_1 s_1, b_2 s_2, \ldots, b_{k-1} s_{k-1})$$

where

$$s_i = \lambda_{i+1} - \sum_{j=i+1}^{k-1} A[i,j]\lambda_{j+1}$$

can be shown to be a bijection from $P_A(n,k)$ to the set of sequences $r_1, r_2, \ldots, r_{k-1}$ of weight n in which r_i is a nonnegative multiple of b_i. In the case that the b_i are distinct positive integers, this can be viewed alternatively as a bijection with partitions of n into parts in $\{b_1, \ldots, b_{k-1}\}$. □

Example 1. The partitions λ of n with at most k parts and with $\lambda_1 = \sum_{i=1}^{k-1} \lambda_{i+1}$ comprise $P_A(n,k)$ where, for $1 \le i \le j \le k-1$, $A[i,j] = 1$ if $i = 1$ or $i = j$ and $A[i,j] = 0$ otherwise. Then $b_i = 2i$ for $1 \le i \le k-1$ so by Theorem 1, the generating function is $\prod_{i=1}^{k-1}(1-q^{2i})^{-1}$.

Example 2. There is a one-to-one correspondence between sequences $\lambda_1, \ldots, \lambda_k$ of weight n satisfying

$$\begin{aligned}
\sum_{j=0}^{k-1} (-1)^j \lambda_{1+j} &= 0 \\
\sum_{j=0}^{k-i} (-1)^j \lambda_{i+j} &\ge 0 \qquad \text{for } 2 \le i \le k
\end{aligned}$$

and the set of compositions of n into $k-1$ even parts. The λ satisfying the constraints are compositions and they comprise the set $P_A(n,k)$ where A is the $(k-1) \times (k-1)$ upper diagonal matrix defined by $A[i,j] = (-1)^{i+j}$ which has $b_1 = b_2 = \cdots = b_{k-1} = 2$.

Corollary 1. *Under the constraints of Theorem 1, if we now allow $\lambda_1 \ge \sum_{j=1}^{k-1} A[1,j]\lambda_{j+1}$ the generating function becomes*

$$\prod_{i=0}^{k-1} \frac{1}{1-q^{b_i}}$$

where $b_0 = 1$.

Linear algebra approach. Our method of proof makes the bijection of Theorem 1 explicit. However, as suggested by one of the referees, a linear algebra proof can also be illuminating. Following the argument of the referee, we embed the constraint matrix, A, into a $k \times k$ matrix, B, where $B[i,j] = 0$ if $i \ge j$ and $B[i,j] = A[i, j-1]$ otherwise. Hence B is strictly upper diagonal and thus, nilpotent : $B^k = 0$. Considering λ as a column vector, the matrix inequality, $\lambda \ge B\lambda$, describes the system (3), with the first constraint changed to inequality. Then

$$\lambda(I - B) = [s_0, s_1, \ldots s_{k-1}]^T$$

for the s_i defined in the proof of Theorem 1. Iterating this identity yields

$$\begin{aligned}\lambda &= (1+B+B^2+\cdots+B^{k-1})[s_0,s_1,\ldots s_{k-1}]^T\\ &= (I-B)^{-1}[s_0,s_1,\ldots s_{k-1}]^T,\end{aligned}$$

since $B^k=0$. So, $I-B$ is invertible, and, in particular, it is an upper diagonal matrix of *integers* whenever B, and therefore, A, is. Hence the recurrences of $A^{(t)}$ are equivalent to taking the inverse of a matrix. Let $C=(I-B)^{-1}$. Then, furthermore, all the elements of P_A are compositions iff all the entries in C are nonnegative. Let $h=(1,1,\ldots,1)$ be the row vector of length k containing only ones. The weight of the composition λ is $|\lambda| = h\lambda = hC[s_0,s_1,\ldots s_{k-1}]^T$. Define $p=hC$. Then the weight generating function of P_A is

$$\sum_{\lambda\in P_A} q^{\lambda_1+\cdots+\lambda_k} = \sum_{s_0,\ldots,s_{k-1}\geq 0} q^{p_1s_0+\cdots+p_ks_{k-1}} = \prod_{j=1}^{k}\frac{1}{1-q^{p_j}}, \tag{5}$$

where p_j (corresponding to b_{j-1}) is $p_j = C[1,j]+C[2,j]+\cdots+C[j,j]$. □

Example 3. The weight generating function of the partitions λ with at most k parts and with $\lambda_1 \geq 2\lambda_2+\sum_{i=2}^{k-1}\lambda_{i+1}$ and $\lambda_2\geq\sum_{i=2}^{k-1}\lambda_{i+1}$ is :

$$\frac{1}{(1-q)(1-q^3)}\prod_{i=1}^{k-2}\frac{1}{1-q^{5i}}.$$

This follows from Corollary 1: since λ is a partition, for $i>2$, $A[i,i]=1$ and $A[i,j]=0$ for $2<j<i$. Also, $A[1,1]=2$ and $A[i,j]=1$ for $i=1,2$ and $j\geq 2$, so $b_0=1$, $b_1=3$, and $b_i=5(i-1)$ for $2\leq i\leq k-1$.

Example 4. The generating function of the partitions λ with at most k parts and $\lambda_i \geq i\sum_{j=i+1}^{k}\lambda_j$, $1\leq i\leq k$, is : $\prod_{i=1}^{k}(1-q^{i!})^{-1}$.

We can generalize Theorem 1 to allow the constraints of the matrix A to be satisfied with equality for any specified set of λ_i. Given a set $S\subseteq\{0,1,\ldots,k-1\}$, let $P_A(k;S)$ be the set of sequences $\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_k)$ satisfying, for $1\leq i\leq k$:

$$\lambda_i = \sum_{j=i}^{k-1} A[i,j]\lambda_{j+1} \quad \text{if } i-1\in S$$

$$\lambda_i \geq \sum_{j=i}^{k-1} A[i,j]\lambda_{j+1} \quad \text{if } i-1\notin S.$$

Let $P_A(n,k;S)$ be the set of sequences in $P_A(k;S)$ of weight n.

Corollary 2. *If all elements of P_A are compositions, the weight generating function for $P_A(k;S)$ is*

$$\sum_{n=0}^{\infty}|P_A(n,k;S)|q^n = \prod_{i=0,i\notin S}^{k-1}\frac{1}{1-q^{b_i}},$$

where $b_0=1$ and for $i\geq 1$, $b_i=A^{(i)}[1,1]+1$. □

As noted by the referee, the inverse problem can be solved completely in the domain of linear algebra if $b_1=1$. Given a sequence $(b_1,\ldots,b_k)$ of positive integers, such that $b_1=1$, you can always construct an upper triangular matrix C with ones on the diagonal and nonnegative entries such that the sum of the entries in the jth column is b_j. Then the matrix $B=1-C^{-1}$ contains in its northeast corner a $(k-1)\times(k-1)$ constraint matrix A such that P_A has generating function $\prod_{i=1}^{k}(1-q^{b_i})^{-1}$.

3. Rational Coefficients

In this section we would like to generalize our results to allow some of the elements of the constraint matrix to be rational. In particular, we will find an explicit form of the generating function for the set of integer sequences $\lambda_1, \ldots, \lambda_k$ satisfying the constraints:
(6)

$$\lambda_1 \geq c_1\lceil\frac{n_1}{d_1}\lambda_2\rceil + \sum_{i=2}^{k} c_i\lambda_i; \quad \lambda_2 \geq \frac{n_2}{d_2}\lambda_3; \quad \lambda_3 \geq \frac{n_3}{d_3}\lambda_4; \quad \ldots; \quad \lambda_{k-1} \geq \frac{n_{k-1}}{d_{k-1}}\lambda_k; \quad \lambda_k \geq 0,$$

where for $1 \leq i \leq k-1$, n_i and d_i are positive integers and the c_i are any integers which make the first constraint strong enough to guarantee that $\lambda_1 \geq 0$.

For $i = 1, \ldots, k$, let $a_i = \prod_{j=1}^{i-1} d_j \prod_{t=i}^{k-1} n_t$. Then $a_i/a_{i+1} = n_i/d_i$, so the system (6) above is equivalent to:

$$\lambda_1 \geq c_1\lceil\frac{a_1}{a_2}\lambda_2\rceil + \sum_{i=2}^{k} c_i\lambda_i \quad \text{and} \tag{7}$$

$$\lambda_2/a_2 \geq \lambda_3/a_3 \geq \cdots \geq \lambda_{k-1}/a_{k-1} \geq \lambda_k/a_k.$$

Theorem 2. *Given a sequence of positive integers $a_1, \ldots, a_k$ and a sequence of integers $c_1, \ldots, c_k$, consider the set of sequences $\lambda_1, \ldots, \lambda_k$ satisfying (7). As long as the c_i are integers which guarantee that $\lambda_1 \geq 0$, the weight generating function is*

$$\frac{\sum_{z_2=0}^{a_2-1}\sum_{z_3=0}^{a_3-1}\cdots\sum_{z_k=0}^{a_k-1} q^{c_1\lceil\frac{a_1z_2}{a_2}\rceil + \sum_{i=2}^{k}(c_i+1)z_i} \prod_{i=1}^{k-2} q^{b_i\lceil\frac{z_{i+2}}{a_{i+2}} - \frac{z_{i+1}}{a_{i+1}}\rceil}}{\prod_{i=0}^{k-1}(1-q^{b_i})}, \tag{8}$$

where $b_0 = 1$, $b_1 = c_1a_1$ and $b_i = c_1a_1 + (c_2+1)a_2 + \cdots + (c_{i+1}+1)a_{i+1}$, for $2 \leq i \leq k-1$.

Sketch of proof. For $2 \leq i \leq k$, let $\lambda_i = a_ix_i + z_i$, where $x_i \geq 0$ and $0 \leq z_i < a_i$. Rearrange the sequence $\lambda_1, \lambda_2, \ldots, \lambda_k$ by decreasing part λ_i by $(a_i-1)x_i + z_i$ and increasing λ_1 by the same amount for $2 \leq i \leq k$ to get a new sequence $x_1, x_2, \ldots, x_k$, of the same weight satisfying

$$\begin{aligned} x_1 &\geq (c_1a_1 + (c_2+1)a_2 - 1)x_2 + \sum_{i=3}^{k}((c_i+1)a_i - 1)x_i + s_0 \\ x_i &\geq x_{i+1} + s_{i-1} \quad \text{for } 2 \leq i \leq k-1 \\ x_k &\geq 0 \end{aligned}$$

where

$$s_0 = c_1\lceil a_1z_2/a_2\rceil + \sum_{i=2}^{k}(c_i+1)z_i,$$

and

$$s_i = \lceil z_{i+2}/a_{i+2} - z_{i+1}/a_{i+1}\rceil$$

for $1 \leq i \leq k-2$.. By the method of Theorem 1, we can show that the generating function for fixed $s_0, s_1, \ldots s_{k-2}$ is

$$\frac{\prod_{i=0}^{k-2} q^{b_is_i}}{\prod_{i=0}^{k-1}(1-q^{b_i})},$$

where $b_0 = 1$, $b_1 = c_1a_1$, and $b_i = c_1a_1 + (c_2+1)a_2 + (c_3+1)a_3 + \cdots + (c_{i+1}+1)a_{i+1}$, for $2 \leq i \leq k-1$. Summing over all possible sequences $s_0, s_1, \ldots s_{k-2}$ as the z_i vary independently from 0 to $a_i - 1$ gives the result. □

Example 5. By Theorem 2, sequences $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ satisfying $\lambda_1 \geq 2\lambda_2 - \lambda_3 + 2\lambda_4$ and $\lambda_2/3 \geq \lambda_3/2 \geq \lambda_4/1 \geq 0$ have generating function

$$\frac{1+q^3+q^6+q^9+q^{12}+q^6}{(1-q)(1-q^9)(1-q^9)(1-q^{12})} = \frac{(1+q^3+q^6)(1+q^6)}{(1-q)(1-q^9)(1-q^9)(1-q^{12})} = \frac{1}{(1-q)(1-q^3)(1-q^6)(1-q^9)}.$$

In some cases, the numerator of the generating function of Theorem 2 can be easily shown to factor, giving results such as the following.

Corollary 3. *Suppose the sequence $a_1, a_2, \ldots, a_k$ has the property that for $1 \leq i \leq k-1$ if $a_i > 1$, then $a_{i+1} = 1$. Then there is a one-to-one correspondence between the compositions $\lambda_1, \ldots, \lambda_k$ of n satisfying*

$$\lambda_1/a_1 \geq \lambda_2/a_2 \geq \lambda_3/a_3 \geq \cdots \geq \lambda_{k-1}/a_{k-1} \geq \lambda_k/a_k$$

and the partitions of n into parts in

$$\{1, b_1, b_1+1, \ldots, b_{k-1}\},$$

where $b_0 = 1$, $b_1 = a_1 + a_2$ and $b_{i+1} = b_i + a_{i+2}$ for $1 \leq i \leq k-2$, such that at most one part can appear from each of the sets

$$S_i = \{p | b_i + 1 \leq p \leq b_{i+1} - 1\}.$$

Proof. The generating function (8) becomes

$$\frac{\prod_{i=0: b_{i+1}-b_i>1}^{k-2} (1 + q^{b_i+1} + q^{b_i+2} + \cdots + q^{b_{i+1}-1})}{\prod_{i=0}^{k-1} (1 - q^{b_i})}.$$

□

Example 6. By Corollary 4, compositions of n satisfying

$$\lambda_1 \geq \lambda_2 \geq \lambda_3/2 \geq \lambda_4 \geq \lambda_5/2 \geq \cdots \geq \lambda_{2k} \geq \lambda_{2k+1}/2$$

are in one-to-one correspondence with the set of partitions of n into parts of size at most $3k+1$ in which parts divisible by 3 can appear at most once.

In some cases, the numerator of the generating function of Theorem 2 does factor, but it is not as easily shown. We consider some examples of this type in Section 5 on lecture hall compositions and in Section 6 on anti-lecture hall compositions.

4. Two Variable Generating Functions

In their study of lecture hall partitions, Bousquet-Mélou and Eriksson found it very useful to consider the 2-variable (odd/even weighted) generating function of the set of partitions satisfying the lecture hall constraints. We show here how our method can be adapted to get multivariable generating functions for compositions satisfying linear constraints, using the two-variable case as an example.

Let $A[1..k-1, 1..k-1]$ be an upper diagonal matrix of integers such that P_A is a set of compositions. For $\lambda \in P_A$, let $|\lambda|_o = \lambda_1 + \lambda_3 + \lambda_5 + \cdots$ and and $|\lambda|_e = \lambda_2 + \lambda_4 + \lambda_6 + \cdots$. Define two sequences of matrices $O^{(t)}$, and $E^{(t)}$, $1 \leq t \leq k-1$ so that $O^{(t)} + E^{(t)} = A^{(t)}$ as follows. $O^{(t)}$ and $E^{(t)}$ are $(k-t) \times (k-t)$ matrices satisfying

$O^{(1)}[i,j] = A^{(1)}[i,j]$ if i is odd, otherwise, $O^{(1)}[i,j] = 0$;

$E^{(1)}[i,j] = A^{(1)}[i,j]$ if i is even, otherwise, $E^{(1)}[i,j] = 0$;

For $t > 1$,

$$\begin{aligned}
O^{(t)}[1,j] &= (O^{(t-1)}[1,1] + 1 - (t \bmod 2))A^{(t-1)}[2,j+1] + O^{(t-1)}[1,j+1], \\
O^{(t)}[i,j] &= O^{(t-1)}[i+1,j+1] \ \text{ if } i \geq 2\ ; \\
E^{(t)}[1,j] &= (E^{(t-1)}[1,1] + (t \bmod 2))A^{(t-1)}[2,j+1] + E^{(t-1)}[1,j+1], \\
E^{(t)}[i,j] &= E^{(t-1)}[i+1,j+1] \ \text{ if } i \geq 2\ ;
\end{aligned}$$

We can prove the following 2-variable version of Theorem 1.

Theorem 3. *The odd/even weighted generating function for the compositions in P_A is:*

$$\sum_{\lambda \in P_A} x^{|\lambda|_o} y^{|\lambda|_e} = \prod_{i=0}^{k-1} \frac{1}{1 - x^{o_i} y^{e_i}},$$

where sequences $o_0, o_1, \ldots, o_{k-1}$ and $e_0, e_1, \ldots, e_{k-1}$ are defined by $o_0 = 1$; $e_0 = 0$ and for $t > 0$,

$$o_t = O^{(t)}[1,1] + 1 - (t \bmod 2) \qquad e_t = E^{(t)}[1,1] + (t \bmod 2).$$

Note from referee. We can refine Theorem 1 via the enumeration in (5) by associating with λ the monomial $\prod q_i^{\lambda_i}$, is possible :

$$\sum_{\lambda \in P_A} \prod_i q_i^{\lambda_i} = \prod_{j=1}^{k} \frac{1}{1 - \prod_{i \leq j} q_i^{C[i,j]}}.$$

From Theorem 3, imitating the bijective proof of Theorem 2, we can get the odd/even generating function $G_k(x,y)$ for the compositions satisfying the constraints:

$$\frac{\lambda_1}{a_1} \geq \frac{\lambda_2}{a_2} \geq \frac{\lambda_3}{a_3} \geq \cdots \geq \frac{\lambda_{k-1}}{a_{k-1}} \geq \frac{\lambda_k}{a_k} \geq 0, \tag{9}$$

Theorem 4.

$$G_k(x,y) = \frac{\sum_{z_2=0}^{a_2-1} \cdots \sum_{z_k=0}^{a_k-1} x^{\lceil \frac{a_1 z_2}{a_2} \rceil + z_3 + z_5 + \cdots} y^{z_2 + z_4 + \cdots} \prod_{i=1}^{k-2} (x^{o_i} y^{e_i})^{\lceil \frac{z_{i+2}}{a_{i+2}} - \frac{z_{i+1}}{a_{i+1}} \rceil}}{\prod_{i=0}^{k-1} (1 - x^{o_i} y^{e_i})}, \tag{10}$$

where $o_0 = 1$, $e_0 = 0$, and for $1 \leq i \leq k-1$ $o_i = a_1 + a_3 + a_5 + \cdots + a_{2\lfloor (i-1)/2 \rfloor + 1}$ and $e_i = a_2 + a_4 + a_6 + \cdots + a_{2\lceil i/2 \rceil}$.

Proof. Omitted. □

We can now add more conditions on the first part when $a_k \geq a_{k-1} \geq \cdots \geq a_1$:

Theorem 5. *If $G_k(x,y) = H_k(x,y)/(1-x)$ is the generating function given in (10), then whenever $a_1 \geq a_2 \geq \cdots \geq a_k$, for any $l \geq 1$ and $j \geq 2 - l$, $H(q^l, q^j)$ is the generating function for the partitions satisfying*

$$\begin{aligned}
\lambda_1 &= l\lceil a_1 \lambda_2 / a_2 \rceil + (j-1)(\lambda_2 + \lambda_4 + \lambda_6 + \cdots) + (l-1)(\lambda_3 + \lambda_5 + \lambda_7 + \cdots) \\
\frac{\lambda_2}{a_2} &\geq \frac{\lambda_3}{a_3} \geq \cdots \geq \frac{\lambda_{k-1}}{a_{k-1}} \geq \frac{\lambda_k}{a_k} \geq 0.
\end{aligned} \tag{11}$$

Proof. If $j = l = 1$, the system (11) is the same as (9) and the generating function is $H(x,y) = G(x,y)/(1-x)$. Suppose λ satisfies (9), with $|\lambda|_o = l$ and $|\lambda|_e = m$. To transform λ into a composition satisfying (11), we increase the first part by $(l-1)|\lambda|_o + (j-1)|\lambda|_e$ to get λ'. The conditions on j and l and the a_i guarantee that this increase is positive. Then

$|\lambda'| = |\lambda| + (l-1)|\lambda|_o + (j-1)|\lambda|_e = |\lambda|_o + |\lambda|_e + (l-1)|\lambda|_o + (j-1)|\lambda|_e = l|\lambda|_o + j|\lambda|_e$. □

We can use Theorem 5 to generalize the Lecture Hall Partition theorem of Bousquet-Mélou and Eriksson.

Corollary 4. *For $l \geq 1$ and $j \geq 2-l$, the generating function for the sequences $\lambda_1, \ldots, \lambda_k$ satisfying $\lambda_1 \geq l\lceil k\lambda_2/(k-1)\rceil + (j-1)(\lambda_2+\lambda_4+\lambda_6+\cdots) + (l-1)(\lambda_3+\lambda_5+\lambda_7+\cdots)$ and $\frac{\lambda_2}{k-1} \geq \frac{\lambda_3}{k-2} \geq \cdots \geq \frac{\lambda_{k-1}}{2} \geq \frac{\lambda_k}{1}$ is*

$$\frac{1}{(1-q)}\prod_{i=0}^{k-1}\frac{1}{(1-q^{il+ij+l})}$$

Proof. Direct, as $G_k(x,y) = \prod_{i=0}^{k-1}\frac{1}{1-x^{i+1}y^i}$ [10]. □

Example 7. There is a one-to-one correspondence between the compositions of n satisfying $\lambda_1 \geq 3\lceil k\lambda_2/(k-1)\rceil + 2\sum_{j=1}^{k-1}(-1)^j\lambda_{1+j}$ and $\frac{\lambda_2}{k-1} \geq \frac{\lambda_3}{k-2} \geq \cdots \geq \frac{\lambda_{k-1}}{2} \geq \frac{\lambda_k}{1} \geq 0$ and the partitions of n into odd parts less than or equal to $2k+1$.

5. Generalized Lecture Hall Compositions

Given integers m, l, define the (m,l)-sequence $a^{(m,l)} = (a_1, a_2, \ldots)$ by the recurrence

$$a_{2i} = la_{2i-1} - a_{2i-2}, \quad a_{2i-1} = ma_{2i-2} - a_{2i-3}, \quad i \geq 2,$$

with the initial conditions $a_1 = 1$ and $a_2 = l$. Let $L_k^{(m,l)}$ be the set of compositions into k nonnegative parts satisfying, for $1 \leq i < k$,

$$\lambda_i \geq \frac{a_{k-i+1}}{a_{k-i}}\lambda_{i+1}.$$

Note that $a^{(2,2)} = (1,2,3,4,\ldots)$ and $L_k^{(2,2)}$ is the set of lecture hall partitions of [10]. When $m, l > 1$, then $a^{(m,l)}$ is nondecreasing and $L_k^{(m,l)}$ is a set of partitions which we call generalized lecture hall partitions. However, if $m = 1$ and $l > 3$ or $l = 1$ and $m > 3$, the sequence $a^{(m,l)}$ is an infinite, but non-monotone, sequence of positive integers and $L_k^{(m,l)}$ is a set of compositions, but not partitions.

Let $G_k^{(m,l)}(x,y)$ be the odd/even weighted generating function for $L_k^{(m,l)}$, that is

$$G_k^{(m,l)}(x,y) = \sum_{\lambda \in L_k^{(m,l)}} = x^{|\lambda_o|}y^{|\lambda_e|}.$$

Theorem 6 below was proved in [11] for $m, l > 1$. We give a short proof and extend it to generalized lecture hall compositions ($m = 1, l > 3$ or $l = 1, m > 3$). We note that it implies a straightforward bijective proof of the theorem.

Theorem 6. *For integers m, l satisfying $m, l > 1$ or $m = 1$ and $l > 3$, or $l = 1$ and $m > 3$, $G_1^{(m,l)}(x,y) = 1/(1-x)$ and*

$$G_{2k+1}^{(m,l)}(x,y) = G_{2k}^{(m,l)}(x^m y, x^{-1})/(1-x), \quad G_{2k}^{(m,l)}(x,y) = G_{2k-1}^{(m,l)}(x^l y, x^{-1})/(1-x), \quad n > 0.$$

Note that this gives for $k > 0$

$$G_{2k}^{(m,l)}(x,y) = \prod_{i=1}^{2k}\frac{1}{1-x^{a_i}y^{b_{i-1}}} \quad \text{and} \quad G_{2k-1}^{(m,l)}(x,y) = \prod_{i=1}^{2n-1}\frac{1}{1-x^{b_i}y^{a_{i-1}}}$$

where $a_0 = 0$, $(a_1, a_2, \ldots) = a^{(m,l)}$, and $(b_1, b_2, \ldots) = a^{(l,m)}$.

Proof of theorem. We follow the method of Bousquet-Mélou and Eriksson, bypassing the detour to reduced lecture hall partitions, as suggested in a note on page 10 of [11]. We fix m, l and drop the superscripts (m,l) to simplify notation. Let $L_{k+1,=}$ denote the partitions

$\lambda \in L_k$ with $\lambda_1 = \lceil a_k \lambda_2 / a_{k-1} \rceil$. We do this by establishing a bijection ϕ_k between partitions $\lambda \in L_k$ and partitions $\gamma \in L_{k+1,=}$ with $|\gamma_e| = m|\lambda_e| - |\lambda_o|$ if k is even and $|\gamma_e| = l|\lambda_e| - |\lambda_o|$ if k is odd, and with $|\gamma_o| = |\lambda_e|$. The bijection and its reverse are presented below :

$$\begin{array}{|l|}\hline \phi_k \ : \ \lambda \to \gamma \\ \gamma_e \leftarrow \lambda_o \\ \gamma_1 \leftarrow \lceil \frac{a_{k+1}}{a_k} \lambda_1 \rceil \\ \text{For } i \text{ from 1 to } \lfloor (k+1)/2 \rfloor \\ \qquad \gamma_{2i+1} \leftarrow \lceil \frac{a_{k-2i+1}}{a_{k-2i}} \lambda_{2i+1} \rceil + \\ \qquad\qquad \lfloor \frac{a_{k+1-2i}}{a_{k+2-2i}} \lambda_{2i-1} \rfloor - \lambda_{2i}. \\ \hline \end{array}$$

$$\begin{array}{|l|}\hline \phi_k^{(-1)} \ : \ \gamma \to \lambda \\ \lambda_o \leftarrow \gamma_e \\ \text{For } i \text{ from 1 to } \lfloor k/2 \rfloor \\ \qquad \lambda_{2i} \leftarrow \lceil \frac{a_{k-2i+1}}{a_{k-2i}} \gamma_{2i+2} \rceil + \\ \qquad\qquad \lceil \frac{a_{k+1-2i}}{a_{k+2-2i}} \gamma_{2i} \rceil - \gamma_{2i+1}. \\ \hline \end{array}$$

As in [11] we can show that for the (m,l) sequence a and for any $j \geq 0$

$$\lfloor \frac{a_{i-1}}{a_i} j \rfloor + \lceil \frac{a_{i+1}}{a_i} j \rceil = \begin{cases} mj & \text{if } i \text{ even} \\ lj & \text{if } i \text{ odd} \end{cases}$$

$|\gamma_o| = m|\lambda_o| - |\lambda_e|$ if k even and $|\gamma_o| = l|\lambda_o| - |\lambda_e|$ if k odd and $|\gamma_e| = |\lambda_o|$.

Also,

$$\gamma_1 = \lceil a_{k+1} \gamma_2 / a_k \rceil.$$

It remains to show that for $k \geq i \geq 2$, $\gamma_i \geq \frac{a_{k+2-i}}{a_{k+1-i}} \gamma_{i-1}$. Note that consecutive parts $\lambda_{2i-1}, \lambda_{2i}, \lambda_{2i+1}$ in λ, map by ϕ_k to the consecutive parts

$$\gamma_{2i} = \lambda_{2i-1}, \qquad \gamma_{2i+1} = \lceil \frac{a_{k+1-2i}}{a_{k-2i}} \lambda_{2i+1} \rceil + \lfloor \frac{a_{k-2i+1}}{a_{k+2-2i}} \lambda_{2i-1} \rfloor - \lambda_{2i}, \qquad , \gamma_{2i+2} = \lambda_{2i+1}$$

in γ. As $\lambda_{2i} \geq \frac{a_{k+1-2i}}{a_{k-2i}} \lambda_{2i+1}$, we get that

$$\lceil a_{k+1-2i} \lambda_{2i+1} / a_{k-2i} \rceil - \lambda_{2i} \leq 0$$

and

$$\gamma_{2i+1} \ \leq a_{k-2i+1} \lambda_{2i-1} / a_{k+2-2i} = a_{k-2i+1} \gamma_{2i} / a_{k+2-2i}.$$

As $\lambda_{2i-1} \geq a_{k+2-2i} \lambda_{2i} / a_{k+1-2i}$, we get that

$$\lfloor a_{k-2i+1} \lambda_{2i-1} / a_{k+2-2i} \lambda_{2i-1} \rfloor - \lambda_{2i} \geq 0,$$

and

$$\gamma_{2i+1} \ \geq a_{k+1-2i} \lambda_{2i+1} / a_{k-2i} = a_{k+1-2i} \gamma_{2i+2} / a_{k-2i}.$$

We have our conditions. □

The bijection Now we show that this gives a straighforward bijection ψ_k between Lecture Hall partitions in L_k and partitions into parts in $c = \{c_1, c_2, \ldots, c_k\}$ with $c_i = a_i + b_{i-1}$ if k is even and $c_i = a_{i-1} + b_i$ otherwise. Let λ be a partition in L_k and μ its image by ψ_k. We denote by $\mu(i)$ the multiplicity of the part c_i in μ. The bijection and its reverse are presented below :

$\psi_k \ : \ \lambda \rightarrow \mu$
$\quad \mu \leftarrow$ empty partition
$\quad$ For i from k downto 1 do
$\quad\quad j \leftarrow \lambda_1 - \lceil a_i/a_{i-1}\lambda_2 \rceil$
$\quad\quad \lambda_1 \leftarrow \lambda_1 - j$
$\quad\quad \mu(k-i+1) \leftarrow j$
$\quad\quad$ If $i > 1$ then
$\quad\quad\quad \lambda \leftarrow \phi_{i-1}^{-1}(\lambda)$.

$\psi_k^{-1} \ : \ \mu \rightarrow \lambda$
$\quad \lambda \leftarrow$ empty partition
$\quad$ For i from 1 to k do
$\quad\quad \lambda_1 \leftarrow \lambda_1 + \mu(k-i+1)$
$\quad\quad$ If $i < k$ then
$\quad\quad\quad \lambda \leftarrow \phi_i(\lambda)$.

Example 8. For $m = l = 2$, $a = (1, 2, 3, 4, \ldots)$ and for $\lambda = (6, 4, 2, 1)$ we get $\mu = (5, 5, 3)$.
Example 9. For $m = 1$ and $l = 4$, $a = (1, 4, 3, 8, \ldots)$ and for $\lambda = (12, 4, 5, 1)$ we get $\mu = (5, 4, 4, 4, 4, 1)$.

6. Anti-Lecture Hall Compositions

In this section we study the sequences $(\lambda_1, \ldots, \lambda_k)$ defined by

$$\frac{\lambda_1}{1} \geq \frac{\lambda_2}{2} \geq \ldots \frac{\lambda_k}{k} \geq 0.$$

We call these sequences *anti-lecture hall* because if they represent the heights of the rows in an amphitheater (as it was done for the Lecture Hall partitions) only the students of the first row are guaranteed to see the professor! We want to show that these compositions have a surprising behavior, in particular, their weight generating function is

$$\prod_{i=1}^{k} \frac{1+q^i}{1-q^{i+1}}. \tag{12}$$

Let A_k be the set of anti-lecture hall compositions into k nonnegative parts. Given $\lambda \in A_k$, we can write λ as $((x_1, \ldots, x_k), (z_1, z_2, \ldots, z_k))$ where $\lambda_i = ix_i + z_i$ with $0 \leq z_i \leq i-1$, $1 \leq i \leq k$. Note that $\lambda \in A_k$ if and only if $x_1 \geq x_2 \geq \ldots x_k \geq 0$ and if $x_i = x_{i+1}$ then $z_i \geq z_{i+1}$. Moreover $|\lambda| = \sum_{i=1}^{k} z_i + ix_i$.

The main result of this section is the following theorem (setting $u = v = 1$ gives (12)).

Theorem 7.

$$\sum_{\lambda \in A_k} q^{|\lambda|} u^{|x|} v^{o(x)} = \prod_{i=1}^{k} \frac{1+uvq^i}{1-u^2q^{i+1}}.$$

where $x = (x_1, \ldots, x_k)$ and $o(x)$ is the number of odd parts of the partition x.

Proof. Let D_k be the set of partitions into distinct parts less than or equal to k. Let E_k be the subset of A_k where all the x_i are even. To show the theorem we must give two bijections :

- A bijection between A_k and $D_k \times E_k$ such that if (α, β) is the image of λ then $|\alpha| + |\beta| = |\lambda|$, $l(\alpha) + |\lfloor \beta \rfloor| = |x|$, and $l(\alpha) = o(x)$, where $\lfloor \beta \rfloor = (\lfloor \beta_1/1 \rfloor, \ldots, \lfloor \beta_k/k \rfloor)$ and $l(\alpha)$ denotes the number of positive parts of α.

- A bijection between the set E_k and the set F_k of partitions into parts in the set $\{2, 3, \ldots, k+1\}$ such that if μ is the image of λ then $|\mu| = |\lambda|$ and $l(\mu) = \sum_{i=1}^{k} x_i/2$.

The first bijection will show that : $H(u, v, q) = \prod_{i=1}^{k}(1 + uvq^i)E_k(u, q)$ where $E_k(u, q) = \sum_{\lambda \in E_k} q^{|\lambda|} u^{|x|}$. The second bijection will show that $E_k(u, q) = \prod_{i=2}^{k+1}(1 - u^2q^i)^{-1}$.

We construct the first bijection :

$A_k \to D_k \times E_k \qquad \lambda \to (\alpha, \beta)$
$\quad \alpha \leftarrow$ empty partition
$\quad \beta \leftarrow \lambda$
$\quad$ While one of the x_i is odd
$\qquad d \leftarrow \max\{i \mid x_i \text{ odd}\}$
$\qquad i \leftarrow \min\{j \geq d \mid j = k \text{ or } x_d - 1 > x_{j+1} \text{ or } z_d \geq z_{j+1}\}$
$\qquad x_d \leftarrow x_d - 1; \quad temp \leftarrow z_d,$
$\qquad z_j \leftarrow z_{j+1} - 1, \quad d \leq j \leq i - 1$
$\qquad z_i \leftarrow temp; \quad \alpha \leftarrow \alpha \cup i$

Consider the f^{th} iteration of the loop. Let d_f and i_f be the indices chosen during that iteration. A careful look at the algorithm shows that $d_f > d_{f+1}$ and that $i_f > i_{f+1}$. As $\alpha_f = i_f$ we get that α is a partition into distinct parts. Moreover it is clear that β is in E_k as each iteration decreases by 1 only the odd x_i. Finally we must note that $|\alpha| + |\beta| = |\lambda|$ and $l(\alpha) = o(x)$. The reverse bijection is easy to construct.

We now give the second bijection between the set E_k and the set F_k.

$E_k \to F_k \qquad \lambda \to \mu$
$m(k+1) = z_k + \frac{kx_k}{2}$
$m(i) = z_{i-1} - z_i + \frac{(i-1)(x_{i-1} - x_i)}{2}, \ 2 \leq i \leq k$
where $m(i)$ is the multiplicity of the part i in μ.

It is easy to see that we can reconstruct λ from μ. Note that $m(i)$ is always nonnegative as $z_{i-1} < i - 1$ and if $z_{i-1} < z_i$ then $x_{i-1} > x_i$, that is, $(x_{i-1} - x_i)(i-1)/2 \geq i - 1$. Now we must show $|\mu| = |\lambda|$ and that the number of parts of μ is equal to the sum of the x_i divided by 2.

$|\mu| = z_k(k+1) + \sum_{i=2}^{k} i(z_{i-1} - z_i) + k(k+1)x_k/2 + \sum_{i=2}^{k}(i-1)i(x_{i-1} - x_i)/2 = \sum_{i=1}^{k}(z_i + ix_i)$

$l(\mu) = \sum_{i=2}^{k+1} m(i) = z_k + \sum_{i=2}^{k}(z_{i-1} - z_i) + kx_k/2 + \sum_{i=2}^{k} = (i-1)(x_{i-1} - x_i)/2 = \sum_{i=1}^{k} x_i/2.$ $\square$

Example 10. Starting with $\lambda = ((7, 6, 4, 3, 3, 2), (0, 1, 1, 2, 2, 3))$ we apply the first bijection and get $\beta = ((6, 6, 4, 2, 2, 2), (0, 0, 1, 2, 2, 2))$ and $\alpha = (6, 4, 2)$. Then we apply the second bijection and get $\mu = (7, 7, 7, 7, 7, 7, 7, 7, 4, 4, 3))$. We can check that $|\lambda| = |\mu| + |\alpha| = 79$, $|x| = 2l(\mu) + l(\alpha) = 25$ and $o(x) = l(\alpha) = 3$.

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