POLYGRAPH ARRANGEMENTS (EXTENDED ABSTRACT)

AXEL HULTMAN

ABSTRACT. A class of subspace arrangements, $Z(n,m)$, known as polygraph arrangements was exploited by Haiman in order to prove the $n!$ theorem. By showing that their intersection lattices, $L(Z(n,m))$, are EL-shellable, we determine the cohomology groups of the complements of the arrangements. Moreover, we generalize the shellability results to a class of lattices which deserve to be called Dowling generalizations of $L(Z(n,m))$. As a consequence, we obtain the cohomology groups of the complements of certain Dowling analogies of polygraph arrangements.

RÉSUMÉ. Une classe d'arrangements de sous-espaces de Z(n, m), connue sous le nom des arrangements de polygraphes, a été exploitée par Haiman afin de prouver le théorme $n!$. En prouvant que leurs treillis d'intersections, $L(Z(n,m))$, sont EL-shellable, nous déterminons les groupes de cohomologies des compléments d'arrangements. De plus, nous généralisons les résultats de shellabilité à une classe de treillis que nous appelons les $L(Z(n,m))$-généralisations de Dowling. Entre autres, nous obtenons les groupes de cohomologies des compléments de certains analogues de Dowling des arrangements de polygraphe.

1. INTRODUCTION

Macdonald [11] introduced a family of polynomials known as *Macdonald polynomials*. They constitute a basis of the algebra of symmetric functions in the variables $x_1, x_2, \ldots$ with coefficients in the field of fractions of $\mathbb{Q}[y,z]$. Transformation to the basis of Schur functions gives rise to transition coefficients that are called *Kostka-Macdonald coefficients*. Until recently, the conjecture that the Kostka-Macdonald coefficients in fact are polynomials in y and z with nonnegative integer coefficients was open. This conjecture was known as the *Macdonald positivity conjecture*.

Garsia and Haiman [6] conjectured that the Kostka-Macdonald coefficients are multiplicities of graded characters of certain S_n-modules. An equivalent (see Haiman [9]) formulation of this has become known as the $n!$ *conjecture*, since it asserts that the said modules are of dimension $n!$. This implies the positivity conjecture.

Recently, Haiman [8] proved the $n!$ conjecture. The proof relies on the fact that a class of subspace arrangements in $(\mathbb{C}^2)^{n+m}$, called *polygraph arrangements*, have coordinate rings that are free modules over the polynomial ring in one coordinate set on $(\mathbb{C}^2)^n$.

In this paper, we show that certain lattices, which deserve to be called Dowling generalizations of the intersection lattices of the polygraph arrangements, are EL-shellable. Via the Goresky-MacPherson formula, this allows us to determine the cohomology groups of the complements of the polygraph arrangements as well as of Dowling analogies of these arrangements. In particular, it turns out that the cohomology is torsion-free and vanishing in "most" dimensions.

The structure of this paper is as follows. After briefly reviewing basic definitions and tools in Section 2, we deal with the case of ordinary polygraph arrangements in Sections 3 and 4. In Section 5, we give Dowling generalizations of the results in Section 4.

The proofs of Section 5 certainly specialize to proofs of the theorems in Section 4. However, they do not quite specialize to the proofs given in Section 4; the latter are simpler and more transparent. This is the reason why we treat ordinary polygraph arrangements and their Dowling generalizations separately.

Acknowledgement. The author is grateful to his advisor Anders Björner who suggested the study of polygraph arrangements. Moreover, his careful reading and remarks have led to substantial improvements in the paper.

2. Tools for investigation of subspace arrangements

We give a brief survey of the techniques that are used in this paper. For basic combinatorial and topological concepts, the reader is referred to the textbooks by Stanley [13] and Munkres [12]. For more on subspace arrangements, see e.g. Björner's survey article [2].

A *subspace arrangement* is a collection $\mathcal{A} = \{A_1, \dots, A_n\}$ of affine subspaces of $\mathbf{k}^m$, where $\mathbf{k}$ is some field. In case $\mathbf{k} \in \{\mathbb{R}, \mathbb{C}\}$, one is often interested in the topological features of the complement $\mathcal{M}_\mathcal{A} := \mathbf{k}^m \setminus (\bigcup_{i=1}^n A_i)$.

2.1. The Goresky-MacPherson formula.

To any poset P, we associate the *order complex* $\Delta(P)$. This is the simplicial complex having the chains of P as simplices. If P has a minimal and/or a maximal element, then the symbols $\hat{0}$ and $\hat{1}$ will be used to denote them, respectively. The *proper part* $\overline{P}$ is the poset $P \setminus \{\hat{0}, \hat{1}\}$.

The *intersection semi-lattice* $L(\mathcal{A})$ of $\mathcal{A}$ is the meet semi-lattice of all nonempty intersections of subsets of $\mathcal{A}$ ordered by reverse inclusion. It is a lattice iff $\bigcap_{i=1}^n A_i \neq \emptyset$. In case $\mathbf{k} \in \{\mathbb{R}, \mathbb{C}\}$, the following result by Goresky and MacPherson [7] relates the reduced cohomology groups of $\mathcal{M}_\mathcal{A}$ and the reduced homology of the lower intervals of $L(\mathcal{A})$:

Theorem 2.1. (The Goresky-MacPherson formula) *Let $\mathcal{A}$ be a real subspace arrangement (i.e. $\mathbf{k} = \mathbb{R}$). Then, for all i,*

$$\widetilde{H}^i(\mathcal{M}_\mathcal{A}; \mathbb{Z}) \cong \bigoplus_{x \in L(\mathcal{A}) \setminus \{\hat{0}\}} \widetilde{H}_{codim_\mathbb{R}(x) - i - 2}(\Delta(\overline{[\hat{0}, x]}); \mathbb{Z}). \qquad \square$$

Note that we can apply Theorem 2.1 to complex arrangements by identifying $\mathbb{C}$ with $\mathbb{R}^2$. Then $codim_\mathbb{R}(\cdot)$ is replaced by $2codim_\mathbb{C}(\cdot)$.

2.2. Lexicographic shellings.

If we are interested in the cohomology of $\mathcal{M}_\mathcal{A}$, then Theorem 2.1 leaves us with the task of determining the homology of $L(\mathcal{A})$ and its lower intervals. To this end, the technique of EL-shellability described below will be useful to us. It was introduced for ranked posets by Björner [1] and later extended to arbitrary posets by Björner and Wachs [4].

For an arbitrary poset P, let $\hat{P}$ denote the poset obtained by adding an additional maximal element $\hat{1}$ and an additional minimal element $\hat{0}$ to P. Let $R(\hat{P}) \subset \hat{P}^2$ denote the covering relation of $\hat{P}$. We write $x \to y$ if y covers x. An *edge-labelling* of P is a map $\lambda : R(\hat{P}) \to \Lambda$, where Λ is some poset of labels. A saturated chain $c = \{c_1 \to \dots \to c_t\} \subseteq \hat{P}$ is *rising* if $\lambda(c_1 \to c_2) < \dots < \lambda(c_{t-1} \to c_t)$. The chain c is *falling* if, instead, $\lambda(c_1 \to c_2) \geq \dots \geq \lambda(c_{t-1} \to c_t)$. Given an interval $[x, y] \subseteq \hat{P}$, we compare two saturated chains $c = \{x = c_1 \to \dots \to c_{t_c} = y\}$ and $d = \{x = d_1 \to \dots \to d_{t_d} = y\}$ using the lexicographic order induced by Λ on the sequences $\lambda(c_1 \to c_2), \dots, \lambda(c_{t_c - 1} \to c_{t_c})$ and $\lambda(d_1 \to d_2), \dots, \lambda(d_{t_d - 1} \to d_{t_d})$.

Definition 2.2. $\lambda : R(\hat{P}) \to \Lambda$ *is an* EL-labelling *of* P *if every* $\hat{P}$*-interval contains a unique rising saturated chain and this chain is lexicographically least of all saturated chains in the interval. If* P *admits an EL-labelling, then* P *is called* EL-shellable.

Clearly, if P is EL-shellable, then so is every interval of P. Moreover, the homotopy type of $\Delta(P)$ can be read off the labelling:

Theorem 2.3. (see [4, Thm. 5.9]) *If* λ *is an EL-labelling of* P*, then* $\Delta(P)$ *is homotopy equivalent to a wedge of spheres. The spheres of dimension* i *in the wedge are indexed by the falling maximal chains of length* $i+2$ *in* $\hat{P}$*. In particular, if* P *is ranked, then* $\Delta(P)$ *is homotopy equivalent to a wedge of spheres in top dimension.* □

3. Polygraph arrangements

Now we describe our main objects of study. Let V be a d-dimensional vector space over $\mathbf{k}$. For $m, n \in \mathbb{N}$ and a function $f : [m] \to [n]$, we let

$$W_f = \{(x_{f(1)}, \ldots, x_{f(m)}, x_1, \ldots, x_n) \in V^{m+n} \mid x_i \in V \ \forall i \in [n]\}.$$

This is a linear subspace of V^{m+n}. The *polygraph arrangement*, $Z_V(n,m)$, is the collection of all such subspaces:

$$Z_V(n,m) := \{W_f \mid f : [m] \to [n]\}.$$

Often, the choice of V (and $\mathbf{k}$) is not important. Therefore, we frequently write $Z(n,m)$ instead of $Z_V(n,m)$. To avoid confusion, we mention that Haiman [8] lets $Z(n,m)$ denote the union, not just the collection, of all W_f.

We need a combinatorial description of the intersection lattice $L(Z(n,m))$. From now on, we let $P = \{p_1 < \cdots < p_m\}$ and $Q = \{q_1 < \cdots < q_n\}$ be fixed disjoint ordered sets. For a subset $S \subseteq P \cup Q$, we use the notation $S^P := S \cap P$ and $S^Q := S \cap Q$, so that $S = S^P \cup S^Q$. Consider the following lattice, which is a join-subsemilattice of the partition lattice $\Pi_{P \cup Q}$.

$$\begin{aligned} L(Q,P) := \ & \{\pi_1|\ldots|\pi_t \in \Pi_{P\cup Q} \mid \pi_i^P = \emptyset \Rightarrow |\pi_i^Q| = 1 \\ & \text{and } \pi_i^P \neq \emptyset \Rightarrow \pi_i^Q \neq \emptyset \ \forall i \in [t], t \text{ arbitrary}\} \cup \{\hat{0}\}. \end{aligned}$$

Proposition 3.1. $L(Z(n,m))$ *and* $L(Q,P)$ *are isomorphic.*

Proof. Pick a subset $F \subseteq \{f : [m] \to [n]\}$. The element $\bigcap_{f \in F} W_f \in L(Z(n,m))$ can be represented by a bipartite graph $G_F = (P \cup Q, E)$, where $\{p_i, q_j\} \in E$ iff $f(i) = j$ for some $f \in F$. Two such graphs represent the same subspace of V^{m+n} precisely if they have the same connected components. Clearly, all bipartite graphs on $P \cup Q$ in which $deg(p_i) \geq 1$ for all $i \in [m]$ occur in this way. Thus, $L(Z(n,m))$ is isomorphic to the lattice of partitions of $P \cup Q$ that correspond to connected components in bipartite graphs on $P \cup Q$ with no isolated vertices in P. This is precisely $L(Q,P)$. □

Corollary 3.2. *The intersection lattice* $L(Z(n,m))$ *is ranked of length* n. □

4. An EL-labelling of $\overline{L(Q,P)}$

We will give an edge-labelling λ of $\overline{L(Q,P)}$. It will turn out to be an EL-labelling. Our poset Λ of labels is

$\Lambda = \{A_2 < \cdots < A_n < B_1 < \cdots < B_n < \underbrace{11\ldots 11}_{m} < \underbrace{11\ldots 12}_{m} < \cdots < \underbrace{nn\ldots n}_{m} < C_2 < \cdots < C_n\}$.

The labelling $\lambda : R(L(Q,P)) \to \Lambda$ is defined by:

- $\lambda(\pi \to \tau) = A_x$ if two non-singleton blocks, π_i and π_j, in π are merged in τ and $q_x = max(\pi_i^Q \cup \pi_j^Q)$.
- $\lambda(\pi \to \tau) = B_x$ if a singleton, q_x, and a non-singleton block, π_i, in π are merged in τ and $q_x < max(\pi_i^Q)$.
- $\lambda(\pi \to \tau) = C_x$ if a singleton, q_x, and a non-singleton block, π_i, in π are merged in τ and $q_x > max(\pi_i^Q)$.
- $\lambda(\hat{0} \to \tau) = f(1)f(2)\ldots f(m)$ (juxtapositioning) if τ corresponds to the subspace W_f.

Theorem 4.1. *The labelling λ is an EL-labelling of $\overline{L(Q,P)}$.*

Proof. Pick an interval $I = [\pi, \tau] \subseteq L(Q,P)$. We must verify that I contains a unique rising chain and that this chain is lexicographically least in I.

Suppose, to begin with, that $\pi \neq \hat{0}$. If there are non-singleton blocks in π which are merged in τ, then any rising chain must begin with the merging of these blocks. This gives rise to A-labels, and for the subscripts of these labels to form a rising sequence, the order in which to merge the blocks is unique. Next, all singletons q_i that are to be merged with non-singleton blocks containing some $q_j > q_i$ must be so. This gives rise to B-labels, and again there is a unique way to make their subscripts form a rising sequence. Finally, the remaining singletons that are larger than all Q-elements in their blocks in τ are to be merged, this giving rise to C-labels. Once again, there is a unique order in which to do this within a rising chain. Thus, we have constructed the unique rising chain in I. Note that if we had replaced the word *rising* with the phrase *lexicographically least*, then the above construction would give us the unique lexicographically least chain. Hence, it coincides with the rising chain.

Now, suppose that $\pi = \hat{0}$. Note that I contains exactly one atom a_f, corresponding to the subspace W_f, with the property that $[a_f, \tau]$ contains a chain with only C-labels. The function f sends $i \in [m]$ to the least $j \in [n]$ such that p_i and q_j are in the same block in τ. As before, the C-labels occur with rising subscripts in exactly one chain in $[a_f, \tau]$. Note that $\lambda(\hat{0} \to a_f) < \lambda(\hat{0} \to a)$ for all atoms $a \in I \setminus \{a_f\}$. Hence, the rising chain is again lexicographically least in I. □

Remark. Let $r = (r_1, \ldots, r_m) \in [n]^m$. Haiman [10] has considered the subarrangement $Z(n,m,r) \subseteq Z(n,m)$ which consists of those W_f that satisfy $f(i) \leq r_i$ for all $i \in [m]$. It is not difficult to see that, with straightforward modifications, the proof of Theorem 4.1 goes through for the appropriate subsemilattice $L(Q,P,r)$ of $L(Q,P)$.

Theorem 2.3 tells us that $\Delta(\overline{L(Q,P)})$ is homotopy equivalent to a wedge of spheres in top dimension, the spheres being indexed by the falling chains in $L(Q,P)$ under the labelling λ. In order to calculate the number of spheres in the wedge, we define an easily counted set of combinatorial objects which is in 1-1 correspondence with the set of falling chains in $L(Q,P)$.

Consider the set $C(Q,P)$ of ordered partitions $(\pi_1, \ldots, \pi_k)$ of $P \cup Q$ such that $q_n \in \pi_1$, $\pi_i^P \neq \emptyset$ and $\pi_i^Q \neq \emptyset$ for all $i \in [k]$, k arbitrary. Define $\Gamma(n,m) := |C(Q,P)|$. Clearly,

$\Gamma(n,m) = \sum_{k=1}^{min(n,m)} S(n,k)S(m,k)k!(k-1)!,$

where the $S(i,j)$ are Stirling numbers of the second kind.

Now we establish the bijection mentioned above.

Theorem 4.2. *$\Delta(\overline{L(Q,P)})$ is homotopy equivalent to a wedge of $(n-2)$-dimensional spheres. The number of spheres in the wedge is*

$\sum_{k=1}^{min(n,m)} S(n,k)S(m,k)k!(k-1)!.$

Proof. We construct a bijection ϕ : {falling chains in $L(Q,P)$} $\to C(Q,P)$ as follows. Let $c = \{\hat{0} \to c_1 \to \ldots \to c_n = \hat{1}\} \subseteq L(Q,P)$ be a falling chain. Then, for some j, all c_i, $i < j$, contain singleton blocks whereas all c_i, $i \geq j$, do not. The blocks in c_j are the blocks in $\phi(c)$. Let π_1 be the block in c_j which contains q_n. Since c is falling, c_{j+1} is obtained by merging π_1 with some other block which we call π_2. Then c_{j+2} is obtained by merging $\pi_1 \cup \pi_2$ with a block which we denote π_3 and so on, until finally $\hat{1}$ is obtained from c_{n-1} by merging $\pi_1 \cup \cdots \cup \pi_{n-j}$ with the only other block, which is then given the name π_{n-j+1}. We define $\phi(c) := (\pi_1, \ldots, \pi_{n-j+1})$.

To check injectivity of ϕ, consider two distinct falling chains $c = \{\hat{0} \to c_1 \to \ldots \to c_n = \hat{1}\}$ and $d = \{\hat{0} \to d_1 \to \ldots \to d_n = \hat{1}\}$ in $L(Q,P)$. Let j be the smallest index such that $c_j \neq d_j$.

If $j = 1$, then c_1 and d_1 correspond to different functions $f_c, f_d : [m] \to [n]$. Note that $\{q_i \in Q \mid i \in f_c([m])\}$ is the set of maximal Q-elements in blocks in $\phi(c)$, since no falling chain possesses C-labels. An analogous statement holds for f_d. Therefore, if $f_c([m]) \neq f_d([m])$, then $\phi(c) \neq \phi(d)$. If, on the other hand, $f_c([m]) = f_d([m])$, then we can pick $i \in [m]$ such that $f_c(i) \neq f_d(i)$ and both $f_c(i)$ and $f_d(i)$ are maximal Q-elements in blocks in both $\phi(c)$ and $\phi(d)$. Therefore, p_i and $q_{f_c(i)}$ belong to the same block in $\phi(c)$ but to different blocks in $\phi(d)$. Hence, $\phi(c) \neq \phi(d)$.

Now suppose $j > 1$. If $\lambda(c_{j-1} \to c_j) = B_x$, for some x, then $\lambda(d_{j-1} \to d_j) = B_x$, too. Since $c_j \neq d_j$, this means that the block containing x in $\phi(c)$ is different from the block containing x in $\phi(d)$. This implies $\phi(c) \neq \phi(d)$.

The only case left is $\lambda(c_{j-1} \to c_j) = \lambda(d_{j-1} \to d_j) = A_n$. This implies that the set of blocks in $\phi(c)$ is equal to the set of blocks in $\phi(d)$. Since $c \neq d$, $\phi(c)$ must differ from $\phi(d)$ in the order of the blocks. Hence $\phi(c) \neq \phi(d)$ in this case too, and ϕ is injective.

To establish surjectivity, choose $\pi = (\pi_1, \ldots, \pi_k) \in C(Q,P)$. We will construct a falling chain $c \subseteq L(Q,P)$ such that $\phi(c) = \pi$. Let $\hat{f} : P \to Q$ be the function mapping all elements in π_i^P to $max(\pi_i^Q)$ for all $i \in [k]$. Define $f : [m] \to [n]$ to be the corresponding function on the indices, i.e. by requiring that $\hat{f}(p_i) = q_{f(i)}$ for all $i \in [m]$. The atom of $L(Q,P)$ corresponding to W_f is c_1. The chain $\{\hat{0} \to c_1 \to \ldots \to c_{n-k+1} = \pi_1|\ldots|\pi_k\}$ is produced by merging the singletons in c_1 one by one with their corresponding non-singleton blocks in the only possible way which ends with $\pi_1|\ldots|\pi_k$ while giving rise to a falling sequence of B-labels. For $l \in [k-1]$, let $c_{n-k+l} = \pi_1 \cup \cdots \cup \pi_l|\pi_{l+1}|\ldots|\pi_k$. Now, $c = \{\hat{0} \to c_1 \to \ldots \to c_{n-1} \to \hat{1}\}$ is mapped to π by ϕ, so ϕ is surjective. □

Note that $\Gamma(n,m) = \Gamma(m,n)$. This implies an unsuspected numerical relationship between the combinatorially very distinct arrangements $Z(n,m)$ and $Z(m,n)$.

The cohomology groups of the complement $\mathcal{M}_{Z_{\mathbb{R}^d}(n,m)}$ are determined by the Goresky-MacPherson formula (Theorem 2.1) and the following corollary:

Corollary 4.3. *Let* $\pi = \pi_1|\dots|\pi_k \in L(Q,P)$. *Then* $\Delta(\overline{[\hat{0},\pi]})$ *is homotopy equivalent to a wedge of* $(n-1-k)$*-dimensional spheres. The number of spheres in the wedge is* $\prod_{j=1}^{k}\Gamma(|\pi_j^Q|,|\pi_j^P|)$.

Proof. $\overline{[\hat{0},\pi]}$ is ranked, and it is EL-shellable since $\overline{L(Q,P)}$ is. Hence, by Theorem 2.3, $\Delta(\overline{[\hat{0},\pi]})$ is homotopy equivalent to a wedge of spheres in top dimension. The number of spheres in the wedge is the absolute value $|\mu(\hat{0},\pi)|$ of the Möbius function. Note that $[\hat{0},\pi] \cong L(\pi_1^Q,\pi_1^P)\times\cdots\times L(\pi_k^Q,\pi_k^P)$. The Möbius function is multiplicative, so $\mu(\hat{0},\pi) = \prod_{j=1}^{k}\mu_j(\hat{0},\pi_j)$, where μ_j is the Möbius function of $L(\pi_j^Q,\pi_j^P)$. The corollary now follows from Theorem 4.2. □

In general dimension, the expression for the cohomology of the complement, although determined by Corollary 4.3, is not pretty. In the following theorem we restrict ourselves to weaker, readable, information. As before, the complex case is obtained by identifying $\mathbb{C}$ and $\mathbb{R}^2$.

Theorem 4.4. *For all* i, $\widetilde{H}^i(\mathcal{M}_{Z_{\mathbb{R}^d}(n,m)};\mathbb{Z})$ *is torsion-free. Let* $\widetilde{\beta}^i$ *denote its rank. We have*

(1) $\widetilde{\beta}^i = 0$, *unless* $i = d(m-1)+j(d-1)$ *for some* $j\in[n]$.

(2) $\widetilde{\beta}^{dm-1} = n^m$, *if* $d\geq 2$.

(3) $\widetilde{\beta}^{d(m+n-1)-n} = \Gamma(n,m)$, *if* $d\geq 2$.

□

Remark. Unlike its complement, the union $\cup\mathcal{A}$ of an arrangement of linear subspaces is topologically not very exciting; it is a cone with apex in the origin. A more interesting object is the *link*, $lk(\mathcal{A}) := S^{l-1}\cap(\cup\mathcal{A})$, where l is the dimension of the space in which the arrangement is embedded. From Ziegler and Živaljević [14, Thm. 2.4], it follows that the link of a real linear subspace arrangement with shellable intersection lattice has the homotopy type of a wedge of spheres. In particular, this applies to the polygraph arrangements $Z_{\mathbb{R}^d}(n,m)$.

5. A Dowling generalization

5.1. **Dowling lattices.** Let G be a finite group and n a positive integer. G acts on the set $([n]\times G)\cup\{0\}$ by $0g := 0$ and $(i,h)g := (i,hg)$ for $i\in[n]$ and $g,h\in G$. For a subset $S\subseteq([n]\times G)\cup\{0\}$, we define $Sg := \{xg \mid x\in S\}$. A partition $\pi=\pi_1|\dots|\pi_t$ of $([n]\times G)\cup\{0\}$ is G*-symmetric* if $\pi_i g\in\pi$ for all $g\in G$ and $i\in[t]$. The block π_i is called g*-symmetric*, for $g\in G$, if $\pi_i g=\pi_i$. If the identity element is the only $g\in G$ which makes π_i g-symmetric, then π_i is called *simple*. Note that if π is G-symmetric, then the block containing 0 is necessarily g-symmetric for all $g\in G$.

Definition 5.1. *Let* G *be a finite group. The* Dowling lattice Π_n^G *is the lattice of all* G*-symmetric partitions* π *of* $([n]\times G)\cup\{0\}$ *such that all blocks not containing* 0 *are simple. The block containing* 0 *is called the* null block *of* π.

Note that $\Pi_n^{\{e\}}\cong\Pi_{n+1}$. Thus, Dowling lattices constitute a generalization of the partition lattice. They were first introduced by Dowling [5]. Two more special cases are worth mentioning. The lattice $\Pi_n^{\mathbb{Z}_2}$ is isomorphic to the partition lattice of type B, i.e. the intersection lattice of the arrangement of reflecting hyperplanes of the Coxeter group B_n. This is a special case of $\Pi_n^{\mathbb{Z}_r}$, which is isomorphic to the intersection lattice of the *Dowling* *arrangement*, i.e. the arrangement in $\mathbb{C}^n$ of complex hyperplanes given by the equations $x_i = \zeta^k x_j$ and $x_l = 0$, where $i < j \in [n]$, $k \in [r]$, $l \in [n]$ and ζ is a primitive r:th root of unity.

For obvious reasons, the notation tends to get horrible when dealing with Dowling lattices. We agree on some conventions to simplify it. We write $i^g := (i, g)$ for $i \in [n]$ and $g \in G$. The G-orbit of a simple block in $\pi \in \Pi_n^G$ has cardinality $|G|$ and is of course completely determined by any representative. When we write out π, we therefore often omit all but one (arbitrary) block in every orbit of a simple block. Thus, $\pi = \pi_1|\ldots|\pi_t \in \Pi_n^G$ should be interpreted as an element with t G-orbits of blocks; hence with $(t-1)|G|+1$ blocks (since the null block is alone in its orbit). When the G-elements in the superscripts are irrelevant, namely in the null block and in singletons, we omit them, too. For example, we write $02|4|1^0 3^1$ for the element $0(2,0)(2,1)(2,2)|(4,0)|(4,1)|(4,2)|(1,0)(3,1)|(1,1)(3,2)|(1,2)(3,0)$ in $\Pi_4^{\mathbb{Z}_3}$.

We view an element in Π_n^G as a "signed" partition of $[n] \cup \{0\}$, where G is the group of "signs". Sometimes we wish to disregard the "signs". Therefore, for $S \subseteq ([n] \times G) \cup \{0\}$, we define $\overline{S} := \{i \in [n] \mid i^g \in S \text{ for some } g \in G\} \cup S^0$, where $S^0 := \{0\}$ if $0 \in S$ and $S^0 := \emptyset$ otherwise. With this, we can define the *absolute value* $\overline{\pi} \in \Pi_{[n]\cup\{0\}}$ of $\pi = \pi_1|\ldots|\pi_t \in \Pi_n^G$ by $\overline{\pi} := \overline{\pi_1}|\ldots|\overline{\pi_t}$. If π_i is the null block of π, then we say that $\overline{\pi_i}$ is the null block of $\overline{\pi}$.

5.2. **Dowling analogies of $L(Q, P)$.** Recall that x is a *modular* element in a ranked lattice L if $rank(x) + rank(y) = rank(x \vee y) + rank(x \wedge y)$ for all $y \in L$. Björner [3] observed that the lattice $L(Q, P)$ can be constructed in the following way, which suggests possible generalizations of the results in Section 3. Consider the modular element $\pi = p_1|\ldots|p_m|Q$ in the partition lattice $\Pi_{P\cup Q}$. Note that the set of complements $Co(\pi) := \{\tau \in \Pi_{P\cup Q} \mid \tau \wedge \pi = \hat{0} \text{ and } \tau \vee \pi = \hat{1}\}$ is precisely the set of atoms in $L(Q, P)$, so that $L(Q, P)$ is the lattice join-generated by $Co(\pi)$.

Now, let G be a finite group and consider the element $\pi = 0Q|p_1|\ldots|p_m$ in the Dowling lattice $\Pi_{P\cup Q}^G$ (meaning that we replace $[n]$ with $P \cup Q$ in Definition 5.1). By [5, Thm. 4], π is modular. Let $L^G(Q, P)$ be the lattice which is join-generated by $Co(\pi)$. Note that $Co(\pi)$ consists of the elements in which every simple block contains exactly one $Q \times G$-element and the null block contains no $Q \times G$-elements. Therefore, $L^G(Q, P)$ consists of those elements in $\Pi_{P\cup Q}^G$ in which every singleton is either 0 or from $Q \times G$ and every non-singleton intersects $P \times G$. In other words, for $\pi \in \Pi_{P\cup Q}^G$, we have $\pi \in L^G(Q, P)$ iff $\overline{\pi} \in L(Q \cup \{0\}, P)$.

We have $L^{\{e\}}(Q, P) \cong L(Q \cup \{0\}, P)$. The cases $G = \mathbb{Z}_2$ and $G = \mathbb{Z}_r$ are also interesting. As before, let V be a vector space over $\mathbf{k}$, and let r be a positive integer.

Definition 5.2. *Suppose that $char(\mathbf{k}) \neq 2$. The* polygraph arrangement of type B, *$Z_V^B(n, m)$, is the collection of all subspaces of the form*

$$\{(\tau_1 x_{f(1)}, \ldots, \tau_m x_{f(m)}, x_1, \ldots, x_n) \in V^{m+n} \mid x_i \in V \ \forall i \in [n]\}$$

over all $f : [m] \to [n]$ and $(\tau_1, \ldots, \tau_m) \in \{-1, 0, 1\}^m$.

Definition 5.3. *Suppose that $\mathbf{k}$ contains a primitive r:th root of unity. The* Dowling polygraph arrangement, *$Z_V^r(n, m)$, is the collection of all subspaces of the form*

$$\{(\tau_1 x_{f(1)}, \ldots, \tau_m x_{f(m)}, x_1, \ldots, x_n) \in V^{m+n} \mid x_i \in V \ \forall i \in [n]\}$$

over all $f : [m] \to [n]$ and $(\tau_1, \ldots, \tau_m) \in \{0, \zeta, \zeta^2, \ldots, \zeta^r = 1\}^m$, where ζ is a primitive r:th root of unity.

As before, we frequently suppress the vector space in the subscript. It is clear that $L(Z^B(n,m)) \cong L^{\mathbb{Z}_2}(Q,P)$ and $L(Z^r(n,m)) \cong L^{\mathbb{Z}_r}(Q,P)$.

It turns out that $L^G(Q,P)$ is EL-shellable. We define an edge-labelling $\omega : R(L^G(Q,P)) \to \Omega$, where Ω is the following poset of labels:

$$\Omega = \{\alpha_1 < \cdots < \alpha_n < A_2 < \cdots < A_n < B_1 < \cdots < B_n < \underbrace{00\ldots00}_{m} < \underbrace{00\ldots01}_{m} < \cdots < \underbrace{nn\ldots n}_{m} < \beta_1 < \cdots < \beta_n < C_2 < \cdots < C_n\}.$$

To simplify notation, we agree that from now on, the term *block* means a block which is neither a singleton nor a null block. Bearing this in mind, we define ω as follows:

- $\omega(\pi \to \tau) = \alpha_x$ if a block, $\overline{\pi_i}$, and the null block in $\overline{\pi}$ are merged in $\overline{\tau}$ and $q_x = max(\overline{\pi_i}^Q)$.
- $\omega(\pi \to \tau) = \beta_x$ if a singleton, q_x, and the null block in $\overline{\pi}$ are merged in $\overline{\tau}$.
- $\omega(\pi \to \tau) = A_x$ if two blocks, $\overline{\pi_i}$ and $\overline{\pi_j}$, in $\overline{\pi}$ are merged in $\overline{\tau}$ and $q_x = max(\overline{\pi_i}^Q \cup \overline{\pi_j}^Q)$.
- $\omega(\pi \to \tau) = B_x$ if a singleton, q_x, and a block, $\overline{\pi_i}$, in $\overline{\pi}$ are merged in $\overline{\tau}$ and $q_x < max(\overline{\pi_i}^Q)$.
- $\omega(\pi \to \tau) = C_x$ if a singleton, q_x, and a block, $\overline{\pi_i}$, in $\overline{\pi}$ are merged in $\overline{\tau}$ and $q_x > max(\overline{\pi_i}^Q)$.
- $\omega(\hat{0} \to \tau) = f(1)f(2)\ldots f(m)$ (juxtapositioning), where f is the function $f : [m] \to [n] \cup \{0\}$ which satisfies that $q_{f(i)}$ is the unique element in $Q \cup \{q_0\}$ sharing block (or null block) with p_i in $\overline{\tau}$. (Here, $q_0 := 0$.)

Given an atom $a \in L^G(Q,P)$, we define $f_a : [m] \to [n] \cup \{0\}$ by requiring that $f_a(1)\ldots f_a(m) = \omega(\hat{0} \to a)$.

We omit the proof of the following theorem; it is along the same lines as the proof of Theorem 4.1. Note, however, that ω does not specialize to the labelling λ of Section 4 when $G = \{e\}$.

Theorem 5.4. *The labelling ω is an EL-labelling of $\overline{L^G(Q,P)}$.* □

As in Section 4, we may exploit the EL-labelling ω to calculate the homotopy type of $\Delta(\overline{L^G(Q,P)})$. The key facts that we need are expressed in Lemma 5.8 and Lemma 5.9. The proof of the former requires three further lemmata. We omit the technical, but reasonably straightforward, proofs of two of them.

Lemma 5.5. *Let R be the set of elements in $L^G(Q,P)$ that contain no non-zero singletons. Fix an atom $a \in L^G(Q,P)$. Define ϕ_a^G to be the number of ω-falling chains $c = \{\hat{0} \to a = c_1 \to \ldots \to c_{t+1}\}$ such that $c_{t+1} \in R$ and $\omega(c_j \to c_{j+1})$ is a B-label for all $j \in [t]$. Then*

$$\phi_a^G = |G|^{n-|f_a([m])\setminus\{0\}|} \prod_{i\in[n]\setminus f_a([m])} |\{j \in f_a([m]) \mid j > i\}|. \qquad \square$$

We define a map $\psi : 2^{[n]} \to \mathbb{N}$ by $\psi(S) = \prod_{i\in[n]\setminus S} |\{j \in S \mid j > i\}|$, so that we obtain $\phi_a^G = |G|^{n-|f_a([m])\setminus\{0\}|}\psi(f_a([m]) \setminus \{0\})$. The Stirling numbers are related to ψ in the following way:

Lemma 5.6. $\sum_{S\in\binom{[n]}{k}} \psi(S) = S(n,k)$.

Proof. $\psi(S)$ counts all partitions $\pi = \pi_1|\ldots|\pi_{|S|}$ of $[n]$ with the property that $S = \{max(\pi_i) \mid i \in [|S|]\}$. □

Lemma 5.7. *Let $S \subseteq [n]$ be fixed. Let Λ_S be the set of atoms $a \in L^G(Q,P)$ such that $f_a([m]) \setminus \{0\} = S$. Then*

$$|\Lambda_S| = |S|! \sum_{j=0}^{m-|S|} \binom{m}{j} S(m-j,|S|)|G|^{m-j}.$$ □

Lemma 5.8. *Let R_k be the set of elements ρ in $L^G(Q,P)$ such that $\overline{\rho}$ consists of a null-block, k blocks and no non-zero singletons. Define $\phi_{\downarrow}^G(k)$ to be the number of ω-falling chains $c = \{\hat{0} \to c_1 \to \ldots \to c_{t+1}\}$ such that $c_{t+1} \in R_k$ and $\omega(c_i \to c_{i+1})$ is a B-label for all $i \in [t]$. Then*

$$\phi_{\downarrow}^G(k) = S(n,k)k! \sum_{j=0}^{m-k} \binom{m}{j} S(m-j,k)|G|^{m+n-j-k}.$$

In particular, this number only depends on k.

Proof. Let c be as in the statement of the lemma. Since all $\overline{c_i}$, $i \in [t]$, have the same number of blocks, k, we obtain

$$\phi_{\downarrow}^G(k) = \sum_{S\in\binom{[n]}{k}} \sum_{a\in\Lambda_S} \phi_a^G =$$

$$\stackrel{(1)}{=} \sum_{S\in\binom{[n]}{k}} \sum_{a\in\Lambda_S} |G|^{n-k}\psi(S) =$$

$$= \sum_{S\in\binom{[n]}{k}} |\Lambda_S| \cdot |G|^{n-k}\psi(S) =$$

$$\stackrel{(2)}{=} |G|^{n-k} S(n,k)|\Lambda_S| =$$

$$\stackrel{(3)}{=} |G|^{n-k} S(n,k)k! \sum_{j=0}^{m-k} \binom{m}{j} S(m-j,k)|G|^{m-j}.$$

Here, (1) follows from Lemma 5.5, (2) follows from Lemma 5.6 and (3) follows from Lemma 5.7. □

Lemma 5.9. *Suppose $\rho \in R_k$. Then the number of ω-falling chains in $[\rho, \hat{1}]$ is $\phi_{\uparrow}^G(k)$, where*

$$\phi_{\uparrow}^G(k) := (1+|G|)(1+2|G|)\ldots(1+(k-1)|G|).$$

In particular, this number only depends on k.

Proof. There is a natural isomorphism between $[\rho, \hat{1}]$ and the Dowling lattice Π_k^G obtained by identifying the blocks of ρ with the set $[k] \times G$ and the null block of ρ with 0. Hence, ω induces an EL-labelling of $\overline{\Pi_k^G}$. As then follows from Dowling's [5] computation of the Möbius function of Π_k^G, $\Delta(\overline{\Pi_k^G})$ is homotopy equivalent to a wedge of $\phi_{\uparrow}^G(k)$ spheres (of top dimension). Therefore, by Theorem 2.3, $\phi_{\uparrow}(k)$ must be the number of ω-falling chains in $[\rho, \hat{1}]$.

□

Now, we are ready to count the falling chains in $L^G(Q,P)$.

Theorem 5.10. *$\Delta(\overline{L^G(Q,P)})$ is homotopy equivalent to a wedge of $(n-1)$-dimensional spheres. Let $\Gamma^G(n,m)$ denote the number of spheres in the wedge. Then,*

$$\Gamma^G(n,m) = \sum_{k=1}^{min(n,m)} S(n,k)k! \sum_{j=0}^{m-k} \binom{m}{j} S(m-j,k)|G|^{m+n-j-k} \prod_{i=1}^{k-1}(1+i|G|).$$

Proof. It is clear that the number of falling chains in $L^G(Q,P)$ under ω is $\sum_{k=0}^{min(n,m)} \phi_{\downarrow}^G(k)\phi_{\uparrow}^G(k)$. The theorem now follows from Lemma 5.8, Lemma 5.9 and Theorem 2.3. □

Below, let $\alpha = min(n+1,m)$. We check that Theorem 5.10 indeed generalizes Theorem 4.2. Note that

$$\begin{aligned}
\Gamma^{\{e\}}(n,m) &= \textstyle\sum_{k=1}^{min(n,m)} S(n,k)k! \sum_{j=0}^{m-k} \binom{m}{j} S(m-j,k)k! = \\
&= \textstyle\sum_{k=1}^{min(n,m)} S(n,k)S(m+1,k+1)(k!)^2 = \\
&= \textstyle\sum_{k=1}^{\alpha} S(n,k)S(m+1,k+1)(k!)^2 = \\
&= \textstyle\sum_{k=1}^{\alpha} \frac{S(n+1,k)-S(n,k-1)}{k}(S(m,k)+(k+1)S(m,k+1))(k!)^2 = \\
&= \Gamma(n+1,m) - \textstyle\sum_{k=1}^{\alpha} S(n,k-1)S(m,k)k!(k-1)!+ \\
&\quad + \textstyle\sum_{k=1}^{\alpha} S(n,k)S(m,k+1)(k+1)!k! = \\
&\overset{(\star)}{=} \Gamma(n+1,m) - S(n,0)S(m,1)1!0! + S(n,\alpha)S(m,\alpha+1)(\alpha+1)!\alpha! = \\
&= \Gamma(n+1,m),
\end{aligned}$$

as required. The identity $(\star)$ follows from substituting $j = k-1$ in the first sum on the left hand side.

Corollary 5.11. *Pick $\pi \in L^G(Q,P)$. Suppose that $\overline{\pi} = 0\pi_0|\pi_1|\pi_2|\ldots|\pi_t$. Then, $\Delta(\overline{[\hat{0},\pi]})$ is homotopy equivalent to a wedge of top-dimensional spheres. The number of spheres in the wedge is $\Gamma^G(|\pi_0^Q|,|\pi_0^P|)\cdot\prod_{i=1}^{t}\Gamma(|\pi_i^Q|,|\pi_i^P|)$.*

Proof. Note that $[\hat{0},\pi] \cong L^G(\pi_0^Q,\pi_0^P)\times L(\pi_1^Q,\pi_1^P)\times\cdots\times L(\pi_t^Q,\pi_t^P)$. The rest of the proof is analogous to the proof of Corollary 4.3. □

As in Section 4, Corollary 5.11 provides the information needed to calculate the cohomology groups of $\mathcal{M}_{Z^r(n,m)}$ with the Goresky-MacPherson formula, thereby obtaining a generalization of Theorem 4.4. We omit the details.

References

[1] A. Björner, Shellable and Cohen-Macaulay partially ordered sets, *Trans. Amer. Math. Soc.* **260** (1980), 159–183.

[2] A. Björner, Subspace arrangements, *First European congress of math.*, Paris, 1992 (A. Joseph et al., eds.), 321–370, *Progr. Math* **119**, *Birkhäuser, Basel*, 1994.

[3] A. Björner, personal communication, 2001.

[4] A. Björner and M. Wachs, Shellable nonpure complexes and posets I, *Trans. Amer. Math. Soc.* **348** (1996), 1299–1327.

[5] T. A. Dowling, A class of geometric lattices based on finite groups, *J. Combin. Theory Ser. B* **14** (1973), 61–86.

[6] A. M. Garsia and M. Haiman, A graded representation model for Macdonald's polynomials, *Proc. Nat. Acad. Sci. U.S.A.* **90** (1993), 3607–3610.

[7] M. Goresky and R. D. MacPherson, Stratified Morse theory, *Ergebnisse der Mathematik und ihrer Grenzgebiete*, vol. 14, Springer-Verlag, Berlin/Heidelberg/New York, 1988.

[8] M. Haiman, Hilbert schemes, polygraphs, and the Macdonald positivity conjecture, *J. Amer. Math. Soc*, to appear.

[9] M. Haiman, Macdonald polynomials and geometry, in *New perspectives in algebraic combinatorics* (L. J. Billera et al., eds.), 207–254, Cambridge Univ. Press, 1999.

[10] M. Haiman, personal communication, 2001.

[11] I. G. Macdonald, A new class of symmetric functions, *Actes du 20e Séminaire Lotharingien*, vol. 372/S-20, Publications I.R.M.A., Strasbourg, 1988, 131–171.

[12] J. R. Munkres, *Elements of algebraic topology*, Addison-Wesley, Menlo Park, 1984.

[13] R. P. Stanley, *Enumerative combinatorics*, vol. 1, Cambridge Univ. Press, 1997.

[14] G. M. Ziegler and R. T. Živaljević, Homotopy types of subspace arrangements via diagrams of spaces, *Math. Ann.*, **295** (1993), 527–548.

Department of Mathematics, Royal Institute of Technology, S-100 44, Stockholm, Sweden
E-mail address: `axel@math.kth.se`