TRIBASIC INTEGRALS AND IDENTITIES OF ROGERS-RAMANUJAN TYPE

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ABSTRACT. Using complex analysis, integrals involving three independent q's are evaluated as infinite products. This leads to identities of Rogers-Ramanujan type.

RÉSUMÉ. En utilisant trois q independant, nous evaluons des integraux en analyse complexe et trouver des produits infinis. Ceci nous donnons des generalisations des identites de Rogers-Ramanujan.

1. Introduction

The purpose of this extended abstract (no proofs are given) is to show the extensive relationship between integrals and identities of Rogers-Ramanujan type. We concentrate on integral evaluations involving infinite products with three independent q's.

In [4, 6], it was shown that the Rogers-Ramanujan identities of modulus 5 follow from evaluating the bibasic integral

$$\frac{(qt^2;q)_\infty}{2\pi}\int_0^\pi \frac{(q^5,e^{2i\theta},e^{-2i\theta};q^5)_\infty}{(te^{i\theta},te^{-i\theta};q)_\infty}d\theta. \tag{1}$$

We generalize the above integral by replacing the three bases of the infinite products q^5, q and q by independent bases s, p and q. The resulting integral can be evaluated as an infinite product for special values of t. By specializing s, p, and q the product sides of identities of Rogers-Ramanujan type appear. The special values of t for which such identities exist can be found by considering the singularities of the integrals as functions of t.

2. Integrals

We use a variant of Schwarz's theorem [1] to evaluate limits of integrals.

Theorem 1. *Let $f(\theta,z)$ be continuous in θ for $\theta\in[0,2\pi]$, and for all z so that $r\geq|z|\geq r-\epsilon$ for some positive ϵ. Assume further that $f(\theta,z)$ converges to $f(\theta,re^{i\phi})$ as $z\to re^{i\phi}$ uniformly in θ, for $\theta\in[0,2\pi]$. Then*

$$\lim_{z\to re^{i\phi}}\int_0^{2\pi}\frac{(r^2-|z|^2)\,f(\theta,z)d\theta}{2\pi|re^{i\theta}-z|^2}=f(\phi,re^{i\phi}).$$

We next apply Theorem 1 to find limiting values of the tribasic integrals

$$G_1(t,p,q,s)=\frac{(-t;q)_\infty}{2\pi}\int_0^\pi \frac{(s,e^{2i\theta},e^{-2i\theta};s)_\infty}{(te^{2i\theta},te^{-2i\theta};p)_\infty}\,d\theta,\quad |t|<1,$$

and

$$G_2(t,p,q,s)=\frac{(pt^2;q)_\infty}{2\pi}\int_0^\pi \frac{(s,e^{2i\theta},e^{-2i\theta};s)_\infty}{(te^{i\theta},te^{-i\theta};p)_\infty}\,d\theta,\quad |t|<1.$$

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Note that $G_2(t,q,q,q^5)$ is the integral (1) of the introduction giving the Rogers-Ramanujan identities modulo 5.

Corollary 2. *We have an analytic continuation of* $G_2(t,p,q,s)$ *to* $p^{1/2} < |t| < p^{-1/2}$ *such that*

$$\begin{aligned}\lim_{t\to -1} G_1(t,p,q,s) &= \frac{(q;q)_\infty}{(p,p;p)_\infty}\,(s,-s,-s;s)_\infty,\\ \lim_{t\to p^{-1/2}} G_2(t,p,q,s) &= \frac{(s,p,s/p;s)_\infty (q;q)_\infty}{(p,p;p)_\infty}.\end{aligned}$$

In particular for the Rogers-Ramanujan identities modulo 5,

$$G_2(q^{-1/2},q,q,q^5) = \frac{1}{(q^2,q^3;q^5)_\infty}.$$

One can also extend the evaluation for G_2 using further analytic continuations. Let $F(z) = (s,z^2,1/z^2;s)_\infty$.

Theorem 3. *For* $k=1,2,\ldots,$ $G_2(t,p,p,s)$ *can be analytically continued to*

$$p^{1-k/2} < |t| < p^{-k/2}, \text{ via}$$

$$G_2(t,p,p,s) = \frac{(pt^2;p)_\infty}{4\pi}\int_0^{2\pi} \frac{F(e^{i\theta}p^{-k/2})}{(te^{i\theta}p^{-k/2}, tp^{k/2}e^{-i\theta};p)_\infty}\,d\theta$$

$$+\frac{(s;s)_\infty}{2(1-t^2)(p;p)_\infty}\sum_{j=0}^{k-1}\frac{(t^2;p)_j (t^2p^{2j},t^{-2}p^{-2j};s)_\infty}{(1/p;1/p)_j}.$$

Furthermore

$$\lim_{t\to p^{-k/2}} G_2(t,p,p,s) = \frac{1}{2(1-p^{-k})(p;p)_\infty}\sum_{j=0}^{k}\begin{bmatrix}k\\ j\end{bmatrix}_p (s,p^{2j-k},p^{k-2j};s)_\infty p^{j(j-k)}.$$

3. Sums

Using techniques from orthogonal polynomials, one can give power series representations in t for several special choices of p in the integrals

$$S_{p,q}(t) = G_2(t,q,q,p) = \frac{(qt^2;q)_\infty (p;p)_\infty}{2\pi}\int_0^{\pi}\frac{(e^{2i\theta},e^{-2i\theta};p)_\infty}{(te^{i\theta},te^{-i\theta};q)_\infty}d\theta,$$

$$H_{p,q}(t) = G_2(t,p,p^2,q^2) = \frac{(q^2;q^2)_\infty (pt^2;p^2)_\infty}{2\pi}\int_0^{\pi}\frac{(e^{2i\theta},e^{-2i\theta};q^2)_\infty}{(te^{i\theta},te^{-i\theta};p)_\infty}d\theta,$$

$$J_{p,q}(t) = G_1(t,p^2,p,q) = \frac{(q;q)_\infty (-t;p)_\infty}{2\pi}\int_0^{\pi}\frac{(e^{2i\theta},e^{-2i\theta};q)_\infty}{(te^{2i\theta},te^{-2i\theta};p^2)_\infty}d\theta.$$

For example

$$\begin{aligned}
S_{q^5,q}(t) &= \sum_{n=0}^{\infty} \frac{q^{n^2+2n}t^{2n}}{(q;q)_n}, \\
S_{q^5,q^2}(t) &= \sum_{n=0}^{\infty} \frac{(q;q^2)_n}{(q^2;q^2)_{2n}} q^{n(3n+2)}(-t^2)^n, \\
S_{q^7,q^2}(t) &= \sum_{n=0}^{\infty} \frac{(q;q^2)_n}{(q^2;q^2)_{2n}} q^{2n(n+2)}t^{2n}, \\
S_{-q^3,q}(t) &= \sum_{n=0}^{\infty} \frac{(-1;q^2)_n}{(q;q)_{2n}} q^{n(n+2)}t^{2n}. \\
S_{\omega q^3,q}(t) &= 1+(1-\omega)\sum_{n=1}^{\infty} \frac{(q^3;q^3)_{n-1}}{(q;q)_{2n}(q;q)_{n-1}} q^{n(n+2)}t^{2n}.
\end{aligned}$$

$$\begin{aligned}
H_{q^2,q}(t) &= \sum_{n=0}^{\infty} \frac{q^{2n^2}(-t^2)^n}{(q^4;q^4)_n} = (q^2t^2;q^4)_\infty, \\
H_{iq,q}(t) &= \sum_{n=0}^{\infty} \frac{(-1;q^4)_n}{(iq;iq)_{2n}}(-qt)^{2n}, \\
H_{q,q^2}(t) &= \sum_{n=0}^{\infty} \frac{(q^2;q^4)_n}{(q;q)_{2n}}(q^2t^2)^n = \frac{(-q^3t^2;q^2)_\infty}{(q^2t;q^2)_\infty}, \\
H_{q,q^3}(t) &= \sum_{n=0}^{\infty} \frac{(-q;q)_{2n}}{(q^2;q^2)_n}(qt)^{2n}, \\
H_{q^2,q^3}(t) &= \sum_{n=0}^{\infty} \frac{(q^4t^2)^n}{(q^4;q^4)_n} = \frac{1}{(q^4t^2;q^4)_\infty},
\end{aligned}$$

$$\begin{aligned}
J_{q^2,q}(t) &= \sum_{n=0}^{\infty} \frac{(q;q^2)_n}{(q^4;q^4)_n} q^{n^2}(-t)^n, \\
J_{-q,q}(t) &= \sum_{n=0}^{\infty} \frac{(-1;q^2)_n}{(q^2;q^2)_n}(-qt)^n = \frac{(qt;q^2)_\infty}{(-qt;q^2)_\infty}, \\
J_{q,q^2}(t) &= \sum_{n=0}^{\infty} \frac{(q;q^2)_n}{(q^2;q^2)_n}(qt)^n = \frac{(q^2t;q^2)_\infty}{(qt;q^2)_\infty}, \\
J_{q,q^3}(t) &= \sum_{n=0}^{\infty} \frac{(q;q)_{2n}}{(q;q)_n(q^2;q^2)_n}(qt)^n, \\
J_{q^2,q^3}(t) &= \sum_{n=0}^{\infty} \frac{(q;q^2)_n}{(q^4;q^4)_n}(q^2t)^n.
\end{aligned}$$

Note that several of these series converge for all t, thus are analytic continuations of the integrals, allowing one to specialize t, and apply Corollary 2 and Theorem 3. These are Rogers-Ramanujan identities. For example, specializing $S_{\omega q^3,q}(t)$, $\omega = e^{2\pi i/3}$, yields

Theorem 4. *We have*

$$1 + (1-\omega)\sum_{n=1}^{\infty} \frac{(q^3;q^3)_{n-1}}{(q;q)_{2n}(q;q)_{n-1}} q^{n(n+1)} = \frac{(\omega q^3, \omega q^2, q; \omega q^3)_\infty}{(q;q)_\infty}$$

$$1 + (1-\omega)\sum_{n=1}^{\infty} \frac{(q^3;q^3)_{n-1}}{(q;q)_{2n}(q;q)_{n-1}} q^{n^2} = \frac{(\omega q^3, q^2, \omega q; \omega q^3)_\infty}{(q;q)_\infty}.$$

4. Integer partition interpretations

Several of the identities are equivalent to integer partition statements, here are three examples.

Corollary 5. *Let $A(n)$ be the number of integer partitions of n into parts congruent to 2,4, 10 or 12 mod 14 and distinct parts congruent to 1,5,7,9, or 13 mod 14. Let $B(n)$ be the number of integer partitions of n*
(1) whose odd parts are consecutive (starting with 1) and have multiplicity one or two,
(2) whose largest even part is at most two more than twice the largest odd part.
Then $A(n) = B(n)$.

Corollary 6. *Let $A(n)$ be the number of integer partitions of n into parts congruent to 2,6,8, or 12 mod 14 and distinct parts congruent to 3,7, or 11 mod 14. Let $B(n)$ be the number of integer partitions of n*
(1) whose odd parts are consecutive (starting with 1) and have multiplicity two or three,
(2) whose largest even part is at most two more than twice the largest odd part.
Then $A(n) = B(n)$.

Corollary 7. *Let $A(n)$ be the number of integer partitions of n into parts congruent to 4 or 6 mod 10 and distinct parts congruent to 3,5, or 7 mod 10. Let $B(n)$ be the number of integer partitions of n*
(1) whose odd parts are consecutive (starting with 1) and have multiplicity three or four,
(2) whose largest even part is at most two more than twice the largest odd part.
Then $A(n) = B(n)$.

5. m-versions

The Rogers-Ramanujan identities have the natural generalization [4]

$$\sum_{n=0}^{\infty} \frac{q^{n^2+mn}}{(q;q)_n} = \frac{a_m(q)}{(q,q^4;q)_\infty} + \frac{b_m(q)}{(q^2,q^3;q)_\infty}, \tag{2}$$

where $a_m(q)$ and $b_m(q)$ are Laurent polynomials in q which are explicitly known. We refer to (2) as an "m-version" of the Rogers-Ramanujan identities.

According to Theorem 3

$$\lim_{t\to q^{-m/2}} S_{p,q}(t) = \frac{1}{2(1-q^{-m})}\sum_{j=0}^{m} \begin{bmatrix} m \\ j \end{bmatrix}_q \frac{(p, q^{2j-m}, q^{m-2j}; p)_\infty}{(q;q)_\infty} q^{j(j-m)}, \tag{3}$$

for $m = 1, 2, \ldots$. Equation (3) generalizes (2) and gives an explicit form for the generalizations of the polynomials $a_m(q)$ and $b_m(q)$. This alternating form is a special case of the hook difference polynomials in [3]. However explicit positive forms may be found using the recurrence relations for the polynomials. In the mod 5 case, this recurrence is three-term and related to orthogonal polynomials. But in other cases higher order recurrences do occur.

For example

$$\begin{aligned} S_{q^7,q^2}(q^{-m}) &= \sum_{n=0}^{\infty} \frac{(q;q^2)_n}{(q^2;q^2)_{2n}}\, q^{2n^2+4n-2mn} \\ &= c_1(m,7,2)\frac{(q^7,q^1,q^6;q^7)_\infty}{(q^2,q^2)_\infty} + c_2(m,7,2)\frac{(q^7,q^2,q^5;q^7)_\infty}{(q^2,q^2)_\infty} \\ &+ c_3(m,7,2)\frac{(q^7,q^3,q^4;q^7)_\infty}{(q^2,q^2)_\infty}. \end{aligned}$$

where $g_m = c_i(m+2,7,2)$ satisfies the four term recurrence relation

$$g_{m+2} + q^{-1}g_{m+1} - (1+q^{-2-2m})g_m - q^{-1}g_{m-1} = 0.$$

The explicit positive forms are $(s = q^{-1})$

$$\begin{aligned} c_1(n+2,7,2) &= (-1)^{n-1} \sum_{2m+2j+k+1=n} \begin{bmatrix} m+j \\ j \end{bmatrix}_{s^4} \begin{bmatrix} m+k \\ k \end{bmatrix}_{s^2} s^{2m(m+1)+k} \\ c_2(n+2,7,2) &= (-1)^{n-1} \sum_{2m+2j+k+1=n} \begin{bmatrix} m+j \\ j \end{bmatrix}_{s^4} \begin{bmatrix} m+k-1 \\ k \end{bmatrix}_{s^2} s^{2m(m+1)+k}, \\ c_3(n+2,7,2) &= (-1)^{n} \sum_{2m+2j+k=n} \begin{bmatrix} m+j \\ j \end{bmatrix}_{s^4} \begin{bmatrix} m+k-1 \\ k \end{bmatrix}_{s^2} s^{2m^2+k}. \end{aligned}$$

One may also find these explicit positive forms for $n < 0$.

Of the identities on Slater's list [5], 39 involve 3-term recurrences and 59 involve 4-term recurrences.

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