ENUMERATION OF UNICURSAL PLANAR MAPS

VALERY A. LISKOVETS* AND TIMOTHY R. S. WALSH†

ABSTRACT. Sum-free enumerative formulae are derived for several classes of rooted eulerian planar maps and maps with two vertices of odd valency. As corollaries we obtain simple formulae for the numbers of unrooted eulerian and unicursal planar maps.

RÉSUMÉ. Nous présentation des formules énumératives sans sommation pour plusieurs classes de cartes planaires eulériennes pointées et cartes unicursalles (avec deux sommets de degré impair). En tant que corollaires nous obtenons des formules simples pour les nombres de cartes planaires eulériennes et unicursalles non-pointées.

1. INTRODUCTION

1.1. Eulerian maps have played a crucial role in enumerative map theory since its beginning in the early sixties. In particular, Tutte's sum-free formula [Tut62] for the number eulerian planar maps, all of whose vertices are labelled and contain a distinguished edge-end, with a given sequence of (even) vertex valencies was an essential step in obtaining his ground-breaking formula for counting rooted planar maps by number of edges [Tut63]. Several new results on the subject have been published since then (see, e.g., [Wal75, Lis85, BouS00]) but a number of natural enumerative problems for eulerian maps have remained unsolved.

Here we consider a slight generalization of eulerian maps: unicursal maps. We continue our previous investigation and count planar maps of several classes. Our main results concern rooted unicursal maps (see Section 2) and unrooted eulerian as well as unicursal maps (see Section 3) specified by the number of edges. We present several new 1-parametric sum-free enumerative formulae.

In the last section, rooted eulerian bipartite and non-separable maps are enumerated. Although these results may be considered as implicitly known, they, apparently, had never been published.

There are some open questions related to the settled ones; they (together with asymptotic corollaries) are to be discussed in the full version of this article.

1.2. Basic definitions. A *map* means a planar map: a 2-cell imbedding of a planar connected graph (loops and multiple edges allowed) in an oriented sphere. A map is *rooted* if one of its edge-ends (variously known as edge-vertex incidence pairs, darts, semi-edges, or "brins" in French) is distinguished as the *root*. The corresponding edge and vertex are called the *root-edge* and *root-vertex*, respectively; the *root-face* is the face incident to the root-edge and on its left as one face away from the root-vertex.

A map (and a graph in general) is *eulerian* if it has an eulerian circuit, that is, a circuit containing all the edges exactly once. It is well-known that a map is eulerian if and only if all its vertices are of even valency.

We call a graph *unicursal* if it possesses an eulerian tour, not necessarily a circuit. A connected graph is unicursal if and only if it contains no more than two vertices of odd valency.

* Supported by the INTAS (Grant INTAS-BELARUS 97-0093).

† Supported by National Science and Engineering Research Council (NSERC) under grant RGPIN-44703.

However, throughout this paper, *par abus de langage*, we will use the term "unicursal maps" in the restricted sense to mean maps with *exactly two* vertices of odd valency.

2. Rooted unicursal maps

2.1. Unicursal maps together with eulerian maps are the very maps considered by Tutte in his seminal paper [Tut62]. But all the enumerative results obtained so far for unicursal maps concerned maps with specified vertex valencies. Accordingly, no formula has been known for the number of n-edged rooted unicursal maps. Evidently, the question cannot be reduced straightforwardly to eulerian maps (by adding an edge connecting the two odd-valent vertices, etc.) since the vertices of odd valency may not be incident to a common face. So, this natural class requires independent consideration.

Let $U'(n)$ denote the number of rooted unicursal maps with n edges. Let also $U'_i(n)$, $i = 0, 1, 2$, denote the number of rooted unicursal maps having i endpoints, i.e. vertices of valency 1. The next theorem is the main result of this paper.

2.2. Theorem.

$$U'(n) = 2^{n-2}\binom{2n}{n}, \qquad n \geq 1, \tag{2.1}$$

and for $n \geq 2$,

$$U'_0(n) = 2^{n-2}\,\frac{n-2}{n}\binom{2n-2}{n-1}, \tag{2.2}$$

$$U'_1(n) = 2^{n-1}\binom{2n-2}{n-1} \tag{2.3}$$

and

$$U'_2(n) = 2^{n-2}\binom{2n-2}{n-1}. \tag{2.4}$$

Proof. The number of unicursal planar maps with n edges and v labelled vertices of valencies $2d_1+1, 2d_2+1, 2d_3, \ldots, 2d_v$, each vertex rooted by distinguishing one of its edge-ends, is given in [Tut62, p. 772] as

$$C(2d_1+1, 2d_2+1, 2d_3, \ldots, 2d_v) = \frac{(n-1)!}{(n-v+2)!}\,\frac{(2d_1+1)!(2d_2+1)!}{d_1!^2 d_2!^2}\prod_{i=3}^{v}\frac{(2d_i)!}{d_i!(d_i-1)!}.$$

The number of rooted planar maps with n edges and v vertices, exactly two of which are of odd valency, is found from the previous equation by multiplying by the number of ways to root a map with n edges and dividing by the number of ways to label and root all the vertices of the same map so that the two vertices of odd valency get labels 1 and 2 (we multiply by $2n$ and divide by the product of the valencies and by $n!$ and then multiply by $n(n-1)/2$ to account for the fact that the two vertices of odd valency get labels 1 and 2) and then summing over the sequences of valencies that add to $2n$:

$$\frac{n!}{(v-2)!(n-v+2)!}\sum_{d_1+\cdots+d_v=n-1}\left\{\frac{(2d_1)!(2d_2)!}{d_1!^2 d_2!^2}\prod_{i=3}^{v}\frac{(2d_i-1)!}{d_i!(d_i-1)!}\right\}.$$

To obtain $U'(n)$ we evaluate the sum and then add over all possible values of v : from 2 to $n+1$.

Since $\sum_{j=0}^{\infty} \frac{(2j)!}{j!^2} x^j = (1-4x)^{-1/2}$ and $\sum_{j=1}^{\infty} \frac{(2j-1)!}{j!(j-1)!} x^j = \frac{(1-4x)^{-1/2}-1}{2}$, we have

$$U'(n) = [x^{n-1}] \sum_{v=2}^{n+1} \frac{n!}{(v-2)!(n-v+2)!} (1-4x)^{-1} \Big[\frac{(1-4x)^{-1/2}-1}{2}\Big]^{v-2},$$

where $[x^n]b$ means the coefficient of x^n in the power series $b = b(x)$.

We set $z := x(z+1)^2$ so that $\frac{(1-4x)^{-1/2}-1}{2} = \frac{z}{1-z}$ and $(1-4x)^{-1} = \left(\frac{1+z}{1-z}\right)^2$. Then

$$\begin{aligned} U'(n) &= [x^{n-1}] \sum_{v=2}^{n+1} \binom{n}{v-2} \left(\frac{1+z}{1-z}\right)^2 \left(\frac{z}{1-z}\right)^{v-2} \\ &= [x^{n-1}] \left(\frac{1+z}{1-z}\right)^2 \sum_{v=0}^{n-1} \binom{n}{v} \left(\frac{z}{1-z}\right)^{v} \\ &= [x^{n-1}] \left(\frac{1+z}{1-z}\right)^2 \Big[\Big(1+\frac{z}{1-z}\Big)^n - \Big(\frac{z}{1-z}\Big)^n\Big] \\ &= [x^{n-1}] \Big[(1+z)^2(1-z)^{-(n+2)} - (1+z)^2 z^n (1-z)^{-(n+2)}\Big]. \end{aligned}$$

By Lagrange's inversion formula (see, e.g., [Lab81]),

$$U'(n) = \frac{1}{n-1}[z^{n-2}]\Big\{(1+z)^{2n-2}\frac{d}{dz}\Big[(1+z)^2(1-z)^{-(n+2)} - (1+z)^2 z^n (1-z)^{-(n+2)}\Big]\Big\}.$$

Now a factor of z^n means that the coefficient of z^{n-2} will be zero even in the derivative. We have $\frac{d}{dz}\Big[(1+z)^2(1-z)^{-(n+2)}\Big] = 2(1+z)(1-z)^{-(n+2)} + (n+2)(1+z)^2(1-z)^{-(n+3)}$, so that

$$\begin{aligned} U'(n) &= \frac{1}{n-1}[x^{n-2}]\Big[2(1+z)^{2n-1}(1-z)^{-(n+2)} + (n+2)(1+z)^{2n}(1-z)^{-(n+3)}\Big] \\ &= \frac{1}{n-1}\Big[2\sum_{i=0}^{n-2}\binom{2n-1}{n-2-i}\binom{i+n+1}{i} + (n+2)\sum_{i=0}^{n-2}\binom{2n}{n-2-i}\binom{i+n+2}{i}\Big] \\ &= \frac{1}{n-1}\Big[2\frac{(2n-1)!}{(n+1)!(n-2)!}\sum_{i=0}^{n-2}\binom{n-2}{i} + (n+2)\frac{(2n)!}{(n+2)!(n-2)!}\sum_{i=0}^{n-2}\binom{n-2}{i}\Big] \\ &= \frac{2^{n-2}}{n-1}\Big[2\frac{(2n-1)!}{(n+1)!(n-2)!} + \frac{(2n)!}{(n+1)!(n-2)!}\Big], \end{aligned}$$

which simplifies to (2.1). This derivation is valid only for $n \geq 2$ since we are taking coefficients of z^{n-2}, but (2.1) turns out to be valid for $n = 1$ as well.

To prove formula (2.4) we set d_1 and d_2 to 0, so that the first and second vertices have valency 1. Proceeding as above, we find that

$$U_2'(n) = [x^{n-1}] \sum_{v=2}^{n+1} \frac{n!}{(v-2)!(n-v+2)!} \Big[\frac{(1-4x)^{-1/2}-1}{2}\Big]^{v-2},$$

which simplifies to (2.4).

If instead we just set d_1 to 0, then the first vertex has valency 1 and the second vertex can have any odd valency $2d_2+1$, including 1. If $d_2 = 0$ then, as before, we multiply by $n(n-1)/2$ to account for the fact that the two vertices of valency 1 get labels 1 and 2, but if $d_2 > 0$, then we instead multiply by $n(n-1)$ to account for the fact the vertex of valency 1 gets label 1 and the other odd-valent vertex gets label 2; so

$$\begin{aligned} 2U_2'(n) + U_1'(n) &= 2[x^{n-1}] \sum_{v=2}^{n+1} \frac{n!}{(v-2)!(n-v+2)!} (1-4x)^{-1/2} \left[\frac{(1-4x)^{-1/2} - 1}{2} \right]^{v-2} \\ &= 2^n \binom{2n-2}{n-1}, \end{aligned}$$

from which (2.3) follows.

Finally, formula (2.2) follows from the other formulae since

$$U_0'(n) + U_1'(n) + U_2'(n) = U'(n).$$

Formulae (2.2), (2.3) and (2.4) are valid only for $n \geq 2$. □

More generally, a simple formula can be deduced in the same way for the number of rooted unicursal maps with n edges and specified valencies of the odd-valent vertices. Moreover, a sum-free formula is valid for the number of n-edged unicursal maps rooted at an odd-valent vertex. In all these cases, in our opinion, the most significant novelty is the existence, per se, of such formulae.

2.3. Rooted eulerian maps. The number $E'(n)$ of rooted eulerian planar maps with n edges is expressed by the following well-known formula [Wal75] (see also the same formula in [Tut63, p. 269] for the number of rooted trivalent planar maps and the bijection between these two classes of maps shown in [Mul66]):

$$E'(n) = \frac{3 \cdot 2^{n-1}}{(n+1)(n+2)} \binom{2n}{n}. \tag{2.5}$$

Denoting $e(z) := \sum_n E'(n) z^n$, it can be easily verified that

$$e(z) = \frac{8z^2 + 12z - 1 + (1-8z)^{3/2}}{32z^2}. \tag{2.6}$$

2.4. Comparing formula (2.5) with (2.1) we obtain the following identity:

$$U'(n) = \tfrac{1}{6}(n+1)(n+2)E'(n).$$

We do not know whether this identity (or anything similar to it) can be proved directly. The same question concerns another curious identity: $U_1'(n) = 2U_2'(n)$.

3. Unrooted eulerian and unicursal maps

Formulae (2.1) and (2.3) – (2.5) enable us to complete the solution of the long-standing problem of the enumeration of unrooted eulerian planar maps. Namely, the formulae obtained in [Lis85] can be transformed into an explicit formula with single sums over the divisors of n.

3.1. Theorem. *The number $E^+(n)$ of non-isomorphic eulerian planar maps with n edges, $n \geq 2$, is expressed as follows:*

$$E^+(n) = \frac{1}{2n}\left[\frac{3\cdot 2^{n-1}}{(n+1)(n+2)}\binom{2n}{n} + 3\sum_{k<n,\,k|n}\phi\Big(\frac{n}{k}\Big)2^{k-2}\binom{2k}{k}\right.$$
$$\left. + \begin{cases} \dfrac{n\cdot 2^{(n+1)/2}}{n+1}\dbinom{n-1}{\frac{n-1}{2}}, & n \text{ odd}, \\ \displaystyle\sum_{k|\frac{n}{2}}\phi\Big(\frac{n}{k}\Big)2^{k-2}\binom{2k}{k} + \frac{n\cdot 2^{(n-2)/2}}{n+2}\binom{n}{\frac{n}{2}}, & n \text{ even}, \end{cases}\right]$$

where $\phi(n)$ is the Euler totient function.

Proof. This is an easy consequence of the following result.

3.2. Theorem [Lis85]**.**

$$E^+(n) = \frac{1}{2n}\left[E'(n) + \frac{1}{2}\sum_{k<n,\,k|n}\phi\Big(\frac{n}{k}\Big)(k+2)(k+1)E'(k)\right.$$
$$\left. + \begin{cases} U'_*\Big(\dfrac{n+1}{2}\Big), & n \text{ odd}, \\ \displaystyle\sum_{k|\frac{n}{2}}\phi\Big(\frac{n}{k}\Big)U'_1(k) + U'_{**}\Big(\frac{n+2}{2}\Big), & n \text{ even}, \end{cases}\right]$$

where U'_* and U'_{**} denote the numbers of rooted unicursal maps with one and two singular vertices respectively; a singular *vertex means a vertex of valency 1 in which the map is not allowed to be rooted.*

It is clear that $U'_{**}(n) = \frac{n-1}{n}U'_2(n)$ since any map with n edges and two singular vertices contains $2n$ edge-ends, of which exactly two are ineligible to be the root.

Likewise $U'_*(n) = \frac{2n-1}{2n}U'_1(n) + \frac{2n-1}{n}U'_2(n)$. Indeed, the first summand reflects the fact that we may take any unrooted unicursal map with a unique 1-valent vertex, declare this vertex to be singular and choose a root in one of $2n-1$ ways. The second summand is obtained by considering the contribution to the set of rooted maps with one singular vertex made by a map Γ with two 1-valent vertices. If Γ has no non-trivial symmetries, then we must declare one of its endpoints to be singular and then choose a root in one of $2n-1$ ways; so Γ contributes $2(2n-1)$ to the set of rooted maps with one singular vertex instead of the usual $2n$ rootings. Now suppose that Γ has a rotational symmetry of order 2 (the only possible non-trivial orientation-preserving automorphism). Both endpoints are equivalent, and after we declare one of them to be singular (which destroys the symmetry), there are $2n-1$ (instead of n) possible rootings. Therefore, in both cases the proportion $(2n-1):n$ is the same.

Finally, taking into account formulae (2.3) and (2.4) we obtain

$$U'_*(n) = \frac{2n-1}{n}U'_1(n). \tag{3.1}$$

□

Similarly we obtain the following.

3.3. Theorem. *Let $U^+(n)$ denote the number of non-isomorphic unicursal planar maps with n edges, $n \geq 2$, then*

$$U^+(n) = \frac{1}{2n} \sum_{k|n,\, n/k \text{ odd}} \phi\left(\frac{n}{k}\right) 2^{k-2} \binom{2k}{k} + \begin{cases} 2^{(n-3)/2} \binom{n-1}{\frac{n-1}{2}}, & n \text{ odd}, \\ 2^{(n-6)/2} \binom{n}{\frac{n}{2}}, & n \text{ even}. \end{cases}$$

Proof. We exploit the method developed in [Lis85]. Unicursal maps are similar to but simpler than eulerian maps with respect to possible rotational symmetries. Namely, only three types of rotations exist:

(I_k) rotations of an odd order k around the two odd-valent vertices (that is, around an axis that intersects the map in the two odd-valent vertices);

(I_2) rotations of order 2 around two even-valent vertices or an even-valent vertex and (the center of) a face;

(T) rotations of order 2 around the middle of an edge and a vertex or a face.

In every case, the quotient map is a unicursal map; it contains one singular vertex in the last case. Now consider the possible liftings. In the first case, the axial cells (the vertices, edges or faces in which the axis of rotation intersects the map) are determined uniquely. For I_2 we choose one odd-valent vertex of the quotient map as axial; the other axial cell is an arbitrary vertex or face except for the second odd-valent vertex. These are possible choices of the second axial vertex for rotations of the type T as well, while the first axial cell is necessarily the singular vertex. Now, by the main theorem of [Lis85] we obtain immediately the formula

$$U^+(n) = \frac{1}{2n}\left[\sum_{k|n,\, n/k \text{ odd}} \phi\left(\frac{n}{k}\right) U'(k) + \begin{cases} nU'\left(\frac{n}{2}\right), & n \text{ odd}, \\ \frac{n+1}{2} U'_*\left(\frac{n+1}{2}\right), & n \text{ even} \end{cases}\right].$$

This, together with formulae (2.1), (2.3) and (3.1), gives rise to Theorem 3.3. □

Specializing this proof to unrooted unicursal maps with two endpoints we obtain the following expression for their number $U_2^+(n)$.

3.4. Proposition. $U_2^+(1) = U_2^+(2) = 1$ *and for* $n \geq 3$,

$$U_2^+(n) = \frac{1}{n} 2^{n-3} \binom{2n-2}{n-1} + 2^{m-3} \binom{2m-2}{m-1}$$

where $m = \lfloor (n+1)/2 \rfloor$.

The first term in this formula can be written as $2^{n-3} C_{n-1}$, where C_n is the n-th Catalan number. Notice also that unicursal maps with one endpoint do not have non-trivial symmetries; therefore by (2.3),

$$U_1^+(n) = 2^{n-2} C_{n-1}.$$

4. Rooted bipartite and non-separable eulerian maps

4.1. Bi-eulerian maps. We call a map *bi-eulerian* if all its vertices and faces are of even valency; thus, both the map and its dual possess eulerian circuits. It is well known (see, e.g., [Wal75]) that the dual of an eulerian planar map is bipartite, and vice versa (so $E'(n)$ is also the number of rooted bipartite maps). Therefore,

4.2. Lemma. *A planar map is bi-eulerian if and only if it is bipartite and eulerian.*

Thus, this is a purely graph-theoretical property, not depending on the embedding.
Since the edges of an eulerian circuit switch alternate between the two parts, from this lemma we obtain the following.

4.3. Corollary. *Any bi-eulerian map contains an even number of edges.*

4.4. According to [KazSW96] (see also [SzaW97]), the cubic equation

$$3z^2y^3 - y + 1 = 0 \tag{4.1}$$

and

$$b(z) = (1 + 3y - y^2)/3$$

determine the generating function $b(z) = 1 + \sum_{n>0} B'(2n)z^{2n}$ of the number of rooted bipartite eulerian planar maps. This remarkable result has been obtained by a strong physical method known as the method of matrix integrals (see [Zvo96]) with the help of character expansion techniques.

From formula (4.1) one can easily obtain the following explicit sum-free formula:

4.5. Proposition.

$$B'(2n) = \frac{3^{n-1}}{n(2n+1)}\binom{3n}{n+1}. \tag{4.2}$$

Proof. Represent (4.1) in the form $w = 3z(w+1)^3$ where $w = y - 1$. Now $b = b(z) = (3 + w - w^2)/3$. Applying Lagrange's inversion formula $[z^n]\,b = [w^{n-1}]\,b'f^n/n$, where $f = f(w) = 3(w+1)^3$, we obtain

$$[z^n]\,b = \frac{1}{n}[w^{n-1}]\frac{1-2w}{3}3^n(1+w)^{3n} = \frac{3^{n-1}}{n}\left\{\binom{3n}{n-1} - 2\binom{3n}{n-2}\right\},$$

which gives rise to (4.2). □

This is, apparently, a new result (announced in [Lis00]); although as we learned not long ago [Sch00], D. Poulalhon and G. Schaeffer deduced formula (4.2) directly, based on the combinatorial technique developed in [BouS00].

4.6. Corollary. *The following identity is valid:*

$$B'(2n) = 3^{n-1}S'(n+1),$$

where $S'(n)$ denotes the number of rooted non-separable *maps with n edges.*

This follows immediately from the well-known formula of Tutte [Tut63] for $S'(n)$. It would be nice to find a direct bijective proof.

4.7. Remarks. 1. There is a simple 1:2:3 correspondence between, resp., rooted bi-eulerian planar maps with $2n$ edges, tetravalent bi-eulerian maps with $4n$ edges and trivalent maps with all face sizes multiple to 3 and with $6n$ edges. This has been established by Szabo and Wheater [SzaW97].

2. It is an easy matter to prove that bi-eulerian maps form a degenerate class of maps in the sense that they cannot be 3-connected (that is, polyhedral).

4.8. Non-separable eulerian and bi-eulerian maps. It is a useful but not difficult matter to count rooted non-separable maps of the classes under consideration. So, we mention the results only sketchily (cf. also [Sch98, p. 45]). Suppose again that $e = e(z) := \sum_n E'(n)z^n$. Then

$$e(z) = d(ze(z)^2)$$

where $d = d(z) := 1 + \sum_{n>0} E'_{\mathrm{NS}}(n)z^n$ is the generating function for non-separable eulerian maps. This functional equation together with expression (2.6) uniquely determines $d(z)$.

Likewise, for $b = b(z) := \sum_n B'(2n)z^{2n}$ we have

$$b(z) = f(zb(z)^2)$$

where $f = f(z) := 1 + \sum_{n>0} B'_{\mathrm{NS}}(2n)z^{2n}$ is the generating function for non-separable bi-eulerian maps.

Here we used the general functional equation for non-separable maps [Tut63, WalL75]: $g(z) = f(zg(z)^2)$. It is applicable in our cases: all 2-connected components of a (bi-)eulerian map are (bi-)eulerian (consider end components in the component tree, and so on).

Unfortunately (and somewhat unexpectedly), the explicit expressions that can be extracted from these equations look very tedious (double sums); so we do not provide them here.

Tables 1 and 2 contain numerical data for maps of the classes under consideration. The values for $n \leq 6$ may be verified by the Atlas of maps [JacV00] (for some quantities, in fact, we first guessed the formulae from data extracted from the Atlas).

Acknowledgement. We are thankful to Gilles Schaeffer and Cedric Chauve for fruitful discussion of various results concerning the enumeration of eulerian maps.

References

[BouS00] Bousquet-Mélou, M., and Schaeffer, G., "Enumeration of planar constellations." *Adv. Appl. Math.* **24** (2000), No 4, 337–368.

[JacV00] Jackson, D. M., and Visentin, T. I., *An Atlas of the Smaller Maps in Orientable and Nonorientable Surfaces.* CRC Press, 2000.

[KazSW96] Kazakov, V. A., Staudacher, M., and Wynter, Th., "Character expansion methods for matrix models of dually weighted graphs." *Commun. Math. Phys.* **177** (1996), No 1–2, 451–468 (also preprint arXiv hep-th/9502132).

[Lab81] Labelle, G., "Une nouvelle démonstration combinatoire des formules d'inversion de Lagrange." *Advances in Math.* **42** (1981), 217–247.

[Lis85] Liskovets, V. A., "Enumeration of nonisomorphic planar maps." *Selecta Math. Soviet.* **4** (1985), No 4, 304–323.

[Lis00] Liskovets, V. A., "Enumeration of planar rooted bipartite eulerian maps." *Abstracts of VIII Belarus. Mathem. Conf., June 2000*, Minsk, Institute of Mathematics of NAS of Belarus (2000), Part 4, 41 (in Russian).

[Mul66] Mullin, R. C., "On the average number of trees in certain maps." *Canad. J. Math.* **18** (1966), No 1, 33–41.

[Sch98] Schaeffer, G., *Conjugaison d'arbes et cartes combinatoires aléatoires.* Thèse, L'Université Bordeaux, 1998, 231 p. (in French).

[Sch00] Schaeffer, G., Private communication (May 2000).

[SzaW97] Szabo, R. J., and Wheater, J. F., "Curvature matrix models for dynamical triangulations and the Itzykson-Di Francesco formula." *Nucl. Phys.* **B 491** (1997), No 3, 689–723 (also preprint arXiv hep-th/9609237).

[Tut62] Tutte, W. T., "A census of slicings." *Canad. J. Math.* **14** (1962), No 4, 708–722.

[Tut63] Tutte, W. T., "A census of planar maps." *Canad. J. Math.* **15** (1963), No 2, 249–271.

[Wal75] Walsh, T. R. S., "Hypermaps versus bipartite maps." *J. Comb. Theory*, Ser. B **18** (1975), No 2, 155–163.

[WalL75] Walsh, T. R. S., and Lehman, A. B., "Counting rooted maps by genus. III: Nonseparable maps." *J. Comb. Theory*, Ser. B **18** (1975), No 3, 222–259.

[Zvo96] Zvonkin, A., "Matrix integrals and map enumeration: An accessible introduction." *Mathem. Comput. Modeling* **26** (1997), 281–304.

Table 1: Unicursal maps

n	$U^+(n)$	$U_2^+(n)$	$U'(n)$	$U_2'(n) = U_1'(n)/2$
2	2	1	6	2
3	9	3	40	12
4	38	11	280	80
5	214	62	2016	560
6	1253	342	14784	4032
7	7925	2152	109824	29568
8	51620	13768	823680	219648
9	346307	91800	6223360	1647360
10	2365886	622616	47297536	12446720
11	16421359	4301792	361181184	94595072
12	115384738	30100448	2769055744	722362368
13	819276830	213019072	21300428800	5538111488
14	5868540399	1521473984	164317593600	42600857600
15	42357643916	10954616064	1270722723840	328635187200
16	307753571520	79420280064	9848101109760	2541445447680
17	2249048959624	579300888960	76467608616960	19696202219520
18	16520782751969	4248201302400	594748067020800	152935217233920
19	121915128678131	31302536066560	4632774416793600	1189496134041600
20	903391034923548	231638727063040	36135640450990080	9265548833587200

Table 2: Eulerian and bi-eulerian maps

n	$E^+(n)$	$E'(n)$	$E'_{\mathrm{NS}}(n)$	$B'(n)$	$B'_{\mathrm{NS}}(n)$
1	1	1	1		
2	2	3	1	1	1
3	4	12	1		
4	12	56	2	6	2
5	34	288	6		
6	154	1584	19	54	8
7	675	9152	64		
8	3534	54912	230	594	54
9	18985	339456	865		
10	108070	2149888	3364	7371	442
11	632109	13891584	13443		
12	3807254	91287552	54938	99144	4032
13	23411290	608583680	228749		
14	146734695	4107939840	967628	1412802	39706
15	934382820	28030648320	4149024		
16	6034524474	193100021760	18000758	21025818	413358
17	39457153432	1341536993280	78905518		
18	260855420489	9390758952960	349037335	323686935	4487693
19	1741645762265	66182491668480	1556494270		
20	11732357675908	469294031831040	6991433386	5120138790	50348500

INSTITUTE OF MATHEMATICS, NATIONAL ACADEMY OF SCIENCES, 220072, MINSK, BELARUS
E-mail address: liskov@im.bas-net.by

DÉPARTEMENT D'INFORMATIQUE, UNIVERSITÉ DU QUÉBEC À MONTRÉAL, MONTRÉAL (QUÉBEC), CANADA, H3C 3P8
E-mail address: walsh.timothy@uqam.ca