THE ART OF NUMBER GUESSING: WHERE COMBINATORICS MEETS PHYSICS

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In my talk I will discuss some surprising and interesting applications of enumeration problems to physics. With the help of known results for enumerations, exact formulae for certain physically interesting quantities can be guessed from small sets of data. An example is given by the following.

Consider the algebra with generators $\{e_i\}_{i=1}^{2n-1}$ subject to the following relations

$$e_i^2 = e_i \quad e_i e_{i\pm 1} e_i = e_i \quad [e_i, e_j] = 0 \text{ for } |i-j| > 1. \tag{1}$$

This algebra is called the Temperley-Lieb (TL) algebra. I will only be concerned with the left ideal generated by the action of TL on $I_0 = \prod_{i=1}^{n} e_{2i-1}$.

From a physical point of view, an important object is formed by the following operator which is called the Hamiltonian,

$$H^{(n)} = \sum_{j=1}^{2n-1} (1 - e_j). \tag{2}$$

In a particular representation of the TL algebra, the Hamiltonian H models the physics of a chain of interacting atoms with a two-valued internal degree of freedom, otherwise known as the XXZ spin chain. In another representation, it is closely related to percolation and also to a one-dimensional stochastic system. The Hamiltonian describes the equations of motion for a physical system. It is important to consider its spectrum and in particular its lowest eigenvalue $\lambda^{(n)}$ and its corresponding eigenstate $\psi^{(n)}$. For the Hamiltonian (2), $\lambda^{(n)} = 0$ and $\psi^{(n)}$ therefore obeys

$$H^{(n)} \psi^{(n)} = 0. \tag{3}$$

Let us consider two examples. For $n = 2$ the left ideal contains only two words, I_0 and $e_2 I_0$. The action of H_2 on these two words is easily calculated and one finds that on the basis of these two words, $\psi^{(2)} = (2, 1)$. For $n = 3$ there are five words in the left ideal: I_0, $e_2 I_0$, $e_4 I_0$, $e_2 e_4 I_0$ and $e_3 e_2 e_4 I_0$. The eigenvector with eigenvalue zero is given by $\psi^{(3)} = (11, 5, 5, 4, 1)$. One can go on like this and, normalising the smallest element to 1, find that the largest element $\psi^{(n)}_{\rm max} = 2, 11, 170 = 2 \cdot 5 \cdot 17, 7429 = 17 \cdot 19 \cdot 23, \ldots$ and that the sum of its elements $Z^{(n)}$ follows the sequence $3, 26 = 2 \cdot 13, 646 = 2 \cdot 17 \cdot 19, 45885 = 3 \cdot 5 \cdot 7 \cdot 19 \cdot 23, \ldots$. These calculations have been done up to $n = 9$ and it is found that all elements of the sequences factorize into small primes, and moreover that they are related to the well known enumeration problem of alternating-sign matrices. It is conjectured that the general formulae for $\psi^{(n)}_{\rm max}$ and $Z^{(n)}$ are given by

$$\psi^{(n)}_{\rm max} = \prod_{j=1}^{n-1} (3j+1) \frac{(2j)!(6j)!}{(4j)!(4j+1)!} \qquad Z^{(n)} = \prod_{j=0}^{n-1} (3j+2) \frac{(2j+1)!(6j+3)!}{(4j+2)!(4j+3)!} \tag{4}$$

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