REFINED POSITIVITY CONJECTURES AND THE MACDONALD POLYNOMIALS

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ABSTRACT. We consider a filtration of the symmetric function space given by the linear span of Hall-Littlewood polynomials indexed by partitions whose first part is not larger than k. We introduce symmetric functions called the k-Schur functions, providing an analog for the Schur functions in these k-subspaces. We give several properties for the k-Schur functions including that they form a basis that reduces to the Schur basis when k is large. We also show that the connection coefficients for the k-Schur function basis with Macdonald polynomials belonging to the k-subspaces are polynomials in q and t with integral coefficients. In fact, we conjecture that these integral coefficients are actually positive, and give several other conjectures generalizing Schur function theory.

RÉSUMÉ. Nous étudions la filtration de l'espace des fonctions symétriques que l'on obtient en considérant l'engendré linéaire des polynômes de Hall-Littlewood indicés par des partitions dont la première entrée n'est pas plus grande que k, pour $k = 1, 2, 3, ...$ Nous introduisons des fonctions symétriques, les k-fonctions de Schur, qui sont en quelque sorte les analogues des fonctions de Schur dans le sous-espace correspondant à k de la filtration. Nous obtenons plusieurs propriétés de ces k-fonctions de Schur, parmi lesquelles le fait qu'elles forment une base se réduisant à la base des fonctions de Schur lorsque k est grand. Nous démontrons aussi que les entrées de la matrice de changement de base entre les polynômes de Macdonald appartenant au k-ième sous-espace de la filtration et les k-fonctions de Schur sont des polynômes en q et t à coefficients entiers. Nous émettons la conjecture que ces coefficients entiers sont en fait positifs et formulons plusieurs autres conjectures se voulant des k-généralisations de propriétés des fonctions de Schur.

1. INTRODUCTION

Let Λ be the ring of symmetric functions in the variables $x_1, x_2, \ldots$, with coefficients in $\mathbb{Q}(q,t)$, for parameters q and t. The Schur functions, $s_\lambda[X]$, form a fundamental basis of Λ, with central roles in fields such as representation theory and algebraic geometry. For example, the Schur functions can be identified with the characters of irreducible representations of the symmetric group, and their products are equivalent to the Pieri formulas for multiplying Schubert varieties in the intersection ring of a Grassmannian. Furthermore, the connection coefficients of the Schur function basis with various bases such as the homogeneous symmetric functions, the Hall-Littlewood polynomials, and the Macdonald polynomials, are positive and have representation theoretic interpretations. In the case of the Macdonald polynomials, $H_\lambda[X;q,t]$, this expansion takes the form

$$(1) \qquad H_\lambda[X;q,t] = \sum_\mu K_{\mu\lambda}(q,t)\, s_\mu[X]\,, \qquad K_{\mu\lambda}(q,t) \in \mathbb{N}[q,t]\,,$$

where $K_{\mu\lambda}(q,t)$ are known as the q,t-Kostka polynomials. The representation theoretic interpretation for these polynomials is given in [1, 2, 3].

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We study the filtration $\Lambda_t^{(1)} \subseteq \Lambda_t^{(2)} \subseteq \cdots \subseteq \Lambda_t^{(\infty)} = \Lambda$, given by the linear span of Hall-Littlewood polynomials indexed by k-bounded partitions. That is,

$$\Lambda_t^{(k)} = \mathcal{L}\{H_\lambda[X;t]\}_{\lambda;\lambda_1\leq k}\,, \qquad k=1,2,3,\dots\,. \tag{2}$$

We introduce a new family of symmetric functions that are indexed by k-bounded partitions, denoted $s_\lambda^{(k)}[X;t]$, and give a number of properties for these functions. In particular, we show that they form bases for the subspaces, $\Lambda_t^{(k)}$. Our functions will be called the k-Schur functions since they appear to play a role for $\Lambda_t^{(k)}$ that is analogous to the role of the Schur functions for Λ. That is, the k-Schur functions give rise to the generalization of many Schur positivity properties. Details are given following a brief outline of our construction for $s_\lambda^{(k)}[X;t]$.

The characterization of $s_\lambda^{(k)}[X;t]$ relies on a t-generalization for Schur function products. More precisely, for any partition sequence $S=(\lambda^{(1)},\dots,\lambda^{(\ell)})$, the t-analog for the product $s_{\lambda^{(1)}}[X]\cdots s_{\lambda^{(\ell)}}[X]$ has been studied in a number of papers including [9, 11, 12, 13, 14]. We find that a very particular subset of these generalized products forms a basis for $\Lambda_t^{(k)}$. The elements of our basis, denoted $G_\lambda^{(k)}[X;t]$, are the generalized Schur products with sequence S obtained by splitting λ into pieces that depend on k. $G_\lambda^{(k)}[X;t]$ are thus called k-split polynomials. These polynomials are essential in our definition for the k-Schur functions as we use a linear operator on $\Lambda_t^{(k)}$ defined by

$$T_i^{(k)}\, G_\lambda^{(k)}[X;t] = \begin{cases} G_\lambda^{(k)}[X;t] & \text{if } \lambda_1 = i \\ 0 & \text{otherwise} \end{cases}\,. \tag{3}$$

The final ingredient needed to define the k-Schur functions is the vertex operator, B_i, introduced in [4] to recursively build the Hall-Littlewood polynomials. More precisely,

$$H_{\lambda_1,\dots,\lambda_\ell}[X;t] = B_{\lambda_1}\cdots B_{\lambda_\ell}\cdot 1\,. \tag{4}$$

Analogously to this relation, we now define the k-Schur function for k-bounded $\lambda = (\lambda_1,\dots,\lambda_\ell)$, by

$$s_\lambda^{(k)}[X;t] = T_{\lambda_1}^{(k)} B_{\lambda_1}\cdots T_{\lambda_\ell}^{(k)} B_{\lambda_\ell}\cdot 1\,. \tag{5}$$

Our work to characterize this basis was originally motivated by two conjectures suggesting that the k-Schur functions play a central role in the understanding of the q,t-Kostka polynomials. Together, these conjectures refine relation (1). That is, for any k-bounded partition λ,

$$i)\qquad s_\lambda^{(k)}[X;t] = \textstyle\sum_{\mu\geq\lambda} v_{\mu\lambda}^{(k)}(t)\, s_\mu[X]\,, \qquad v_{\mu\lambda}^{(k)}(t)\in\mathbb{N}[t]\,, \tag{6}$$

$$ii)\quad H_\lambda[X;q,t] = \textstyle\sum_{\mu;\mu_1\leq k} K_{\mu\lambda}^{(k)}(q,t)\, s_\mu^{(k)}[X;t]\,, \qquad K_{\mu\lambda}^{(k)}(q,t)\in\mathbb{N}[q,t]\,. \tag{7}$$

Both conjectures hold when $k=2$ and we prove that $v_{\mu\lambda}^{(k)}(t)\in\mathbb{Z}[t]$ and that $K_{\mu\lambda}^{(k)}(q,t)\in\mathbb{Z}[q,t]$ for all k. Tables of coefficients $v_{\mu\lambda}^{(k)}(t)$ and $K_{\mu\lambda}^{(k)}(q,t)$ are included in section 5 to illustrate these conjectures. Our examples suggest an even stronger property,

$$0 \subseteq K_{\mu\lambda}^{(k)}(q,t) \subseteq K_{\mu\lambda}(q,t) \tag{8}$$

where for two polynomials $P,Q\in\mathbb{Z}[t]$, $P\subseteq Q$ means $Q-P\in\mathbb{N}[q,t]$.

More generally, it develops that properties of the k-Schur functions, with a number of conjectures, provide a k-generalization for the properties that make the Schur functions important to the theory of symmetric functions. In particular, we prove that the $s_\lambda^{(k)}[X;t]$ form a basis for $\Lambda_t^{(k)}$ and that the k-Schur functions of $\Lambda_t^{(\infty)} = \Lambda$ are indeed the Schur functions themselves. Conjectural evidence for the significance of the k-Schur functions includes a k-analog of partition conjugation and generalizations of Pieri and Littlewood-Richardson rules. Consequently, in the case of the multiplicative action of $h_1[X]$, a k-analog of the Young Lattice is induced. Further, we have observed that the k-Schur functions, expanded in terms of k-Schur functions in two sets of variables, have coefficients in $\mathbb{N}[t]$. This is a special property of Schur functions that is not shared by the Hall-Littlewood or Macdonald functions. Finally, the k-Schur functions of $\Lambda_t^{(k)}$, when embedded in $\Lambda_t^{(k')}$ for $k' > k$, seem to decompose positively in terms of k'-Schur functions:

$$s_\lambda^{(k)}[X;t] = s_\lambda^{(k')}[X;t] + \sum_{\mu > \lambda} v_{\mu\lambda}^{(k \to k')}(t)\, s_\mu^{(k')}[X;t]\,, \quad \text{where } v_{\mu\lambda}^{(k\to k')}(t) \in \mathbb{N}[t]\,. \tag{9}$$

Remarkably, not all of the k-Schur functions need to be constructed using (5). For each k, there is a subset of $s_\lambda^{(k)}[X;t]$, called the irreducible k-Schur functions, from which all other k-Schur functions may be constructed [7] by simply applying a succession of certain operators. The elements of this set are the k-Schur functions indexed by partitions with no more than i parts equal to $k-i$, and the operators are vertex operators [14] associated to rectangularly shaped partitions $(\ell^{k+1-\ell})$ for $\ell = 1, \ldots, k$. That is,

$$s_\lambda^{(k)}[X;t] = t^c\, B_{R_1} B_{R_2} \ldots B_{R_\ell}\, s_\mu^{(k)}[X;t] \quad \text{with} \quad c \in \mathbb{N}\,, \tag{10}$$

for an irreducible $s_\mu^{(k)}$, and vertex operators B_R, where R is a partition of rectangular shape.

Since the Hall-Littlewood polynomials at $t = 1$ are the complete symmetric functions

$$H_\lambda[X;1] = h_{\lambda_1}[X]\, h_{\lambda_2}[X] \cdots h_{\lambda_\ell}[X]\,, \tag{11}$$

we see that $\Lambda_t^{(k)}$ reduces to the polynomial ring $\Lambda^{(k)} = \mathbb{Q}[h_1, \ldots, h_k]$. Each of the properties held by the k-Schur functions has a specialization in this subring. In particular, since B_R is simply multiplication by the Schur function s_R when $t = 1$, relation (10) reduces to

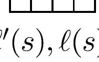
$$s_\lambda^{(k)}[X] = s_{R_1}[X] s_{R_2}[X] \ldots s_{R_\ell}[X] s_\mu^{(k)}[X]\,. \tag{12}$$

The irreducible k-Schur functions thus constitute a natural basis for the quotient ring $\Lambda^{(k)}/\mathcal{I}_k$, where $\mathcal{I}_k$ is the ideal generated by Schur functions indexed by partitions of the form $(\ell^{k+1-\ell})$.

2. Definitions

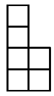
2.1. **Partitions.** Symmetric polynomials are indexed by partitions, sequences of non-negative integers $\lambda = (\lambda_1, \lambda_2, \ldots)$ with $\lambda_1 \geq \lambda_2 \geq \ldots$. The number of non-zero parts in λ is denoted $\ell(\lambda)$ and the degree of λ is $|\lambda| = \lambda_1 + \cdots + \lambda_{\ell(\lambda)}$. We use the dominance order on partitions with $|\lambda| = |\mu|$, where $\lambda \leq \mu$ when $\lambda_1 + \cdots + \lambda_i \leq \mu_1 + \cdots + \mu_i$ for all i.

Any partition λ has an associated Ferrers diagram with λ_i lattice squares in the i^{th} row, from the bottom to top. For example,

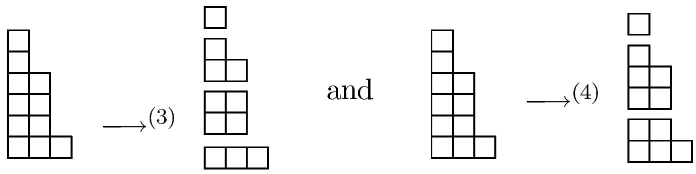
(13) $\lambda = (4,2) =$  .

For each cell $s = (i,j)$ in the diagram of λ, let $\ell'(s), \ell(s), a(s)$ and $a'(s)$ be respectively the number of cells in the diagram of λ to the south, north, east and west of the cell s. The hook-length of any cell in λ, is $h_s(\lambda) = \ell(s) + a(s) + 1$. In the example, $h_{(1,2)}(4,2) = 2+1+1$. The *main hook-length* of λ, $h_M(\lambda)$, is the hook-length of the cell $s = (1,1)$ in the diagram of

λ. Therefore, $h_M\big((4,2)\big) = 5$. The conjugate λ' of a partition λ is defined by the reflection of the Ferrers diagram about the main diagonal. For example, the conjugate of (4,2) is

(14) $\lambda' =$  $= (2,2,1,1)\,.$

A partition λ is said to be *k-bounded* if its first part is not larger than k, i.e, if $\lambda_1 \le k$. We associate to any k-bounded partition λ a sequence of partitions, $\lambda^{\to k}$, called the *k-split* of λ. $\lambda^{\to k} = (\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(r)})$ is obtained by partitioning λ (without rearranging the entries) into partitions $\lambda^{(i)}$ where $h_M(\lambda^{(i)}) = k$, for all $i < r$. For example, $(3,2,2,2,1,1)^{\to 3} = \big((3),(2,2),(2,1),(1)\big)$ and $(3,2,2,2,1,1)^{\to 4} = \big((3,2),(2,2,1),(1)\big)$. Equivalently, the diagram of λ is cut horizontally into partitions with main hook-length k.

(15) 

The last partition in the sequence $\lambda^{\to k}$ may have main hook- length less than k.

2.2. **Symmetric functions.** The power sum $p_i(x_1, x_2, \ldots)$ is

(16) $$p_i(x_1, x_2, \ldots) = x_1^i + x_2^i + \cdots\,,$$

and for a partition $\lambda = (\lambda_1, \lambda_2, \dots)$,

(17) $$p_\lambda(x_1, x_2, \ldots) = p_{\lambda_1}(x_1, x_2, \ldots)\, p_{\lambda_2}(x_1, x_2, \ldots) \cdots\,.$$

We employ the notation of λ-rings, needing only the formal ring of symmetric functions Λ to act on the ring of rational functions in $x_1, \ldots, x_N, q, t$, with coefficients in $\mathbb{R}$. The action of a power sum p_i on a rational function is, by definition,

(18) $$p_i\left[\frac{\sum_\alpha c_\alpha u_\alpha}{\sum_\beta d_\beta v_\beta}\right] = \frac{\sum_\alpha c_\alpha u_\alpha^i}{\sum_\beta d_\beta v_\beta^i},$$

with $c_\alpha, d_\beta \in \mathbb{R}$ and u_α, v_β monomials in $x_1, \ldots, x_N, q, t$. Since the power sums form a basis of the ring Λ, any symmetric function has a unique expression in terms of power sums, and (18) extends to an action of Λ on rational functions. In particular $f[X]$, the action of a symmetric function f on the monomial $X = x_1 + \cdots + x_N$, is simply $f(x_1, \ldots, x_N)$. In the remainder of the article, we will always consider the number of variables N to be infinite, unless otherwise specified.

The monomial symmetric function $m_\lambda[X_n]$ is

(19) $$m_\lambda[X_n] = \sum_{\sigma \in S_n;\sigma(\lambda)\text{ distinct}} x^{\sigma(\lambda)}\,.$$

The complete symmetric function $h_r[X]$ is

(20) $$h_r[X] = \sum_{1 \le i_1 \le i_2 \le \cdots \le i_r} x_{i_1} x_{i_2} \cdots x_{i_r}\,,$$

and $h_\lambda[X]$ stands for the homogeneous symmetric function

(21) $$h_\lambda[X] = h_{\lambda_1}[X]\, h_{\lambda_2}[X] \cdots\,.$$

Although the Schur functions may be characterized in many ways, here it will be convenient to use the Jacobi-Trudi determinantal expression:

$$s_\lambda[X] = \det\Big|h_{\lambda_i+j-1}[X]\Big|_{1\le i,j\le \ell(\lambda)} \tag{22}$$

where $h_r[X]=0$ if $r<0$. Note, in particular, $s_r[X]=h_r[X]$.

We recall that the Macdonald scalar product, $\langle\ ,\ \rangle_{q,t}$, on $\Lambda\otimes\mathbb{Q}(q,t)$ is defined by setting

$$\langle p_\lambda[X], p_\mu[X]\rangle_{q,t} = \delta_{\lambda\mu}\, z_\lambda \prod_{i=1}^{\ell(\lambda)} \frac{1-q^{\lambda_i}}{1-t^{\lambda_i}}\,, \tag{23}$$

where for a partition λ with $m_i(\lambda)$ parts equal to i, we associate the number

$$z_\lambda = 1^{m_1} m_1!\, 2^{m_2} m_2! \cdots \tag{24}$$

If $q=t$, this expression no longer depends on a parameter and is then denoted $\langle,\rangle$, satisfying

$$\langle s_\lambda[X], s_\mu[X]\rangle = \delta_{\lambda\mu}\,. \tag{25}$$

The Macdonald integral forms $J_\lambda[X;q,t]$ are uniquely characterized [10] by

$$\text{(i) } \langle J_\lambda, J_\mu\rangle_{q,t} = 0, \qquad \text{if } \lambda\neq\mu, \tag{26}$$

$$\text{(ii) } J_\lambda[X;q,t] = \sum_{\mu\le\lambda} v_{\lambda\ mu}(q,t) s_\mu[X] \quad \text{with} \quad v_{\lambda\mu}(q,t)\in\mathbb{Q}(q,t)\,, \tag{27}$$

$$\text{(iii) } v_{\lambda\lambda}(q,t) = \prod_{s\in\lambda}(1-q^{a(s)}t^{\ell(s)+1}), \tag{28}$$

Here, we use a modification of the Macdonald integral forms that is obtained by setting

$$H_\lambda[X;q,t] = J_\lambda[X/(1-t);q,t] = \sum_\mu K_{\mu\lambda}(q,t)\, s_\mu[X]\,, \tag{29}$$

with the coefficients $K_{\mu\lambda}(q,t)\in\mathbb{N}[q,t]$ known as the q,t-Kostka polynomials. When $q=0$, $J_\lambda[X;q,t]$ reduces to the Hall-Littlewood polynomial, $J_\lambda[X;0,t]=Q_\lambda[X;t]$. Again, we use a modification;

$$H_\lambda[X;t] = H_\lambda[X;0,t] = Q_\lambda[X/(1-t);t] = s_\lambda[X] + \sum_{\mu>\lambda} K_{\mu\lambda}(t)\, s_\mu[X]\,, \tag{30}$$

with the coefficients $K_{\mu\lambda}(t)\in\mathbb{N}[t]$ known as the Kostka-Foulkes polynomials.

3. Results

The ring of symmetric polynomials over rational functions in an extra parameter t has proven to be of interest in many fields of mathematics and physics. One natural basis of this space is given by the Hall-Littlewood polynomials, $H_\lambda[X;t]$, which provide t-analogs of the homogeneous symmetric functions $h_\lambda[X]$. Our approach employs vertex operators that arise in the recursive construction for the Hall-Littlewood polynomials [4]. These operators can be defined [14] for $\ell\in\mathbb{Z}$, by

$$B_\ell = \sum_{i=0}^{\infty} s_{i+\ell}[X]\, s_i[X(t-1)]^\perp\,, \tag{31}$$

where for f, g and h arbitrary symmetric functions, $f^\perp$ is such that on the scalar product (25),

$$\langle f^\perp g, h\rangle = \langle g, fh\rangle\,. \tag{32}$$

The operators add an entry to the Hall-Littlewood polynomials, that is,

$$H_\lambda[X;t] = B_{\lambda_1} H_{\lambda_2,\dots,\lambda_\ell}[X;t]\,, \qquad \text{for} \qquad \lambda_1 \ge \lambda_2\,. \tag{33}$$

We study the subspaces given by

$$\Lambda_t^{(k)} = \mathcal{L}\,\{H_\lambda[X;t]\}_{\lambda;\lambda_1\le k}\,. \tag{34}$$

It is clear that $\Lambda_t^{(1)} \subseteq \Lambda_t^{(2)} \subseteq \cdots \subseteq \Lambda_t^{(\infty)} = \Lambda$ and thus that these subspaces provide a filtration for Λ. Note that $\Lambda_t^{(k)}$ can be equivalently defined as

$$\Lambda_t^{(k)} = \mathcal{L}\,\{H_\lambda[X;q,t]\}_{\lambda;\lambda_1\le k}\,. \tag{35}$$

We seek elements that play the role in $\Lambda_t^{(k)}$ that the Schur functions play in Λ. In particular, since $\Lambda_t^{(\infty)} = \Lambda$, we want a basis for $\Lambda_t^{(k)}$ that reduces to the Schur functions when k is large.

3.1. **k-split polynomials.** Important in our work to find such functions is a family of polynomials, studied in many recent papers such as [9, 11, 12, 13, 14], that give a t-analog of the product of Schur functions. These functions, indexed by a sequence of partitions, can be built recursively using vertex operators [14]. For a partition λ of length m, define

$$B_\lambda \equiv \prod_{1\le i<j\le m} (1 - t e_{ij}) B_{\lambda_1}\cdots B_{\lambda_m}\,, \tag{36}$$

where e_{ij} acts by

$$e_{ij}\,(B_{\lambda_1}\cdots B_{\lambda_m}) = B_{\lambda_1}\cdots B_{\lambda_i+1}\cdots B_{\lambda_j-1}\cdots B_{\lambda_m}\,. \tag{37}$$

For any sequence of partitions $(\lambda^{(1)},\lambda^{(2)},\dots)$, the generalized Schur function product can then be defined recursively by

$$\mathcal{H}_{(\lambda^{(1)},\lambda^{(2)},\lambda^{(3)},\dots)}[X;t] = B_{\lambda^{(1)}} \mathcal{H}_{(\lambda^{(2)},\lambda^{(3)},\dots)}[X;t]\,, \tag{38}$$

starting with $\mathcal{H}_{()} = 1$. Note that since $B_\lambda \cdot 1 = s_\lambda[X]$, we have that

$$\mathcal{H}_{(\lambda)}[X;t] = s_\lambda[X]\,. \tag{39}$$

For our purposes, we consider only the $\mathcal{H}_S[X;t]$ indexed by a dominant sequence S. That is, sequences of partitions $S = (\lambda^{(1)},\lambda^{(2)},\dots)$ such that the concatenation of $\lambda^{(1)},\lambda^{(2)},\dots,$ denoted $\bar{S}$, forms a partition. In this case, we prove that $\mathcal{H}_S[X;t]$ obeys important unitriangular relations.

Property 1. [6] *If S is dominant, with $\bar{S} = \lambda$, then*

$$\mathcal{H}_S[X;t] = s_\lambda[X] + \sum_{\mu>\lambda} K_{\mu;S}(t) s_\mu[X]\,, \quad \textit{where} \quad K_{\mu;S}(t) \in \mathbb{Z}[t]\,, \tag{40}$$

$$\mathcal{H}_S[X;t] = H_\lambda[X] + \sum_{\mu>\lambda} C_{\mu;S}(t) H_\mu[X]\,, \quad \textit{where} \quad C_{\mu;S}(t) \in \mathbb{Z}[t]\,. \tag{41}$$

Furthermore, we discovered that a particular subset of the $\mathcal{H}_S$ not only form a basis for $\Lambda_t^{(k)}$, but are essential in the construction of the k-Schur functions. This subset consists only of the elements that are indexed by the *k-split* of a k-bounded partition. More precisely,

Definition 1. *The k-split polynomials are defined, for a k-bounded partition λ, by*

$$G_\lambda^{(k)}[X;t] = \mathcal{H}_S[X;t] \quad \textit{where} \quad S = \lambda^{\to k} \quad \textit{is the } k\textit{-split of } \lambda\,. \tag{42}$$

We give a number of properties for the k-split polynomials, starting with the fact that the $G_\lambda^{(k)}[X;t]$ actually lie in the space $\Lambda_t^{(k)}$.

Property 2. [6] *For any k-bounded partition λ, we have that*

$$G_\lambda^{(k)}[X;t] \in \Lambda_t^{(k)} . \tag{43}$$

It also happens that the k-split polynomials are triangularly related to the Hall-Littlewood polynomials.

Property 3. [6] *We have*

$$G_\lambda^{(k)}[X;t] = H_\lambda[X;t] + \sum_{\mu>\lambda;\mu_1\leq k} g_{\lambda\mu}^{(k)}(t)\, H_\mu[X;t] , \quad \textit{where} \quad g_{\lambda\mu}^{(k)}(t) \in \mathbb{Z}[t] . \tag{44}$$

In fact, k-split polynomials reduce to usual Schur functions when k is large enough.

Property 4. [6] *Let λ be such that $h_M(\lambda) \leq k$. Then,*

$$G_\lambda^{(k)}[X;t] = s_\lambda[X] . \tag{45}$$

Given these properties, we are able to prove that the k-split polynomials form a basis for the k-subspaces.

Theorem 1. [6] *The k-split polynomials form a basis of $\Lambda_t^{(k)}$.*

3.2. **k-Schur functions.** Although the k-split polynomials form a basis for $\Lambda_t^{(k)}$, they do not play the fundamental role that the Schur functions do for Λ. That is, the positivity properties stated in Section 4 do not hold for the k-split polynomials. However, these polynomials are needed in the construction of our Schur analog, $s_\lambda^{(k)}[X;t]$. Using a projection operator, $T_j^{(k)}$, for $j \leq k$, that acts linearly on $\Lambda_t^{(k)}$ by

$$T_j^{(k)}\, G_\lambda^{(k)}[X;t] = \begin{cases} G_\lambda^{(k)}[X;t] & \text{if } \lambda_1 = j \\ 0 & \text{otherwise} \end{cases} , \tag{46}$$

we can define our functions that play an important role in the refinement of symmetric function theory.

Definition 2. *For k-bounded partition λ, the k-Schur functions are recursively defined*

$$s_\lambda^{(k)}[X;t] = T_{\lambda_1}^{(k)} B_{\lambda_1} s_{(\lambda_1,\lambda_2,\dots)}^{(k)}[X;t] , \quad \textit{where} \quad s_{()}^{(k)}[X;t] = 1 . \tag{47}$$

Tables of k-Schur functions in terms of Schur functions can be found in Section 5.1.

We prove several properties satisfied by the k-Schur functions, including that they are triangularly related to the k-split polynomials and that they form a basis for the k-subspaces.

Property 5. [6] *For λ a k-bounded partition, we have*

$$s_\lambda^{(k)}[X;t] = G_\lambda^{(k)}[X;t] + \sum_{\substack{\mu>\lambda \\ \mu_1=\lambda_1}} u_{\mu\lambda}^{(k)}(t)\, G_\mu^{(k)}[X;t] , \quad \textit{where} \quad u_{\mu\lambda}^{(k)}(t) \in \mathbb{Z}[t] . \tag{48}$$

Theorem 2. [6] *The k-Schur functions form a basis of $\Lambda_t^{(k)}$.*

We can thus refine the expansion of Hall-Littlewood polynomials in terms of Schur functions (30). That is, for any k-bounded partition λ,

$$H_\lambda[X;t] = s_\lambda^{(k)}[X;t] + \sum_{\mu>\lambda:\mu_1\le k} K_{\mu\lambda}^{(k)}(t)\, s_\mu^{(k)}[X;t]\,, \quad \text{where} \quad K_{\mu\lambda}^{(k)}(t) \in \mathbb{Z}[t]\,. \tag{49}$$

The integrality of $K_{\mu\lambda}^{(k)}(t)$ follows from the unitriangularity and integrality in Properties 3 and 5. Moreover, by our triangularity and integrality properties and (30), we also have integrality of the coefficients in Conjecture 6:

Property 6. [6] *For any k-bounded partition λ,*

$$s_\lambda^{(k)}[X;t] = s_\lambda[X] + \sum_{\mu>\lambda} v_{\mu\lambda}^{(k)}(t)\, s_\mu[X]\,, \quad \textit{where} \quad v_{\mu\lambda}^{(k)}(t) \in \mathbb{Z}[t]\,. \tag{50}$$

This unitriangularity property, given that the coefficients in the Schur function expansion of the Macdonald polynomials are polynomials in q and t with integral coefficients, implies

Property 7. [6] *For any k-bounded partition λ,*

$$H_\lambda[X;q,t] = \sum_{\mu:\mu_1\le k} K_{\mu\lambda}^{(k)}(q,t)\, s_\mu^{(k)}[X;t]\,, \quad \textit{where} \quad K_{\mu\lambda}^{(k)}(q,t) \in \mathbb{Z}[q,t]\,. \tag{51}$$

Now, to further support the idea that the $s_\lambda^{(k)}[X;t]$ provide a refinement for Schur function theory, we show that they reduce to the usual $s_\lambda[X]$ when $k \to \infty$.

Property 8. [6] *For any k-bounded partition λ,*

$$s_\lambda^{(k)}[X;t] = s_\lambda[X] \qquad \textit{if} \quad h_M(\lambda) \le k\,. \tag{52}$$

3.3. **Irreducibility.** Further results with the k-Schur functions naturally impose an irreducible structure on the space $\Lambda_t^{(k)}$. We prove that the action of such an operator on a k-Schur function produces only one k-Schur function. Namely, for $\ell = 1, 2, \ldots, k$,

$$B_{\ell^{k+1-\ell}}\, s_\lambda^{(k)}[X;t] = t^d s_\mu^{(k)}[X;t]\,, \tag{53}$$

where μ is the partition rearrangement of the entries in $(\ell^{k+1-\ell})$ and λ, and t^d is a positive power of t given explicitly in [7]. This result has the important consequence of simplifying the construction of the k-Schur functions. In effect, for each k, there is a subset of $k!$ k-Schur functions called the irreducible k-Schur functions, from which all other $s_\lambda^{(k)}[X;t]$ may be constructed by successive application of operators indexed by rectangular partitions. That is,

Property 9. [7] *For any k-bounded partition λ,*

$$s_\lambda^{(k)}[X;t] = t^c B_{R_1}\cdots B_{R_j}\, s_\mu^{(k)}[X;t] \qquad c \in \mathbb{N}\,, \tag{54}$$

where $s_\mu^{(k)}[X;t]$ is an irreducible k-Schur function and $R_1, \ldots, R_j$ are rectangular partitions.

Since the Hall-Littlewood polynomials at $t = 1$ are the homogeneous symmetric functions, $h_\lambda[X]$, $\Lambda_t^{(k)}$ reduces to the polynomial ring $\Lambda^{(k)} = \mathbb{Q}[h_1, \ldots, h_k]$. Since B_R is simply multiplication by the Schur function s_R when $t = 1$, relation (54) reduces to

$$s_\lambda^{(k)}[X] = s_{R_1}[X] s_{R_2}[X] \ldots s_{R_\ell}[X] s_\mu^{(k)}[X]\,. \tag{55}$$

It follows that the irreducible k-Schur functions thus constitute a natural basis for the quotient ring $\Lambda_t^{(k)}/\mathcal{I}_k$, where $\mathcal{I}_k$ is the ideal generated by Schur functions indexed by rectangular shapes of the type $(\ell^{k+1-\ell})$.

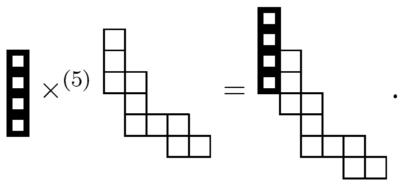
4. Positivity conjectures

Computer experimentation reveals that many of the properties making the Schur function basis so important are generalized by the k-Schur functions. We now state several of these properties.

4.1. **The k-conjugation of a partition.** We give a generalization of partition conjugation that is an involution on k-bounded partitions, and reduces to usual conjugation of partitions for large k.

A skew diagram D has hook-lengths bounded by k if the hook-length of any cell in D is not larger than k. For a positive integer $m \leq k$, the k-multiplication $m \times^{(k)} D$ is the skew diagram $\overline{D}$ obtained by prepending a column of length m to D such that the number of rows of $\overline{D}$ is as small as possible while ensuring that its hook-lengths are bounded by k. For example,

(56)



Definition 3. *The k-conjugate of a k-bounded partition $\lambda = (\lambda_1, \ldots, \lambda_n)$, denoted λ^{ω_k}, is the vector obtained by reading the number of boxes in each row of the skew diagram,*

$$D = \lambda_1 \times^{(k)} \cdots \times^{(k)} \lambda_n \,, \tag{57}$$

arising by k-multiplying the entries of λ from right to left.

When $k \to \infty$, $\lambda^{\omega_k} = \lambda' = D$ since each k-multiplication step reduces to concatenating a column of height λ_i. Further, the k-conjugate is an involution on k-bounded partitions:

Theorem 3. [5] *ω_k is an involution on partitions bounded by k. That is, for λ with $\lambda_1 \leq k$,*

$$(\lambda^{\omega_k})^{\omega_k} = \lambda \,. \tag{58}$$

We have observed that the k-conjugation of a partition plays a natural role in the generalization of classical Schur function properties. We now give two examples.

4.2. **The involution ω.** It is known that the involution in (25) preserves the space Λ and acts on a Schur function by

$$\omega s_\lambda[X] = s_{\lambda'}[X] \,. \tag{59}$$

We prove that a simple generalization for this involution,

$$\omega_t f = (\omega f)\big|_{t \to 1/t} \quad \text{for} \quad f \in \Lambda \tag{60}$$

preserves the space $\Lambda_t^{(k)}$. Note also that ω_t is an involution since ω is an involution. This given, many examples support the following natural generalization of (59):

Conjecture 1. [6] *For any k-bounded partition λ,*

$$\omega_t \, s_\lambda^{(k)}[X;t] = t^{-c(\lambda)} s_{\lambda^{\omega_k}}^{(k)}[X;t] \,, \tag{61}$$

where $c(\lambda)$ is some nonnegative integer.

The conjecture holds when $h_M(\lambda) \leq k$ since then, $\lambda^{\omega_k} = \lambda'$ and by Property 8, $s_\lambda^{(k)}[X;t] = s_\lambda[X]$.

4.3. **Pieri Rules.** Beautiful combinatorial algorithms are known for the Littlewood-Richardson coefficients that appear in a product of Schur functions;

$$s_\lambda[X]\, s_\mu[X] = \sum_\nu c_{\lambda\mu}^\nu\, s_\nu[X] \quad \text{where} \quad c_{\lambda\mu}^\nu \in \mathbb{N}\,. \tag{62}$$

Since $\Lambda^{(k)} \equiv \Lambda_{t=1}^{(k)}$ is a ring, and $s_\lambda^{(k)}$ forms a basis for this space, a similar expression holds for the product of two k-Schur functions. That is, for k-bounded partitions λ and μ,

$$s_\lambda^{(k)}[X]\, s_\mu^{(k)}[X] = \sum_\nu c_{\lambda\mu}^{\nu\,(k)} s_\nu^{(k)}[X] \quad \text{where} \quad c_{\lambda\mu}^{\nu\,(k)} \in \mathbb{Z}\,, \tag{63}$$

Further, Property 8 says that the k-Schur functions are simply the Schur functions when k is large, and therefore $c_{\lambda\mu}^{\nu\,(k)} = c_{\lambda\mu}^\nu$ for $k \geq |\nu|$. In fact, we believe the coefficients are nonnegative for all k.

Conjecture 2. [6] *For all k-bounded partitions λ, μ, ν, we have $0 \leq c_{\lambda\mu}^{\nu\,(k)} \leq c_{\lambda\mu}^\nu$.*

In particular, (63) reduces to a k-generalization of the Pieri rule when λ is a row (resp. column) of length $\ell \leq k$ since $s_\lambda^{(k)}[X]$ reduces to $h_\ell[X]$ (resp. $e_\ell[X]$). That is, for $\ell \leq k$,

$$h_\ell[X]\, s_\lambda^{(k)}[X] = \sum_{\mu \in E_{\lambda,\ell}^{(k)}} s_\mu^{(k)}[X] \quad \text{and} \quad e_\ell[X]\, s_\lambda^{(k)}[X] = \sum_{\mu \in \bar{E}_{\lambda,\ell}^{(k)}} s_\mu^{(k)}[X]\,, \tag{64}$$

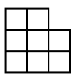
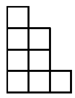
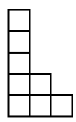
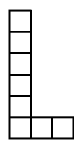
for some sets of partitions $E_{\lambda,\ell}^{(k)}$ and $\bar{E}_{\lambda,\ell}^{(k)}$, which we believe naturally extend the Pieri rules by:

Conjecture 3. [6] *For any positive integer $\ell \leq k$,*

$$\begin{aligned} E_{\lambda,\ell}^{(k)} &= \{\mu \,|\, \mu/\lambda \textit{ is a horizontal } \ell\textit{-strip and } \mu^{\omega_k}/\lambda^{\omega_k} \textit{ is a vertical } \ell\textit{-strip}\}\ , \\ \bar{E}_{\lambda,\ell}^{(k)} &= \{\mu \,|\, \mu/\lambda \textit{ is a vertical } \ell\textit{-strip and } \mu^{\omega_k}/\lambda^{\omega_k} \textit{ is a horizontal } \ell\textit{-strip}\}\ . \end{aligned} \tag{65}$$

For example, to obtain the indices of elements occurring in $e_2\, s_{3,2,1}^{(4)}$, we find $(3,2,1)^{\omega_4} = (2,2,1,1)$ by definition. Adding a horizontal 2-strip to (2,2,1,1) in all ways, we obtain (2,2,2,1,1),(3,2,1,1,1),(3,2,2,1) and (4,2,1,1) of which all are 4-bounded. Our set then consists of all the 4-conjugates of these partitions that leave a vertical 2-strip when $(3,2,1)$ is extracted from them. The 4-conjugates are

(66)

$(2,2,2,1,1)^{\omega_4} =$ , $(3,2,1,1,1)^{\omega_4} =$ , $(3,2,2,1)^{\omega_4} =$ , $(4,2,1,1)^{\omega_4} =$ ,

and of these, the first three are such that a vertical 2-strip remains when $(3,2,1)$ is extracted. Thus

$$e_2[X]\, s_{3,2,1}^{(4)}[X] \;=\; s_{3,3,2}^{(4)}[X] + s_{3,2,2,1}^{(4)}[X] + s_{3,2,1,1,1}^{(4)}[X]\,. \tag{67}$$

4.4. **Coproduct expansion.** Another unique property of the Schur functions is the expansion,

$$s_\lambda[X+Y] = \sum_{|\mu|+|\rho|=|\lambda|} c_{\mu\rho}^{\lambda}\, s_\mu[X]\, s_\rho[Y] \quad \text{where} \quad c_{\mu\rho}^{\lambda} \in \mathbb{N}\,. \tag{68}$$

We have found by experimentation that the k-Schur functions also satisfy a similar relation,

Conjecture 4. [6] *For any k-bounded partition,*

$$s_\lambda^{(k)}[X+Y;t] = \sum_{|\mu|+|\rho|=|\lambda|} g_{\mu\rho}^{\lambda}(t)\, s_\mu^{(k)}[X;t]\, s_\rho^{(k)}[Y;t] \quad \textit{where} \quad g_{\mu\rho}^{\lambda} \in \mathbb{N}[t]\,. \tag{69}$$

5. Tables

In the tables below, we have not included the cases when $k \geq |\lambda|$, which, from Property 8, simply correspond to the trivial cases $s_\lambda[X;t] = s_\lambda[X]$.

5.1. k-Schur functions in terms of Schur functions.

$k=2$	1^3	21	3
1^3	1	t	
21		1	t

$k=2$	1^4	21^2	2^2	31	4
1^4	1	t	t^2		
21^2		1		t	
2^2			1	t	t^2

$k=2$	1^5	21^3	2^21	31^2	32	41	5
1^5	1	$t+t^2$	t^2+t^3	t^3	t^4		
21^3		1	t	$t+t^2$	t^2	t^3	
2^21			1	t	$t+t^2$	t^2+t^3	t^4

$k=3$	1^4	21^2	2^2	31	4
1^4	1	t			
21^2		1		t	
2^2			1		
31				1	t

$k=3$	1^5	21^3	2^21	31^2	32	41	5
1^5	1	t	t^2				
21^3		1		t			
2^21			1		t		
31^2				1		t	
32					1	t	t^2

$k=3$	1^6	21^4	2^21^2	2^3	31^3	321	3^2	41^2	42	51	6
1^6	1	t	t^2	t^3							
21^4		1	t		t	t^2					
2^21^2			1			t	t^2				
2^3				1		t			t^2		
31^3					1			t			
321						1		t	t	t^2	
3^3							1		t	t^2	t^3

5.2. Macdonald polynomials in terms of k-Schur functions.

$k=2$	1^3	21
1^3	1	t^2
21	q	1

$k=2$	1^4	21^2	2^2
1^4	1	t^2+t^3	t^4
21^2	q	$1+qt^2$	t
2^2	q^2	$q+qt$	1

$k=2$	1^5	21^3	2^21
1^5	1	t^3+t^4	t^6
21^3	q	$1+qt^3$	t^2
2^21	q^2	$q+qt$	1

$k=2$	1^6	21^4	2^21^2	2^3
1^6	1	$t^3+t^4+t^5$	$t^6+t^7+t^8$	t^9
21^4	q	$1+qt^3+qt^4$	$t^2+t^3+qt^6$	t^4
2^21^2	q^2	$q+qt+q^2t^3$	$1+qt^2+qt^3$	t
2^3	q^3	$q^2+q^2t+q^2t^2$	$q+qt+qt^2$	1

$k=3$	1^4	21^2	2^2	31
1^4	1	t^2+t^3	t^2+t^4	t^5
21^2	q	$1+qt^2$	$t+qt^2$	t^2
2^2	q^2	$q+qt$	$1+q^2t^2$	t
31	q^3	$q+q^2$	$q+q^2t$	1

$k=3$	1^5	21^3	2^21	31^2	32
1^5	1	$t^2+t^3+t^4$	$t^3+t^4+t^5+t^6$	$t^5+t^6+t^7$	t^8
21^3	q	$1+qt^2+qt^3$	$t+t^2+qt^3+qt^4$	$t^2+t^3+qt^5$	t^4
2^21	q^2	$q+qt+q^2t^2$	$1+qt+qt^2+q^2t^3$	$t+qt^2+qt^3$	t^2
31^2	q^3	$q+q^2+q^3t^2$	$q+qt+q^2t+q^2t^2$	$1+qt^2+q^2t^2$	t
32	q^4	$q^2+q^3+q^3t$	$q+q^2+q^2t+q^3t$	$q+qt+q^2t$	1

$k=4$	1^5	21^3	2^21	31^2	32	41
1^5	1	$t^2+t^3+t^4$	$t^2+t^3+t^4+t^5+t^6$	$t^5+t^6+t^7$	$t^4+t^5+t^6+t^7+t^8$	t^9
21^3	q	$1+qt^2+qt^3$	$t+t^2+qt^2+qt^3+qt^4$	$t^2+t^3+qt^5$	$t^2+t^3+t^4+qt^4+qt^5$	t^5
2^21	q^2	$q+qt+q^2t^2$	$1+qt+qt^2+q^2t^2+q^2t^3$	$t+qt^2+qt^3$	$t+t^2+qt^2+qt^3+q^2t^4$	t^3
31^2	q^3	$q+q^2+q^3t^2$	$q+qt+q^2t+q^2t^2+q^3t^2$	$1+qt^2+q^2t^2$	$t+qt+qt^2+q^2t^2+q^2t^3$	t^2
32	q^4	$q^2+q^3+q^3t$	$q+q^2+q^2t+q^3t+q^4t^2$	$q+qt+q^2t$	$1+qt+q^2t+q^2t^2+q^3t^2$	t
41	q^6	$q^3+q^4+q^5$	$q^2+q^3+q^4+q^4t+q^5t$	$q+q^2+q^3$	$q+q^2+q^2t+q^3t+q^4t$	1

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