SATURATED SIMPLICIAL COMPLEXES
(EXTENDED ABSTRACT)

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ABSTRACT. Among shellable complexes a certain class has maximal modular homology, and these are the so-called *saturated* complexes. We give a brief survey of their properties and characterize saturated complexes via p-ranks of incidence matrices and via structure of links.

RÉSUMÉ. Parmi les complexes analysables, une certaine catégorie, que l'on appelle les *complexes saturés*, a une homologie modulaire maximale; nous donnons un bref aperçu des propriétés des complexes analysables et décrivons les complexes saturés grâce aux p-rangs d'incidence des matrices et à la structure de leur liens.

1. INTRODUCTION

The standard homology for a simplicial complex Δ is concerned with the $\mathbb{Z}$-module $\mathbb{Z}\Delta$ with basis Δ and the boundary map

$$\tau \mapsto \sigma_1 - \sigma_2 + \sigma_3 - \ldots \pm \sigma_k$$

which assigns to the face τ the alternating sum of the co-dimension 1 faces of τ. This defines a homological sequence over $\mathbb{Z}$ and hence over any domain with identity.

In [12] we started to investigate the same module with respect to a different homomorphism. This is the *inclusion map* $\partial : \mathbb{Z}\Delta \to \mathbb{Z}\Delta$ given by

$$\partial : \quad \tau \mapsto \sigma_1 + \sigma_2 + \sigma_3 + \ldots + \sigma_k.$$

Clearly, $\partial^2 \neq 0$. However, when coefficients are taken modulo an integer p then a simple calculation shows that in fact $\partial^p = 0$. One may attempt therefore to build a generalized *modular homology theory* of simplicial complexes, in particular when p is a prime. This kind of homology appears to be mentioned first in W Mayer [9] in 1947, further historical remarks and references can be found in [1, 12]. More recent papers on nilpotent homomorphisms include Dubois-Violette [6] and Kapranov [8].

We showed in [12] that in general modular homology does not behave nicely: It is not homotopy invariant and there are shellable complexes with the same h-vector but with different modular homology. Nevertheless, homology of any shellable complex can be embedded into a well-understood module constructed purely from the shelling of the complex. It follows in particular that the modular Betti numbers for an arbitrary shellable complex are bounded by functions of its h-vector only.

Shellable complexes which attain these bounds are of special interest and are called *saturated*. Here we investigate conditions which guarantee saturation. Our main results are Theorems 4.1 and 5.3 which characterize saturated complexes via p-ranks of incidence matrices and via structure of links respectively. As a corollary we prove that rank-selected subcomplexes of a saturated complex are saturated.

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2. Modular Homology of Shellable Complexes

Let F be a field, Ω be a finite set and k a non-negative integer. Let then M_k denote the F-vector space with k-element subsets of Ω as basis and put $M := \oplus_{0\leq k} M_k$. The *inclusion map* is the linear map $\partial : M_k \to M_{k-1}$ defined on a basis by mapping each k-element subset of Ω to the sum of all its $(k-1)$-element subsets. If $\Delta \subseteq 2^\Omega$ is a simplicial complex, denote by M^Δ the subspace of M with basis Δ and let $M_k^\Delta := M^\Delta \cap M_k$. Then ∂ restricts to maps $M_k^\Delta \to M_{k-1}^\Delta$ for all k, and so we can attach to the complex Δ the sequence

$$\mathcal{M}^\Delta : \quad 0 \xleftarrow{\partial} M_0^\Delta \xleftarrow{\partial} M_1^\Delta \xleftarrow{\partial} ... \xleftarrow{\partial} M_{k-1}^\Delta \xleftarrow{\partial} M_k^\Delta \xleftarrow{\partial} ...$$

of submodules of M.

Throughout we suppose that p is a fixed prime and that F is a field of characteristic p. For any j and $0 < i < p$ consider the sequence

$$\ldots \xleftarrow{\partial^*} M_{j-p}^\Delta \xleftarrow{\partial^*} M_{j-i}^\Delta \xleftarrow{\partial^*} M_j^\Delta \xleftarrow{\partial^*} M_{j+p-i}^\Delta \xleftarrow{\partial^*} M_{j+p}^\Delta \xleftarrow{\partial^*} \ldots$$

in which ∂^* is the appropriate power of ∂. This sequence is determined uniquely by any arrow $M_l^\Delta \leftarrow M_r^\Delta$ in it, and so is denoted by $\mathcal{M}_{(l,r)}^\Delta$. The unique arrow $M_a^\Delta \leftarrow M_b^\Delta$ in it for which $0 \leq a+b < p$ is the *initial arrow*. We regard M_b^Δ as the 0-*position* of $\mathcal{M}_{(l,r)}^\Delta$ and while a may be negative b is always positive. The position of any other module in $\mathcal{M}_{(l,r)}^\Delta$ will be counted from this 0-position and (a, b) is referred to as the *type* of $\mathcal{M}_{(l,r)}^\Delta$.

As F has characteristic $p > 0$ it follows immediately that $\partial^p = 0$. In particular, in $\mathcal{M}_{(l,r)}^\Delta$ we have $(\partial^*)^2 = 0$ and so this sequence is homological. The homology at $M_{j-i}^\Delta \leftarrow M_j^\Delta \leftarrow M_{j+p-i}^\Delta$ is referred as the *p-modular homology* and is denoted by

$$H_{j,i}^\Delta := (\operatorname{Ker} \partial^i \cap M_j^\Delta)/\partial^{p-i}(M_{j+p-i}^\Delta).$$

with the corresponding Betti number $\beta_{j,i}^\Delta := \dim H_{j,i}^\Delta$.

If $\mathcal{M}_{(l,r)}^\Delta$ has at most one non-vanishing homology then it is said to be *almost exact* and the only non-trivial homology then is denoted by $H_{(l,r)}^\Delta$. In general, when referring to a particular sequence $\mathcal{M}_{(l,r)}^\Delta$, the homology at position t is denoted by H_t^Δ and $\beta_t^\Delta := \dim H_t^\Delta$ is the corresponding Betti number. It is useful to allow the possibility $t = \infty$ so that an almost exact sequence $\mathcal{M}_{(l,r)}^\Delta$ is exact iff either $\beta_t^\Delta := \beta_{(l,r)}^\Delta = 0$ or $t = \infty$. Finally, if $\mathcal{M}_{(l,r)}^\Delta$ is almost exact for every choice of l and r, then $\mathcal{M}^\Delta$ is *almost p-exact*.

To formulate further results we shall need the following functions on sequences $\mathcal{M}_{(l,r)}^\Delta$: If Δ is any complex of dimension $n-1$ suppose that $\mathcal{M}_{(l,r)}^\Delta$ has type (a, b). We put

$$d_{(l,r)}^n := \begin{cases} \left\lfloor \frac{n-a-b}{p} \right\rfloor & \text{if } n-a-b \not\equiv 0 \pmod p\,, \\ \infty & \text{if } n-a-b \equiv 0 \pmod p \end{cases} \tag{1}$$

and let the *weight* of $\mathcal{M}_{(l,r)}^\Delta$ be the integer $0 < w \leq p$ with $w \equiv l+r-n \pmod p$. It is useful to call the finite number $d := \min\{ d_{(l,r)}^n , d_{(l,r)}^{n+1} \}$ the *middle position* or just the *middle* of $\mathcal{M}_{(l,r)}^\Delta$.

Now we are in position to formulate a result from [10] and [1] about the p-modular homology of the $(n-1)$-dimensional simplex Σ^n on n vertices. For this we shall throughout use the notation $\mathcal{M}_{(l,r)}^n := \mathcal{M}_{(l,r)}^{\Sigma^n}$ and $H_{(l,r)}^n := H_{(l,r)}^{\Sigma^n}$.

Theorem 2.1. *The sequence $\mathcal{M}^n$ is almost p-exact. For any l, r with $0 < r - l < p$ the Betti number of $\mathcal{M}^n_{(l,r)}$ is*

$$\beta^n_{(l,r)} := \left| \sum_{t=-\infty}^{+\infty} \binom{n}{l-pt} - \binom{n}{r-pt} \right| \tag{2}$$

at position $d^n_{(l,r)}$.

For $p = 3$ the numbers $\beta^n_{(l,r)}$ could be 0 or 1 only, while for $p = 5$ these are Fibonacci numbers. The structure of $H^n_{(l,r)}$ as a $Sym(n)$-module has been determined in [1] and [2].

The structure of modular homology of shellable complexes has been determined in [12]. Note that in this paper shellable complexes are always pure, see [3, 4] for standard notions of shellability and h-vector.

Theorem 2.2. *Let Δ be an $(n-1)$-dimensional shellable complex with h-vector $(h_0,\ldots,h_n)$. For a fixed sequence $\mathcal{M}^\Delta_{(l,r)}$ let d be its middle position and let w be its weight. Then $H^\Delta_t = 0$ for $t < d$ and for all $s \geq 0$ there is an embedding*

$$H^\Delta_{d+s} \hookrightarrow \bigoplus_{j=w+(s-1)p+1}^{w+sp} \left[H^{n-j}_{(l-j,r-j)}\right]^{h_j} . \tag{3}$$

Note that in (3) we use the convention that $[H]^0$ is the zero module. The result of Theorem 2.2 cannot be improved in general: there are examples of 7-dimensional complexes with the same h-vector which have the same 3-modular homologies but different 5-modular homologies, see [12].

3. Saturated Complexes

The result of Theorem 2.2 motivates the following definition:

Definition 3.1. A shellable complex Δ is (l, r)-*saturated* in characteristic p if the embedding (3) is an isomorphism for all $s \geq 0$. The complex Δ is *saturated* if it is (l, r)-saturated for all (l, r).

Thus, saturation is defined with respect to a prime p and it is not clear if there are complexes which are saturated for some primes but not for others. Note that there are examples of complexes which are (l, r)-saturated for certain values of (l, r) but not for others.

It follows immediately from Theorem 2.2 that for a fixed p the saturated complexes have the maximal possible modular homology, in the following sense:

Proposition 3.2. *Let Δ' and Δ be shellable complexes of the same dimension and with the same h-vector. Suppose that Δ is (l, r)-saturated. Then the Betti numbers of $\mathcal{M}^{\Delta'}_{(l,r)}$ and $\mathcal{M}^\Delta_{(l,r)}$ satisfy $\beta^{\Delta'}_t \leq \beta^\Delta_t$ for all $t \in \mathbb{Z}$. Furthermore, Δ' is (l, r)-saturated if and only if $\beta^{\Delta'}_t = \beta^\Delta_t$ for each $t \in \mathbb{Z}$.*

Thus, for a saturated complex all Betti numbers are determined entirely by the h-vector. For instance, if Δ is a 5-dimensional complex with $h = (h_0, h_1, \ldots, h_6)$ which is saturated for $p = 3$ then its Betti numbers are the following:

(l,r)	w			
(1,2)	3	$\beta_{4,2} = h_1 + h_2$;		$\beta_{5,1} = h_4 + h_5$
(1,3)	1	$\beta_{3,2} = h_0$;	$\beta_{4,1} = h_2 + h_3$;	$\beta_{6,2} = h_5 + h_6$
(2,3)	2	$\beta_{3,1} = h_0 + h_1$;	$\beta_{5,2} = h_3 + h_4$;	$\beta_{6,1} = h_6$

If the same Δ is saturated for $p = 5$ then its Betti numbers are the following:

(l,r)	w		
(1,2)	2	$\beta_{2,1} = 8h_0 + 3h_1$;	$\beta_{6,4} = h_3 + h_4 + h_5 + h_6$
(1,3)	3	$\beta_{3,2} = 13h_0 + 8h_1 + 3h_2$;	$\beta_{6,3} = h_4 + h_5 + h_6$
(1,4)	4	$\beta_{4,3} = 8h_0 + 8h_1 + 5h_2 + 2h_3$;	$\beta_{6,2} = h_5 + h_6$
(1,5)	5	$\beta_{5,4} = 3h_1 + 3h_2 + 2h_3 + h_4$;	$\beta_{6,1} = h_6$
(2,3)	4	$\beta_{3,1} = 5h_0 + 5h_1 + 3h_2 + h_3$;	
(2,4)	5	$\beta_{4,2} = 5h_1 + 5h_2 + 3h_3 + h_4$;	
(2,5)	1	$\beta_{2,2} = 8h_0$;	$\beta_{5,3} = 3h_2 + 3h_3 + 2h_4 + h_5$
(3,4)	1	$\beta_{3,4} = 5h_0$;	$\beta_{4,1} = 2h_2 + 2h_3 + h_4$
(3,5)	2	$\beta_{3,3} = 13h_0 + 5h_1$;	$\beta_{5,2} = 2h_3 + 2h_4 + h_5$
(4,5)	3	$\beta_{4,4} = 8h_0 + 5h_1 + 2h_2$;	$\beta_{5,1} = h_4 + h_5$

A number of examples of saturated complexes have been found in [12] and [13]:

Example 1: Let Δ be a $(n-1)$-dimensional complex with m facets and with h-vector of the form $(1, m-1, 0, \ldots, 0)$. Every such Δ is saturated for every p. Moreover, every sequence $\mathcal{M}^{\Delta}_{(l,r)}$ is almost p-exact with homology

$$H^{\Delta}_{(l,r)} \simeq H^{n}_{(l,r)} \oplus \left[H^{n-1}_{(l-1,r-1)}\right]^{m-1}$$

in the middle. In particular, a simplex Σ^n is trivially saturated.

Example 2: The $(n-1)$-dimensional *hyperoctahedron* or *cross-polytope* is obtained by performing successive suspensions over vertex pairs α_i, β_i, or alternatively, as the dual of the $(n-1)$-dimensional cube. It is shellable and it follows from results of [13] that it is saturated for all primes.

Example 3: Finite Coxeter complexes and spherical buildings are saturated for every prime, see [13].

4. Saturated Complexes and Ranks of Incidence Matrices

Here we give an alternative definition of saturated complexes. Let $\mathrm{rk}^{\Delta}_{p}(s,t)$ be the p-rank of the incidence matrix of s-faces versus t-faces of a complex Δ. When Δ is a simplex Σ^n, we denote corresponding ranks by $\mathrm{rk}^{n}_{p}(s,t)$. It is well-known [7, 10, 15] that for $s + t < n$,

$$\mathrm{rk}^{n}_{p}(s,t) = \sum_{k=0} \binom{n}{s-pk} - \binom{n}{t-p-pk} \tag{4}$$

A similar relation holds for arbitrary shellable complexes:

Theorem 4.1. *Let Δ be a shellable $(n-1)$-dimensional complex and $p > 2$ be a prime. Let $s < t \leq n$ be non-negative integers such that $t - s < p$. If $s + t < n$ then*

$$\mathrm{rk}^{\Delta}_{p}(s,t) = \sum_{i=0}^{n} h_i \mathrm{rk}^{n-i}_{p}(s-i, t-i) \tag{5}$$

Moreover, a shellable complex Δ is saturated if and only if the relation (5) holds also for $s + t \geq n$.

Proof. First, let Δ be an arbitrary shellable complex with f-vector $(f_0, f_1, \ldots, f_n)$. In view of the condition $0 < t - s < p$ we may look at $\mathrm{rk}_p^\Delta(s,t)$ as the p-rank of the map $\partial^{t-s} : M_t^\Delta \to M_s^\Delta$. According to Theorem 2.2, in the sequence $\mathcal{M}_{(s,t)}^\Delta$ all homologies on the left from the middle are trivial. Equivalently (see [12, Corollary 5.6]), for $s + t < n$,

$$\mathrm{rk}_p^\Delta(s,t) = f_s - f_{t-p} + f_{s-p} - f_{t-2p} + f_{s-2p} - f_{t-2p} + \ldots \;.$$

The result follows now from the well-known formula

$$h_k = \sum_{i=0}^{n} (-1)^{i+k} f_i \binom{n-i}{k-i}. \tag{6}$$

after substituting it into (4).

Now let Δ be saturated, so that its Betti numbers are defined by (3). For $s + t \geq n$ we need to take these into account when evaluating rank:

$$\mathrm{rk}_p^\Delta(s,t) = \sum_{k=0} (f_{s-kp} - f_{t-p-kp}) - (\beta^\Delta_{s-kp,p-t+s} - \beta^\Delta_{t-p-kp,t-s}). \tag{7}$$

Also

$$\mathrm{rk}_p^n(s,t) = \sum_{k=0} \binom{n}{s-pk} - \binom{n}{t-p-pk} \pm \beta^n_{(s,t)} \;, \tag{8}$$

where the sign of Betti number is determined by its position in the sequence $\mathcal{M}^n_{(s,t)}$. Now put (6) and (8) into right-hand side of (5). After transforming dimensions into positions we obtain (7). Thus, for saturated Δ the relation (5) holds also for $s + t \geq n$.

Finally, since Betti numbers are completely determined by ranks, (5) implies saturation of Δ in view of Proposition 3.2. □

We note that by using the *r-step modular homology* [1] it can be shown that the condition $t - s < p$ in Theorem 4.1 is redundant.

5. Combinatorial Characterization of Saturated Complexes

Two previous definitions of saturated complexes were algebraic. Now we shall state a combinatorial description of saturated complexes. We show that certain conditions on the links of the complex imply saturation.

Let Γ be an $(n-1)$-dimensional complex and let $\Delta = \Gamma \overset{k}{\cup} \Sigma^n$ be obtained by gluing Σ^n onto Γ along some k facets of Σ^n.

Definition 5.1. We say that $\Delta := \Gamma \overset{k}{\cup} \Sigma^n$ is (l,r)-*saturated over* Γ, if Δ has the same homologies as Γ in all positions but $u := d^{n+k}_{(l,r)}$, where $H_u^\Delta \simeq H_u^\Gamma \oplus H^{n-k}_{(l-k,r-k)}$. We say that Δ is *saturated over* Γ, if Δ is (l,r)-saturated over Γ for all (l,r).

Proposition 5.2. *Let Γ be (l,r)-saturated. Then $\Delta = \Gamma \overset{k}{\cup} \Sigma^n$ is (l,r)-saturated iff Δ is (l,r)-saturated over Γ.*

In particular, a shellable complex Δ is (l,r)-*saturated* if and only if Δ has a shelling $\Delta_1, \Delta_2, \ldots, \Delta_m = \Delta$ in which Δ_i is (l,r)-saturated over Δ_{i-1} for every $2 < i \leq m$.

Let σ denote the vertex set of Σ^n and let $\Delta = \Gamma \overset{k}{\cup} \Sigma^n$. Then the *restriction* $\mathrm{res}(\sigma)$ is the set of all vertices $\beta \in \sigma$ such that $\sigma \setminus \{\beta\}$ is contained in Γ, see Björner [5]. So $\mathrm{res}(\sigma)$ is a $(k-1)$-face of Σ^n and one may regard it as the 'outer face' under gluing. Its complement $t(\sigma) := \sigma \setminus \mathrm{res}(\sigma)$ is the 'inner face' under gluing. If x is a face of Δ then the subcomplex

$\text{star}_\Delta(x)$ is generated by all facets that contain x and $\text{link}_\Delta(x)$ is the subcomplex of all faces of $\text{star}_\Delta(x)$ that do not contain x. So the dimension of $\text{link}_\Delta(x)$ is $n - |x| - 1$.

The next result gives combinatorial characterization of saturated complexes. Note that its sufficiency has been proved in [13].

Theorem 5.3. *Let Γ be a complex and let $\Delta = \Gamma \overset{k}{\cup} \Sigma^n$. Then Δ is saturated over Γ if and only if* $\text{res}(\sigma)$ *is a 1-cycle of Δ relative to* $\text{link}_\Gamma(t(\sigma))$.

Note that when saying that $\text{res}(\sigma)$ is a 1-*cycle of* Δ *relative to* $\text{link}_\Gamma(t(\sigma))$ we mean, as usual, that there is some $f \in M_k^\Gamma \subset M^\Delta$ such that $\text{supp}(f) \cap t(\sigma) = \emptyset$, $f \cup t(\sigma) \in M^\Gamma$ and $\partial(\text{res}(\sigma) + f) = 0$.

There is a simple geometrical condition which implies saturation.

Definition 5.4. Let Δ be a pure $(n-1)$-dimensional complex with facets $\sigma_1, \ldots, \sigma_m$. Then Δ is *null over F with respect to ∂*, or just *null* for short, if there are non-zero $c_1, \ldots, c_m \in F$ such that $\partial(c_1\sigma_1 + \ldots + c_m\sigma_m) = 0$.

We say that a complex is 2-*colourable* if its facets can be 2-coloured in such a way that facets with a common co-dimension 1 face have different colours. Further, in a *pseudomanifold without boundary*, see Definition 3.15 in [14], each co-dimension 1 face is contained in exactly 2 facets. Therefore a 2-colourable pseudomanifold without boundary is null: Choose all $c_i = \pm 1$, suitably according to the 2-colouring. In particular, even cyclic graphs are null over every field, and odd cyclic graphs are null only over fields of characteristic 2.

Corollary 5.5. *Let Γ be a complex and let $\Delta = \Gamma \overset{k}{\cup} [\sigma]$ for some $k \geq 1$. Suppose that* $\text{link}_\Delta(t(\sigma))$ *is null. (In particular, suppose that* $\text{link}_\Delta(t(\sigma))$ *is a 2-colourable triangulation of a sphere, or a 2-colourable pseudomanifold without boundary.) Then Δ is saturated over Γ.*

The next result follows from Theorem 5.3:

Theorem 5.6. *Let Δ be a pure $(n-1)$-dimensional completely balanced complex. For every $R \subseteq \{0, \ldots, n-1\}$ let Δ_R be the type-selected subcomplex. If Δ is saturated then Δ_R is also saturated.*

In particular, let P be a poset of a finite rank with saturated order complex $\Delta(P)$. Then all rank-selected subcomplexes $\Delta(P)_R$ are saturated.

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