

# A DETERMINANTAL FORMULA FOR SUPERSYMMETRIC SCHUR POLYNOMIALS

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ABSTRACT. We derive a new formula for the supersymmetric Schur polynomial  $s_\lambda(x/y)$ . The origin of this formula goes back to representation theory of the Lie superalgebra  $\mathfrak{gl}(m/n)$ . In particular, we show how a character formula due to Kac and Wakimoto can be applied to covariant representations, leading to a new expression for  $s_\lambda(x/y)$ . This new expression gives rise to a determinantal formula for  $s_\lambda(x/y)$ . In particular, the denominator identity for  $\mathfrak{gl}(m/n)$  corresponds to a determinantal identity combining Cauchy's double alternant with Vandermonde's determinant. We also provide an independent proof of the new determinantal formula.

RÉSUMÉ Nous dérivons une nouvelle formule pour les polynômes de Schur supersymétriques  $s_\lambda(x/y)$ . Cette formule remonte à la théorie de représentation de la superalgèbre de Lie  $\mathfrak{gl}(m/n)$ . En fait, nous démontrons de quelle façon l'application de la formule de caractère de Kac et Wakimoto aux représentations covariantes mène à une nouvelle expression de  $s_\lambda(x/y)$ . Cette expression peut être reformulée uniquement en termes de déterminants. En particulier, dans le domaine de l'algèbre  $\mathfrak{gl}(m/n)$  l'identité du dénominateur correspond à une identité de déterminants dans laquelle une combinaison d'un déterminant double alterné de Cauchy et d'un déterminant de Vandermonde peut être retrouvée. Une preuve directe de l'expression en déterminants de  $s_\lambda(x/y)$  est également présentée.

## 1. INTRODUCTION

This paper deals with a new formula for supersymmetric Schur polynomials  $s_\lambda(x/y)$ , parametrized by a partition  $\lambda$ , and symmetric in two sets of variables  $x = (x_1, \dots, x_m)$  and  $y = (y_1, \dots, y_n)$ . Supersymmetric Schur polynomials, or S-functions, were studied in detail in the work of Berele and Regev [3], who also showed that these polynomials are closely related to characters of certain representations of the Lie superalgebra  $\mathfrak{gl}(m/n)$  or  $\mathfrak{sl}(m/n)$ . More precisely, the characters of irreducible covariant tensor representations of  $\mathfrak{gl}(m/n)$  are supersymmetric S-functions (just as the characters of irreducible covariant tensor representations of  $\mathfrak{gl}(m)$  are ordinary Schur functions). It is in the context of  $\mathfrak{gl}(m/n)$  representations that our new formula for supersymmetric S-functions was discovered, as a consequence of a certain character formula of Kac and Wakimoto [6]. In their general study of characters for classical Lie superalgebras and affine superalgebras, Kac and Wakimoto [6] developed the idea of so-called tame representations. These tame representations allow for the construction of an explicit character formula. Unfortunately, it is not easy to give a simple characterization of which representations are tame. In this paper, we show that the covariant tensor representations of  $\mathfrak{gl}(m/n)$  are indeed tame. As a consequence we can apply the character formula of Kac and Wakimoto, and obtain a new expression for supersymmetric S-functions. This new expression is not particularly elegant, but after certain manipulations we show that it is equivalent to a nice determinantal formula. Once

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this determinantal formula for  $s_\lambda(x/y)$  was found, we realized that its validity could also be proved independently. Indeed, in [10, §I.3, exercise 23] or [13] it is shown that the supersymmetric S-functions are characterized by four properties. Using our determinantal expression, it is possible to show that these four properties are indeed satisfied. Secondly, one can also prove the determinantal formula using a double Laplace expansion and a special case of the so-called Sergeev-Pragacz formula (we are due to the referee for pointing this out).

The structure of this paper is as follows. In the first section, we fix the notation and recall some known formulas for ordinary and supersymmetric Schur functions. The simplest form of our new determinantal formula for  $s_\lambda(x/y)$  is already given in (1.17). Furthermore, we point out that the case  $\lambda = 0$  gives rise to an interesting determinantal identity, which can be called “the denominator identity for  $\mathfrak{gl}(m/n)$ ”. Section 2 is devoted to showing that covariant tensor representations of the Lie superalgebra  $\mathfrak{gl}(m/n)$  are tame, and to applying the Kac-Wakimoto character formula. This section uses a lot of representation theory, and closes with a formula for the character, i.e. a formula for  $s_\lambda(x/y)$ . This formula is not in an optimal form, and in Section 3 we use a number of intricate but elementary manipulations to derive from this form our main result, Theorem 3.2, giving the determinantal expression for  $s_\lambda(x/y)$ . Finally, we provide a straightforward proof, without using representation theory, based on the formula of Sergeev-Pragacz and Laplace’s theorem. So in fact we provide two proofs : one in the context of representation theory of  $\mathfrak{gl}(m/n)$ , and one by means of the formula of Sergeev-Pragacz. A reader who is not so familiar with representation theory can easily skip Section 2 (apart from Definition 2.1). In fact, we could have presented our main result without any reference to representation theory. However, we did not want to hide the natural background of this new formula, and therefore we have chosen to present also its representation theoretic origin. This is also clear from the determinantal formula itself : in this expression, the so-called  $(m, n)$ -index  $k$  of  $\lambda$  is crucial; this definition of  $k$  has a natural interpretation in representation theory of  $\mathfrak{gl}(m/n)$ .

Let  $\lambda = (\lambda_1, \dots, \lambda_p, 0, 0, \dots)$  be a partition of the non-negative integer  $N$ , with  $\lambda_1 \geq \dots \geq \lambda_p > 0$  and  $\sum_i \lambda_i = |\lambda| = N$ . The number  $p = \ell(\lambda)$  is the length of  $\lambda$ . The Young diagram  $F^\lambda$  of shape  $\lambda$  is the set of left-adjusted rows of squares with  $\lambda_i$  squares (or boxes) in the  $i$ th row (reading from top to bottom). For example, the Young diagram of  $(5, 2, 1, 1)$  is given by :

$$(1.1) \quad F^\lambda = \begin{array}{cccccc} \square & \square & \square & \square & \square & \\ \square & \square & & & & \\ \square & & & & & \\ \square & & & & & \end{array}$$

As usual,  $\lambda'$  denotes the conjugate to  $\lambda$ ; e.g. if  $\lambda = (5, 2, 1, 1)$  then  $\lambda' = (4, 2, 1, 1, 1)$ . Denote by  $\mathcal{S}(x)$  the ring of symmetric functions in  $m$  independent variables  $x_1, \dots, x_m$  [10]. The Schur functions or S-functions  $s_\lambda$  form a  $\mathbb{Z}$ -basis of  $\mathcal{S}(x)$ . There are various ways to define the S-functions [10]. For a partition  $\lambda$  with  $\ell(\lambda) \leq m$ , there is the determinantal formula (as a quotient of two alternants) :

$$(1.2) \quad s_\lambda(x) = \frac{\det(x_i^{\lambda_j+m-j})_{1 \leq i, j \leq m}}{\det(x_i^{m-j})_{1 \leq i, j \leq m}}.$$

The numerator can be rewritten as

$$(1.3) \quad \det(x_i^{\lambda_j+m-j})_{1 \leq i, j \leq m} = \det(x^{\lambda+\delta_m}) = \sum_{w \in S_m} \varepsilon(w) w(x^{\lambda+\delta_m}),$$

where  $S_m$  is the symmetric group acting on  $x = (x_1, \dots, x_m)$ ,  $\varepsilon(w)$  the signature of  $w$ , and

$$(1.4) \quad \delta_m = (m - 1, m - 2, \dots, 1, 0).$$

Thus (1.2) is essentially equivalent to Weyl's character formula for the Lie algebra  $\mathfrak{gl}(m)$ , where an irreducible representation of  $\mathfrak{gl}(m)$  is characterized by a partition  $\lambda$  and its character is given by  $s_\lambda(x)$ . The denominator on the right hand side of (1.2) is the Vandermonde determinant, equal to the product  $\prod_{1 \leq i < j \leq m} (x_i - x_j)$ ; this is Weyl's denominator formula for  $\mathfrak{gl}(m)$ .

When  $\lambda = (r)$ ,  $s_\lambda(x)$  is the complete symmetric function  $h_r(x)$ , and when  $\lambda = (1^r)$ ,  $s_\lambda(x)$  is the elementary symmetric function  $e_r(x)$ . The Jacobi-Trudi formula and the N\"agelsbach-Kostka formula give  $s_\lambda$  in terms of these [10] :

$$(1.5) \quad s_\lambda(x) = \det\left(h_{\lambda_i - i + j}(x)\right)_{1 \leq i, j \leq \ell(\lambda)} = \det\left(e_{\lambda'_i - i + j}(x)\right)_{1 \leq i, j \leq \ell(\lambda')}.$$

Other determinantal formulas for  $s_\lambda$  are Giambelli's formula [10] and the ribbon formula [9]. Finally, recall there is a combinatorial formula for  $s_\lambda(x)$  as a sum of monomials summed over all column-strict Young tableaux of shape  $\lambda$  [10].

All these formulas, except (1.2), have their extensions to skew Schur functions  $s_{\lambda/\mu}(x)$ , where  $\lambda$  and  $\mu$  are two partitions with  $\lambda_i \geq \mu_i$  for all  $i$ .

Let us now recall some notions of *supersymmetric S-functions* [3, 7, 17]. The ring of doubly symmetric polynomials in  $x = (x_1, \dots, x_m)$  and  $y = (y_1, \dots, y_n)$  is  $\mathcal{S}(x, y) = \mathcal{S}(x) \otimes_{\mathbb{Z}} \mathcal{S}(y)$ . An element  $p \in \mathcal{S}(x, y)$  has the *cancellation property* if it satisfies the following : when the substitution  $x_1 = t, y_1 = -t$  is made in  $p$ , the resulting polynomial is independent of  $t$ . We denote  $\mathcal{S}(x/y)$  the subring of  $\mathcal{S}(x, y)$  consisting of all elements satisfying the cancellation property. The elements of  $\mathcal{S}(x/y)$  are the supersymmetric polynomials [17].

The complete supersymmetric functions  $h_r(x/y)$  belong to  $\mathcal{S}(x/y)$ , and are defined by

$$(1.6) \quad h_r(x/y) = \sum_{k=0}^r h_{r-k}(x)e_k(y).$$

The following gives a first formula for the supersymmetric S-functions :

$$(1.7) \quad s_\lambda(x/y) = \det\left(h_{\lambda_i - i + j}(x/y)\right)_{1 \leq i, j \leq \ell(\lambda)}.$$

The polynomials  $s_\lambda(x/y)$  are identically zero when  $\lambda_{m+1} > n$ . Denote by  $\mathcal{H}_{m,n}$  the set of partitions with  $\lambda_{m+1} \leq n$ , i.e. the partitions (with their Young diagram) inside the  $(m, n)$ -hook. Stembridge [17] showed that the set of  $s_\lambda(x/y)$  with  $\lambda \in \mathcal{H}_{m,n}$  form a  $\mathbb{Z}$ -basis of  $\mathcal{S}(x/y)$ .

There exist a number of other formulas for the supersymmetric S-functions. One is a combinatorial formula in terms of supertableaux of shape  $\lambda$ , see [3]. From the combinatorial formula, one finds expansions of  $s_\lambda(x/y)$  in terms of ordinary S-functions :

$$(1.8) \quad s_\lambda(x/y) = \sum_{\mu} s_{\mu}(x)s_{(\lambda/\mu)'}(y) = \sum_{\mu, \nu} c_{\mu, \nu}^{\lambda} s_{\mu}(x)s_{\nu}(y),$$

where  $c_{\mu, \nu}^{\lambda}$  are the Littlewood-Richardson coefficients [10].

Just as the functions  $s_\lambda(x)$  are characters of simple modules of the Lie algebra  $\mathfrak{gl}(m)$ , the supersymmetric S-functions are characters of (a class of) simple modules of the Lie superalgebra  $\mathfrak{gl}(m/n)$  [3]. In this context, a different formula for  $s_\lambda(x/y)$  was found by

Sergeev (see [12]) and in [19]; the first proof of this formula was given by Pragacz [12]. To describe the so-called Sergeev-Pragacz formula, let  $\lambda$  be a partition with  $\lambda_{m+1} \leq n$ . Consider the Young diagram  $F^\lambda$ , let  $F^\kappa$  be the part of  $F^\lambda$  that falls within the  $m \times n$  rectangle, and let  $F^\tau$ , resp.  $F^\eta$ , be the remaining part to the right, resp. underneath this rectangle; i.e.  $\lambda = (\kappa + \tau) \cup \eta$ . This is illustrated, for  $m = 5$ ,  $n = 8$  and  $\lambda = (11, 9, 5, 3, 2, 2, 1)$ , as follows :

$$(1.9) \quad F^\lambda = \begin{array}{cccccccc} \square & \square & \square & \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square & \square & \square & \square \end{array} \quad \text{hence} \quad \begin{array}{l} \kappa = (8, 8, 5, 3, 2) \\ \tau = (3, 1) \\ \eta = (2, 2, 1) \end{array}$$

Then, the Sergeev-Pragacz formula for  $s_\lambda(x/y)$  is given by

$$(1.10) \quad s_\lambda(x/y) = D_0^{-1} \sum_{w \in S_m \times S_n} \varepsilon(w) w \left( x^{\tau + \delta_m} y^{\eta' + \delta_n} \prod_{(i,j) \in F^\kappa} (x_i + y_j) \right),$$

where  $(i, j) \in F^\kappa$  if and only if the box with row-index  $i$  (read from left to right) and column-index  $j$  (read from top to bottom) belongs to  $F^\kappa$ , and

$$(1.11) \quad D_0 = \prod_{1 \leq i < j \leq m} (x_i - x_j) \prod_{1 \leq i < j \leq n} (y_i - y_j).$$

This formula is useful for the computation of  $s_\lambda(x/y)$ , and even for the computation of Littlewood-Richardson coefficients [2, 18]. Note that for the special case that  $\lambda_m \geq n$ , (1.10) becomes :

$$(1.12) \quad s_\lambda(x/y) = s_\tau(x) s_{\eta'}(y) \prod_{i=1}^m \prod_{j=1}^n (x_i + y_j).$$

This is the case of *factorization*. Furthermore, from (1.10) one easily deduces duality :

$$(1.13) \quad s_\lambda(x/y) = s_{\lambda'}(y/x).$$

Observe, on the other hand, that  $D_0$  is just Weyl’s denominator for  $\mathfrak{gl}(m) \oplus \mathfrak{gl}(n)$ . So when  $\lambda = 0$ , (1.10) does not yield a new denominator identity related to  $\mathfrak{gl}(m/n)$ ; it only gives the denominator identity for  $\mathfrak{gl}(m)$  and  $\mathfrak{gl}(n)$ .

In this paper, we shall give a new formula for  $s_\lambda(x/y)$ . In its simplest form, this yields a new determinantal formula for supersymmetric S-functions. Furthermore, this formula yields a genuine denominator identity related to  $\mathfrak{gl}(m/n)$ . Let us briefly describe one of the forms of the new formula. First, we introduce some new notations. Let

$$(1.14) \quad D(x) = \prod_{1 \leq i < j \leq m} (x_i - x_j) \quad \text{and} \quad R(x, y) = \prod_{i=1}^m \prod_{j=1}^n (x_i - y_j)$$

and define  $D$  by

$$(1.15) \quad D = \frac{\prod_{1 \leq i < j \leq m} (x_i - x_j) \prod_{1 \leq i < j \leq n} (y_i - y_j)}{\prod_{i=1}^m \prod_{j=1}^n (x_i + y_j)} = \frac{D(x)D(y)}{R(x, -y)}.$$

Let  $\lambda$  be a partition with  $\lambda_{m+1} \leq n$ , i.e.  $\lambda \in \mathcal{H}_{m,n}$ , and put

$$(1.16) \quad k = \min\{j | \lambda_j + m + 1 - j \leq n\};$$

since  $\lambda_{m+1} \leq n$ , we have that  $1 \leq k \leq m + 1$ . Then the new formula reads

$$(1.17) \quad s_\lambda(x/y) = (-1)^{mn-m+k-1} D^{-1} \det \begin{pmatrix} R & X_\lambda \\ Y_\lambda & 0 \end{pmatrix},$$

where the (rectangular) blocks of the determinant are given by

$$R = \left( \frac{1}{x_i + y_j} \right)_{1 \leq i \leq m, 1 \leq j \leq n},$$

$$X_\lambda = \left( x_i^{\lambda_j + m - n - j} \right)_{1 \leq i \leq m, 1 \leq j \leq k-1}, \quad Y_\lambda = \left( y_j^{\lambda'_i + n - m - i} \right)_{1 \leq i \leq n-m+k-1, 1 \leq j \leq n}.$$

When  $\lambda = 0$  it follows from (1.7) or (1.10) that  $s_\lambda(x/y) = 1$ . The new formula (1.17) gives rise to a denominator identity for  $\mathfrak{gl}(m/n)$ . Suppose  $m \leq n$  ( $m \geq n$  is similar); when  $\lambda = 0$ , it follows from (1.16) that  $k = 1$ . So the  $X_\lambda$ -block and 0-block disappear in (1.17). Changing the order of the  $R$ -block and  $Y_\lambda$ -block, implies

$$(1.18) \quad \det \begin{pmatrix} y_1^{n-m-1} & \cdots & y_n^{n-m-1} \\ \vdots & & \vdots \\ y_1^0 & \cdots & y_n^0 \\ \frac{1}{x_1+y_1} & \cdots & \frac{1}{x_1+y_n} \\ \vdots & & \vdots \\ \frac{1}{x_m+y_1} & \cdots & \frac{1}{x_m+y_n} \end{pmatrix} = \frac{\prod_{1 \leq i < j \leq m} (x_i - x_j) \prod_{1 \leq i < j \leq n} (y_i - y_j)}{\prod_{i=1}^m \prod_{j=1}^n (x_i + y_j)}.$$

Clearly, when  $m = n$ , this is simply Cauchy's double alternant; when  $m = 0$ , it is just Vandermonde's determinant. When  $0 < m < n$ , it is a combination of the two. These type of determinants have already been encountered in a different context [1] (we found this reference in [8]); here they are for the first time related to a denominator identity.

## 2. COVARIANT MODULES FOR THE LIE SUPERALGEBRA $\mathfrak{gl}(m/n)$

For general theory on classical Lie superalgebras and their representations, we refer to [4, 5, 14]; for representations of the Lie superalgebra  $gl(m/n)$ , see [19, 20, 21].

Let  $\mathfrak{g} = \mathfrak{gl}(m/n)$ ,  $\mathfrak{h} \subset \mathfrak{g}$  its Cartan subalgebra, and  $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{+1}$  the consistent  $\mathbb{Z}$ -grading. Note that  $\mathfrak{g}_0 = \mathfrak{gl}(m) \oplus \mathfrak{gl}(n)$ , and put  $\mathfrak{g}^+ = \mathfrak{g}_0 \oplus \mathfrak{g}_{+1}$  and  $\mathfrak{g}^- = \mathfrak{g}_0 \oplus \mathfrak{g}_{-1}$ . The dual space  $\mathfrak{h}^*$  has a natural basis  $\{\epsilon_1, \dots, \epsilon_m, \delta_1, \dots, \delta_n\}$ , and the roots of  $\mathfrak{g}$  can be expressed in terms of this basis. Let  $\Delta$  be the set of all roots,  $\Delta_0$  the set of even roots, and  $\Delta_1$  the set of odd roots. One can choose a set of simple roots (or, equivalently, a triangular decomposition), but note that contrary to the case of simple Lie algebras not all such choices are equivalent. The so-called *distinguished choice* [4] for a triangular decomposition  $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$  is such that  $\mathfrak{g}_{+1} \subset \mathfrak{n}^+$  and  $\mathfrak{g}_{-1} \subset \mathfrak{n}^-$ . Then  $\mathfrak{h} \oplus \mathfrak{n}^+$  is the corresponding distinguished Borel subalgebra, and  $\Delta_+$  the set of positive roots. For this choice we have explicitly :

$$(2.1) \quad \begin{aligned} \Delta_{0,+} &= \{\epsilon_i - \epsilon_j | 1 \leq i < j \leq m\} \cup \{\delta_i - \delta_j | 1 \leq i < j \leq n\}, \\ \Delta_{1,+} &= \{\beta_{ij} = \epsilon_i - \delta_j | 1 \leq i \leq m, 1 \leq j \leq n\}, \end{aligned}$$

and the corresponding set of simple roots (the distinguished set) is given by

$$(2.2) \quad \Pi = \{\epsilon_1 - \epsilon_2, \dots, \epsilon_{m-1} - \epsilon_m, \epsilon_m - \delta_1, \delta_1 - \delta_2, \dots, \delta_{n-1} - \delta_n\}.$$

Thus in the distinguished basis there is only one simple root which is odd. As usual, we put

$$(2.3) \quad \rho_0 = \frac{1}{2} \left( \sum_{\alpha \in \Delta_{0,+}} \alpha \right), \quad \rho_1 = \frac{1}{2} \left( \sum_{\alpha \in \Delta_{1,+}} \alpha \right), \quad \rho = \rho_0 - \rho_1.$$

There is a symmetric form  $(\ , \ )$  on  $\mathfrak{h}^*$  induced by the invariant symmetric form on  $\mathfrak{g}$ , and in the natural basis it takes the form  $(\epsilon_i, \epsilon_j) = \delta_{ij}$ ,  $(\epsilon_i, \delta_j) = 0$  and  $(\delta_i, \delta_j) = -\delta_{ij}$ . The odd roots are isotropic :  $(\alpha, \alpha) = 0$  if  $\alpha \in \Delta_1$ .

The Weyl group of  $\mathfrak{g}$  is the Weyl group  $W$  of  $\mathfrak{g}_0$ , hence it is the direct product of symmetric groups  $S_m \times S_n$ . For  $w \in W$ , we denote by  $\varepsilon(w)$  its signature.

Let  $\Lambda \in \mathfrak{h}^*$ ; the *atypicality* of  $\Lambda$ , denoted by  $\text{atyp}(\Lambda)$ , is the maximal number of linearly independent roots  $\beta_i$  such that  $(\beta_i, \beta_j) = 0$  and  $(\Lambda, \beta_i) = 0$  for all  $i$  and  $j$  [6]. Such a set  $\{\beta_i\}$  is called a  $\Lambda$ -maximal isotropic subset of  $\Delta$ .

Given a set of positive roots  $\Delta_+$  of  $\Delta$ , and a simple odd root  $\alpha$ , one may construct a new set of positive roots [11, 6] by

$$(2.4) \quad \Delta'_+ = (\Delta_+ \cup \{-\alpha\}) \setminus \{\alpha\}.$$

The set  $\Delta'_+$  is called a simple reflection of  $\Delta_+$ . Since we use only simple reflections with respect to simple odd roots,  $\Delta_{0,+}$  remains invariant, but  $\Delta_{1,+}$  will change and the new  $\rho$  is given by :

$$(2.5) \quad \rho' = \rho + \alpha.$$

Let  $V$  be a finite-dimensional irreducible  $\mathfrak{g}$ -module. Such modules are  $\mathfrak{h}$ -diagonalizable with weight decomposition  $V = \bigoplus_{\mu} V(\mu)$ , and the character is defined to be  $\text{ch } V = \sum_{\mu} \dim V(\mu) e^{\mu}$ , where  $e^{\mu}$  ( $\mu \in \mathfrak{h}^*$ ) is the formal exponential. Consider such a module  $V$ . If we fix a set of positive roots  $\Delta_+$ , we may talk about the highest weight  $\Lambda$  of  $V$  and about the corresponding  $\rho$ . If  $\Delta'_+$  is obtained from  $\Delta_+$  by a simple  $\alpha$ -reflection, where  $\alpha$  is odd, and  $\Lambda'$  denotes the highest weight of  $V$  with respect to  $\Delta'_+$ , then [6]

$$(2.6) \quad \Lambda' = \Lambda - \alpha \text{ if } (\Lambda, \alpha) \neq 0; \quad \Lambda' = \Lambda \text{ if } (\Lambda, \alpha) = 0.$$

From this, one deduces that for the  $\mathfrak{g}$ -module  $V$ ,  $\text{atyp}(\Lambda + \rho)$  is independent of the choice of  $\Delta_+$ ; then  $\text{atyp}(\Lambda + \rho)$  is referred to as the atypicality of  $V$  (if  $\text{atyp}(\Lambda + \rho) = 0$ ,  $V$  is typical, otherwise it is atypical). If one can choose a  $(\Lambda + \rho)$ -maximal isotropic subset  $S_{\Lambda}$  in  $\Delta_+$  such that  $S_{\Lambda} \subset \Pi \subset \Delta_+$  ( $\Pi$  is the set of simple roots with respect to  $\Delta_+$ ), then the  $\mathfrak{g}$ -module  $V$  is called *tame*, and a character formula is known due to Kac and Wakimoto [6]. It reads :

$$(2.7) \quad \text{ch } V = j_{\Lambda}^{-1} e^{-\rho} R^{-1} \sum_{w \in W} \varepsilon(w) w \left( e^{\Lambda + \rho} \prod_{\beta \in S_{\Lambda}} (1 + e^{-\beta})^{-1} \right),$$

where

$$(2.8) \quad R = \prod_{\alpha \in \Delta_{0,+}} (1 - e^{-\alpha}) / \prod_{\alpha \in \Delta_{1,+}} (1 + e^{-\alpha})$$

and  $j_{\Lambda}$  is a normalization coefficient to make sure that the coefficient of  $e^{\Lambda}$  on the rhs of (2.7) is 1.

The rest of this section is now devoted to a particular class of finite-dimensional irreducible  $\mathfrak{g}$ -modules, the *covariant* modules, and to showing that these modules are tame.

Berele and Regev [3], and Sergeev [15], showed that the tensor product of  $N$  copies of the natural  $(m + n)$ -dimensional representation of  $\mathfrak{g} = \mathfrak{gl}(m/n)$  is completely reducible, and that the irreducible components  $V_\lambda$  can be labeled by a partition  $\lambda$  of  $N$  such that  $\lambda$  is inside the  $(m, n)$ -hook, i.e. such that  $\lambda_{m+1} \leq n$ . These representations  $V_\lambda$  are known as covariant modules. Let us first consider  $V_\lambda$  in the distinguished basis fixed by (2.2). Then, the highest weight  $\Lambda_\lambda$  of  $V_\lambda$  in the standard  $\epsilon$ - $\delta$ -basis is given by [19]

$$(2.9) \quad \Lambda_\lambda = \lambda_1 \epsilon_1 + \cdots + \lambda_m \epsilon_m + \nu'_1 \delta_1 + \cdots + \nu'_n \delta_n,$$

where  $\nu'_j = \max\{0, \lambda'_j - m\}$  for  $1 \leq j \leq n$ . Let us consider the atypicality of  $V_\lambda$ , in the distinguished basis. For this purpose, it is sufficient to compute the numbers  $(\Lambda_\lambda + \rho, \beta_{ij})$ , with  $\beta_{ij} = \epsilon_i - \delta_j$ , for  $1 \leq i \leq m$  and  $1 \leq j \leq n$ , and count the number of zeros. It is convenient to put the numbers  $(\Lambda_\lambda + \rho, \beta_{ij})$  in a  $(m \times n)$ -matrix (the atypicality matrix [19, 20]), and give the matrix entries in the  $(m, n)$ -rectangle together with the Young frame of  $\lambda$ . This is illustrated here for example (1.9) :

$$(2.10) \quad \begin{array}{cccccccc} 18 & 16 & 13 & 12 & 11 & 10 & 9 & 8 & & & \\ 15 & 13 & 10 & 9 & 8 & 7 & 6 & 5 & & & \\ 10 & 8 & 5 & 4 & 3 & 2 & 1 & 0 & & & \\ 7 & 5 & 2 & 1 & 0 & -1 & -2 & -3 & & & \\ 5 & 3 & 0 & -1 & -2 & -3 & -4 & -5 & & & \\ & & & & & & & & & & \\ & & & & & & & & & & \\ & & & & & & & & & & \\ & & & & & & & & & & \end{array}$$

So for this example,  $\text{atyp}(\Lambda_\lambda + \rho) = 3$ , since the atypicality matrix contains three zeros. In the following, we shall sometimes refer to  $(i, j)$  as the position of  $\beta_{ij}$ ; the position is also identified by a box in the  $(m, n)$ -rectangle.

The maximal isotropic subset  $S_{\Lambda_\lambda}$  consist now of those odd roots  $\beta_{ij}$  with  $(\Lambda_\lambda + \rho, \beta_{ij}) = 0$ , i.e. it corresponds to the zeros in the atypicality matrix. From the combinatorics of atypicality matrices [20], it follows that

$$S_{\Lambda_\lambda} = \{\beta_{i, \lambda_i + m + 1 - i} \mid 1 \leq i \leq m, 1 \leq \lambda_i + m + 1 - i \leq n\}.$$

That is to say, one finds the zeros in the atypicality matrix as follows : on row  $m$  in column  $\lambda_m + 1$ ; on row  $m - 1$  in column  $\lambda_{m-1} + 2$ ; etc. (as long as these column indices are not exceeding  $n$ ); see also in (2.10).

Clearly,  $S_{\Lambda_\lambda}$  is in general not a subset of the set of simple roots  $\Pi$  (since  $\Pi$  contains only one odd root,  $\epsilon_m - \delta_1$ ). So formula (2.7) cannot be applied. The purpose is to show that there exists a sequence of simple odd  $\alpha$ -reflections such that for the new  $\Delta'_+$ , where the module  $V_\lambda$  has highest weight  $\Lambda'$ , there exists a  $(\Lambda' + \rho')$ -maximal isotropic subset  $S_{\Lambda'}$  with  $S_{\Lambda'} \subset \Pi' \subset \Delta'_+$ .

**Definition 2.1.** For  $\lambda \in \mathcal{H}_{m,n}$ , the  $(m, n)$ -index of  $\lambda$  is the number

$$(2.11) \quad k = \min\{i \mid \lambda_i + m + 1 - i \leq n\}, \quad (1 \leq k \leq m + 1).$$

In what follows,  $k$  will always denote this number; it will be a crucial entity for our developments. If  $k = m + 1$ , then  $S_{\Lambda_\lambda} = \emptyset$  and  $V_\lambda$  is typical and trivially tame. Thus in the following, we shall assume that  $k \leq m$ . To begin with,  $\Delta_+$  corresponds to the distinguished choice, and  $\Pi$  is the distinguished set of simple roots (2.2). The highest weight of  $V_\lambda$  is given by  $\Lambda_\lambda$ . Denote  $\Lambda^{(1)} = \Lambda_\lambda$ ,  $\rho^{(1)} = \rho$  and  $\Pi^{(1)} = \Pi$ . Now we perform a sequence of

simple odd  $\alpha^{(i)}$ -reflections; each of these reflections preserve  $\Delta_{0,+}$  but may change  $\Lambda^{(i)} + \rho^{(i)}$  and  $\Pi^{(i)}$ . Denote the sequence of reflections by :

$$(2.12) \quad \Lambda^{(1)} + \rho^{(1)}, \Pi^{(1)} \xrightarrow{\alpha^{(1)}} \Lambda^{(2)} + \rho^{(2)}, \Pi^{(2)} \xrightarrow{\alpha^{(2)}} \dots \xrightarrow{\alpha^{(f)}} \Lambda' + \rho', \Pi'$$

where, at each stage,  $\alpha^{(i)}$  is an odd root from  $\Pi^{(i)}$ . For given  $\lambda$ , consider the following sequence of odd roots (with positions on row  $m$ , row  $m-1$ , ..., row  $k$ ) :

$$(2.13) \quad \begin{array}{ll} \text{row } m : & \beta_{m,1}, \beta_{m,2}, \dots, \beta_{m,\lambda_k-k+m} \\ \text{row } m-1 : & \beta_{m-1,1}, \beta_{m-1,2}, \dots, \beta_{m-1,\lambda_k-k+m-1} \\ \vdots & \vdots \\ \text{row } k : & \beta_{k,1}, \beta_{k,2}, \dots, \beta_{k,\lambda_k} \end{array}$$

in this particular order (i.e. starting with  $\beta_{m,1}$  and ending with  $\beta_{k,\lambda_k}$ ). Then we have :

**Lemma 2.2.** *The sequence (2.13) is a proper sequence of simple odd reflections for  $\Lambda_\lambda$ , i.e.  $\alpha^{(i)}$  is a simple odd root from  $\Pi^{(i)}$ . At the end of the sequence, one finds :*

$$(2.14) \quad \begin{aligned} \Pi' = & \{ \epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_3, \dots, \epsilon_{k-2} - \epsilon_{k-1}, \epsilon_{k-1} - \delta_1, \delta_1 - \delta_2, \delta_2 - \delta_3, \dots, \delta_{\lambda_k-1} - \delta_{\lambda_k}, \\ & \delta_{\lambda_k} - \epsilon_k, \epsilon_k - \delta_{\lambda_k+1}, \delta_{\lambda_k+1} - \epsilon_{k+1}, \epsilon_{k+1} - \delta_{\lambda_k+2}, \dots, \delta_{\lambda_k+m-k} - \epsilon_m, \epsilon_m - \delta_{\lambda_k+m+1-k}, \\ & \delta_{\lambda_k+m+1-k} - \delta_{\lambda_k+m+2-k}, \dots, \delta_{n-1} - \delta_n \}. \end{aligned}$$

Furthermore,

$$(2.15) \quad \Lambda' + \rho' = \Lambda_\lambda + \rho + \sum_{i=k+1}^m \sum_{j=\lambda_i+1}^{\lambda_k-k+i} \beta_{i,j}.$$

*Proof.* Let us consider, in the first stage, the reflections with respect to the roots in row  $m$ . Clearly,  $\alpha^{(1)} = \beta_{m,1} = \epsilon_m - \delta_1$  is an odd root from  $\Pi^{(1)} = \Pi$ . Performing the reflection with respect to  $\beta_{m,1}$  implies that  $\Pi^{(2)}$  contains  $\epsilon_{m-1} - \delta_1, \delta_1 - \epsilon_m, \epsilon_m - \delta_2$  as simple odd roots. Thus  $\Pi^{(2)}$  contains  $\alpha^{(2)} = \beta_{m,2}$ . A reflection with respect to  $\beta_{m,2}$  implies that  $\Pi^{(3)}$  contains  $\epsilon_{m-1} - \delta_1, \delta_2 - \epsilon_m, \epsilon_m - \delta_3$  as simple odd roots. So this process continues, and after  $\lambda_k - k + m$  such reflections (i.e. at the end of row  $m$ ), we have

$$(2.16) \quad \begin{aligned} \Pi^{(\lambda_k-k+m+1)} = & \{ \epsilon_1 - \epsilon_2, \dots, \epsilon_{m-2} - \epsilon_{m-1}, \epsilon_{m-1} - \delta_1, \delta_1 - \delta_2, \dots, \delta_{\lambda_k-k+m-1} - \delta_{\lambda_k-k+m}, \\ & \delta_{\lambda_k-k+m} - \epsilon_m, \epsilon_m - \delta_{\lambda_k-k+m+1}, \delta_{\lambda_k-k+m+1} - \delta_{\lambda_k-k+m+2}, \dots, \delta_{n-1} - \delta_n \}. \end{aligned}$$

Observe that this process can continue since  $\lambda_k - k + m < n$  by definition of the  $(m, n)$ -index  $k$  of  $\lambda$ . So after the first stage (i.e. after the reflections with respect to odd roots of row  $m$ ) there are three odd roots in  $\Pi^{(\lambda_k-k+m+1)}$ , and the set is ready to continue the reflections with respect to the elements of row  $m-1$ , since  $\beta_{m-1,1}$  belongs to  $\Pi^{(\lambda_k-k+m+1)}$ . From (2.5) and (2.6) we have that

$$\begin{aligned} \Lambda^{(i+1)} + \rho^{(i+1)} &= \Lambda^{(i)} + \rho^{(i)} \quad \text{if } (\Lambda^{(i)} + \rho^{(i)}, \alpha^{(i)}) \neq 0, \\ \Lambda^{(i+1)} + \rho^{(i+1)} &= \Lambda^{(i)} + \rho^{(i)} + \alpha^{(i)} \quad \text{if } (\Lambda^{(i)} + \rho^{(i)}, \alpha^{(i)}) = 0. \end{aligned}$$

Examining this explicitly for the elements of row  $m$  yields

$$(2.17) \quad \Lambda^{(\lambda_k-k+m+1)} + \rho^{(\lambda_k-k+m+1)} = \Lambda_\lambda + \rho + \sum_{j=\lambda_m+1}^{\lambda_k-k+m} \beta_{m,j}.$$

If  $k = m$  the lemma follows. If  $k < m$  the process continues; suppose this is the case. But now we are in a situation where the elements of row  $m-1$  play completely the same role



as those of row  $m$  in the first stage. This means that at the end of the second stage, the new set of simple roots is given by

$$(2.18) \quad \Pi^{(i)} = \{\epsilon_1 - \epsilon_2, \dots, \epsilon_{m-3} - \epsilon_{m-2}, \epsilon_{m-2} - \delta_1, \delta_1 - \delta_2, \dots, \delta_{\lambda_k - k + m - 2} - \delta_{\lambda_k - k + m - 1}, \\ \delta_{\lambda_k - k + m - 1} - \epsilon_{m-1}, \epsilon_{m-1} - \delta_{\lambda_k - k + m}, \delta_{\lambda_k - k + m} - \epsilon_m, \epsilon_m - \delta_{\lambda_k - k + m + 1}, \\ \delta_{\lambda_k + m + 1 - k} - \delta_{\lambda_k + m + 2 - k}, \dots, \delta_{n-1} - \delta_n\},$$

and the new  $\Lambda^{(i)} + \rho^{(i)}$  by

$$(2.19) \quad \Lambda^{(i)} + \rho^{(i)} = \Lambda_\lambda + \rho + \sum_{j=\lambda_m+1}^{\lambda_k-k+m} \beta_{m,j} + \sum_{j=\lambda_{m-1}+1}^{\lambda_{m-1}-k+m-1} \beta_{m-1,j}$$

(the last addition follows by inspecting the atypicality matrix). Continuing with the remaining stages (i.e. rows in (2.13)) leads to (2.14) and (2.15).  $\square$

**Corollary 2.3.** *The covariant module  $V_\lambda$  is tame.*

*Proof.* Having performed the simple odd reflections (2.13), one can compute the atypicality matrix for  $\Lambda' + \rho'$  using (2.15). This gives :

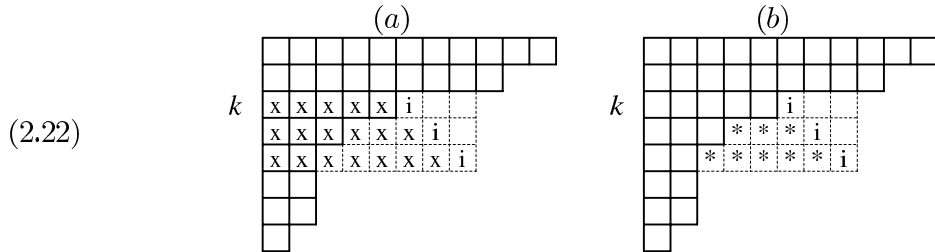
$$(2.20) \quad (\Lambda' + \rho', \beta_{ij}) = 0 \text{ for all } (i, j) \text{ with } k \leq i \leq m, \lambda_k + 1 \leq j \leq \lambda_k + m + 1 - k.$$

Therefore the set

$$(2.21) \quad S'_\Lambda = \{\epsilon_k - \delta_{\lambda_k+1}, \epsilon_{k+1} - \delta_{\lambda_k+2}, \dots, \epsilon_m - \delta_{\lambda_k+m+1-k}\}$$

is a  $(\Lambda' + \rho')$ -maximal isotropic subset. Furthermore,  $S'_\Lambda \subset \Pi'$ , see (2.14).  $\square$

Let us illustrate some of these notions for example (1.9).



In (2.22)(a) the positions marked with “i” refer to the  $(\Lambda' + \rho')$ -maximal isotropic set (2.21). The first element, with row index  $k$ , is simply the position of the box just to the right of the Young diagram  $F^\lambda$  in row  $k$ . For the remaining positions one continues in the direction of the diagonal until one reaches row  $m$ . For convenience, let us refer to these positions as “the isotropic diagonal.” The positions of the odd roots that have been used for the sequence of reflections to go from  $\Lambda_\lambda$  and  $\Pi$  to  $\Lambda'$  and  $\Pi'$  are marked by “x” in (2.22)(a). So, they are simply all positions to the left of the isotropic diagonal. Finally, (2.22)(b) shows the positions of those  $\beta_{ij}$  that appear on the rhs of (2.15); they are marked by “\*”. These are all positions to the left of the isotropic diagonal that are not inside  $F^\lambda$ . One can see from these examples that the  $(m, n)$ -index  $k$  determines all other necessary ingredients.

We are now in a position to evaluate the character formula (2.7),

$$(2.23) \quad \text{ch } V_\lambda = j_{\Lambda'}^{-1} e^{-\rho'} R'^{-1} \sum_{w \in W} \varepsilon(w) w \left( e^{\Lambda' + \rho'} \prod_{\beta \in S_{\Lambda'}} (1 + e^{-\beta})^{-1} \right),$$

where  $R'$  is given by (2.8) with  $\Delta_{1,+}$  replaced by  $\Delta'_{1,+}$  ( $\Delta_{0,+}$  remains unchanged). The  $mn$  elements of  $\Delta'_{1,+}$  are  $\pm\beta_{ij}$ , where one must take the minus-sign if  $\beta_{ij}$  appears in the

list (2.13) (i.e. if its position is marked by “x” in (2.22)(a)) and the plus-sign otherwise. However, all this information is not necessary here, since by definition of  $\rho$  and  $R$

$$e^{-\rho'} R'^{-1} = e^{-\rho} R^{-1}.$$

Putting, as usual in this context,

$$(2.24) \quad x_i = e^{\epsilon_i}, \quad y_j = e^{\delta_j} \quad (1 \leq i \leq m, 1 \leq j \leq n),$$

one has

$$e^{-\rho} R^{-1} = D^{-1} \prod_{i=1}^m x_i^{(m-n-1)/2} \prod_{j=1}^n y_j^{(n-m-1)/2},$$

with  $D$  given in (1.15). Using (2.15) and (2.21), one finds

$$(2.25) \quad e^{\Lambda + \rho'} \prod_{\beta \in S_{\Lambda'}} (1 + e^{-\beta})^{-1} = \prod_{i=1}^m x_i^{(1-m+n)/2} \prod_{j=1}^n y_j^{(1+m-n)/2} \prod_{i=1}^{k-1} x_i^{\lambda_i + m - i - n} \prod_{i=k}^m x_i^{\lambda_k + m + 1 - k - n} \\ \prod_{j=1}^{\lambda_k} y_j^{\lambda_j' + n - j - m} \prod_{j=\lambda_k + 1}^{\lambda_k + m + 1 - k} y_j^{n - \lambda_k - m - 1 + k} \prod_{j=\lambda_k + m + 2 - k}^n y_j^{n-j} / \prod_{i=k}^m (x_i + y_{\lambda_k + i + 1 - k}).$$

In order to rewrite this in a more appropriate form, let us introduce some further notation related to the partition  $\lambda \in \mathcal{H}_{m,n}$ . Clearly, the  $(m, n)$ -index  $k$  defined in (2.11) plays again an essential role. Related to this, let us also put

$$(2.26) \quad l = \lambda_k + 1, \quad r = n - m + k - l.$$

Now we have

$$\text{ch } V_{\lambda} = j_{\Lambda'}^{-1} D^{-1} \sum_{w \in W} \varepsilon(w) w(t_{\lambda}),$$

with

$$(2.27) \quad t_{\lambda} = \prod_{i=1}^{k-1} x_i^{\lambda_i + m - i - n} \prod_{j=1}^{l-1} y_j^{\lambda_j' + n - j - m} \prod_{i=k}^m \frac{y_{l+i-k}^r}{x_i^r (x_i + y_{l+i-k})} \prod_{j=l+m+1-k}^n y_j^{n-j}.$$

This form also allows us to deduce  $j_{\Lambda'} = j_{\Lambda}$ . Indeed, consider in  $W = S_m \times S_n$  the subgroup  $H$  of elements  $w = \sigma_x \times \sigma_y$ , where  $\sigma_x$  is a permutation of  $(x_k, x_{k+1}, \dots, x_m)$  and  $\sigma_y$  is the same permutation of  $(y_l, y_{l+1}, \dots, y_{l+m-k})$ . Each element of  $H$  leaves  $t_{\lambda}$  invariant. Furthermore,  $\varepsilon(w) = 1$  for  $w \in H$ . Since  $H$  is isomorphic to  $S_{m-k+1}$ ,

$$\sum_{w \in H} \varepsilon(w) w(t_{\lambda}) = (m - k + 1)! t_{\lambda}.$$

So, in order to have multiplicity one for the highest weight term, one must take  $j_{\Lambda'} = (m - k + 1)!$ .

We can now conclude this section by the following two alternative formulas for the computation of  $\text{ch } V_{\lambda}$  (i.e. of  $s_{\lambda}(x/y)$ ) :

$$(2.28) \quad \text{ch } V_{\lambda} = \frac{D^{-1}}{(m - k + 1)!} \sum_{w \in S_m \times S_n} \varepsilon(w) w(t_{\lambda})$$

$$(2.29) \quad = D^{-1} \sum_{w \in (S_m \times S_n) / S_{m-k+1}} \varepsilon(w) w(t_{\lambda}),$$

where the second sum is over the cosets of  $H = S_{m-k+1}$  in  $S_m \times S_n$ .

3. A DETERMINANTAL FORMULA FOR  $s_\lambda(x/y)$

Let  $\lambda \in \mathcal{H}_{m,n}$ ; the important quantities related to  $\lambda$  are given by the  $(m, n)$ -index  $k$  (see (2.11)) and the related numbers  $l$  and  $r$  (see (2.26)). Since  $\text{ch } V_\lambda = s_\lambda(x/y)$ , (2.28) or (2.29), together with the expression (2.27) for  $t_\lambda$ , yield a (new) expression for the super-symmetric S-function. In this section we shall rewrite (2.28) in a nicer form; in particular we shall show that it is equivalent to a determinantal form for  $s_\lambda(x/y)$ . Then we shall end with an alternative proof.

The first step in this process is the following :

**Lemma 3.1.** *Let  $t_\lambda$  be given by (2.27). Then*

$$(3.1) \quad \frac{1}{(m-k+1)!} \sum_{w \in S_m \times S_n} \varepsilon(w) w(t_\lambda) = (-1)^{(k-1)(l-1)} \det(C),$$

where  $C$  is the following square matrix of order  $n+k-1$  :

$$(3.2) \quad C = \begin{pmatrix} 0 & Y_\lambda^{(1)} \\ X_\lambda & R_\lambda^{(r)} \\ 0 & Y^{(r)} \end{pmatrix}$$

with

$$(3.3) \quad X_\lambda = \left( x_i^{\lambda_j+m-n-j} \right)_{1 \leq i \leq m, 1 \leq j \leq k-1}, \quad R^{(r)} = \left( \frac{y_j^r}{x_i^r(x_i+y_j)} \right)_{1 \leq i \leq m, 1 \leq j \leq n}$$

$$(3.4) \quad Y_\lambda^{(1)} = \left( y_j^{\lambda_i+n-m-i} \right)_{1 \leq i \leq l-1, 1 \leq j \leq n}, \quad Y^{(r)} = \left( y_j^{r-i} \right)_{1 \leq i \leq r, 1 \leq j \leq n}$$

*Proof.* The result is obtained by applying Laplace's theorem [16, Section 1.8] for the expansion of  $\det(C)$  with respect to columns  $1, 2, \dots, k-1$ . □

This determinantal formula can be rewritten, leading to our main result :

**Theorem 3.2.** *Let  $\lambda \in \mathcal{H}_{m,n}$  and  $k$  be the  $(m, n)$ -index of  $\lambda$ . Then*

$$(3.5) \quad s_\lambda(x/y) = (-1)^{mn-m+k-1} D^{-1} \det \left( \begin{array}{cc} \left( \frac{1}{x_i+y_j} \right)_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} & \left( x_i^{\lambda_j+m-n-j} \right)_{\substack{1 \leq i \leq m \\ 1 \leq j \leq k-1}} \\ \left( y_j^{\lambda_i+n-m-i} \right)_{\substack{1 \leq i \leq n-m+k-1 \\ 1 \leq j \leq n}} & 0 \end{array} \right).$$

*Proof.* Consider  $C$  as defined in the previous lemma and let  $R_i$  denote its row  $i$ . Let  $C'$  denote the modified matrix  $C$  where every column  $R_{l-1+i}$  has been multiplied by  $x_i^r$  (for all  $i$  with  $1 \leq i \leq m$ ). The result can be found by applying the following row operations on the modified matrix  $C'$  :

$$R'_{l-1+i} \longrightarrow R'_{l-1+i} - \sum_{s=0}^{r-1} (-1)^s x_i^s R'_{l+m+s} \quad (1 \leq i \leq m)$$

taking into account that  $\lambda_{k-1} - \lambda_k \geq r \geq 0$  and  $\lambda'_{l-1+i} = k-1$  for  $1 \leq i \leq r$  ( $k, l$  and  $r$  defined as in (2.11) and (2.26)). □

In Section 1 we have already pointed out that for  $\lambda = 0$  and  $m \leq n$ , (3.5) gives rise to the determinantal identity (1.18) combining Cauchy's double alternant with Vandermonde's

determinant. It is easy to verify that for  $\lambda = 0$  and  $m > n$  we have that  $k = n - m + 1$ , and then (3.5) gives rise to an identity equivalent to (1.18).

Finally, let us consider the case with  $k = m + 1$  (corresponding to a typical representation  $V_\lambda$  in terms of the previous section). Then the blocks  $X_\lambda$  and  $Y_\lambda$  are square matrices, and one finds :

$$\begin{aligned}
 s_\lambda(x/y) &= D^{-1} \det(x_i^{\lambda_j+m-n-j})_{1 \leq i,j \leq m} \det(y_j^{\lambda'_i+n-m-i})_{1 \leq i,j \leq n} \\
 (3.6) \qquad &= s_\tau(x) s_{\eta'}(y) \prod_{i=1}^m \prod_{j=1}^n (x_i + y_j) = s_\tau(x) s_{\eta'}(y) R(x, -y),
 \end{aligned}$$

where  $\tau$  and  $\eta$  are the parts of  $\lambda$  defined in (1.9). This is in agreement with (1.12).

Once the determinantal expression has been established, there are several ways to prove that the rhs of (3.5) is indeed a supersymmetric S-polynomial. A first proof is based on the fact that, in [10, §I.3, exercise 23], Macdonald showed that the supersymmetric S-polynomials satisfy four properties (see also [13]) which also characterize these polynomials.<sup>1</sup> We have shown that the polynomials defined by means of the rhs of (3.5) do indeed satisfy these four properties (see an earlier version of this paper, available from the authors).

Another and more direct proof was outlined by a referee and is based on the formula (1.10) of Sergeev-Pragacz (see [12] and [19]) and Laplace's theorem [16, Section 1.8].

Let us write  $x = x' + x''$  for a decomposition of  $x = (x_1, x_2, \dots, x_m)$  into two disjoint subsets of fixed size, say  $|x'| = p$  and  $|x''| = q$  with  $p + q = m$ . Recall the definition of  $R(x', x'')$  in (1.14).

**Lemma 3.3.** *For  $m = p + q$ , let  $\mu = (\mu_1, \dots, \mu_p)$ ,  $\nu = (\nu_1, \dots, \nu_q)$  be two partitions and  $\lambda = (\mu_1, \dots, \mu_p, \nu_1, \dots, \nu_q)$ . Then*

$$(3.7) \qquad \sum_{x'+x''} \frac{s_{\mu+(q^p)}(x') s_\nu(x'')}{R(x', x'')} = s_\lambda(x)$$

where the sum is over all possible decompositions  $x = x' + x''$  with the size of  $x'$  equal to  $p$  and the size of  $x''$  equal to  $q$ .

*Proof.* We can rewrite the lhs of expression (3.7) using the determinantal formula for S-functions and the equality  $D(x) = (-1)^{\frac{p(p+1)}{2}+r_1+\dots+r_p} D(x') D(x'') R(x', x'')$  with the elements of  $x'$  denoted by  $x_{r_1}, \dots, x_{r_p}$  and those of  $x''$  by  $x_{s_1}, \dots, x_{s_q}$  :

$$(3.8) \qquad \sum_{x'+x''} \frac{s_{\mu+(q^p)}(x') s_\nu(x'')}{R(x', x'')} = \frac{(-1)^{\frac{p(p+1)}{2}}}{D(x)} \sum_{x'+x''} (-1)^{r_1+\dots+r_p} |x_{r_i}^{\mu_j+q+m-j}| |x_{s_i}^{\nu_j+m-j}|.$$

The numerator of this sum is the Laplace expansion of the following determinant with respect to columns  $1, 2, \dots, p$  :

$$(3.9) \qquad \left| \begin{array}{cc} x^{\mu+(q^p)+\delta_p} & x^{\nu+\delta_q} \end{array} \right| = \left| x^{\lambda+\delta_m} \right|$$

with  $\lambda = (\mu_1, \dots, \mu_p, \nu_1, \dots, \nu_q)$  and  $\delta_l$  given in (1.4). The result follows. □

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<sup>1</sup>Our convention for the  $s_\lambda(x/y)$  is slightly different from that of Macdonald's  $s_\lambda^{\text{Mac}}(x/y)$  :  $s_\lambda(x/y) = s_\lambda^{\text{Mac}}(x/-y)$ .

**Lemma 3.4.** *Let  $m = p+q$  and  $\lambda = (\lambda_1, \lambda_2, \dots)$  be a partition. Denote  $\lambda^{(p)} = (\lambda_1, \dots, \lambda_p) + (q^p)$  and  $\lambda_*^{(p)} = (\lambda_{p+1}, \lambda_{p+2}, \dots)$ . Then,*

$$(3.10) \quad s_\lambda(x/y) = \sum_{x'+x''} \frac{s_{\lambda^{(p)}}(x'/-y) s_{\lambda_*^{(p)}}(x''/-y)}{R(x', x'')}$$

where the sum is over all possible decompositions  $x = x' + x''$  with the size of  $x'$  equal to  $p$  and the size of  $x''$  equal to  $q$ .

*Proof.* One can use, e.g., induction on  $n$ , i.e. the number of variables  $y = (y_1, \dots, y_n)$ . If  $n = 0$  then we are reduced to the symmetric case and the result follows from Lemma 3.3. Otherwise, one separates the variable  $y_n$  and sums over all partitions  $\mu$  such that  $\lambda - \mu$  is a vertical strip (v.s.) :

$$\begin{aligned} s_\lambda(x/y^{(n)}) &= \sum_{\mu \subseteq \lambda, (\text{v.s.})} s_\mu(x/y^{(n-1)}) s_{(r)}(-y_n) \\ &= \sum_{\mu \subseteq \lambda, (\text{v.s.})} \sum_{x'+x''} \frac{s_{\mu^{(p)}}(x'/-y^{(n-1)}) s_{\mu_*^{(p)}}(x''/-y^{(n-1)})}{R(x', x'')} (-y_n)^r \\ &= \sum_{x'+x''} \left[ \left( \sum_{\mu^{(p)} \subseteq \lambda^{(p)}, (\text{v.s.})} s_{\mu^{(p)}}(x'/-y^{(n-1)}) (-y_n)^s \right) \left( \sum_{\mu_*^{(p)} \subseteq \lambda_*^{(p)}, (\text{v.s.})} s_{\mu_*^{(p)}}(x''/-y^{(n-1)}) (-y_n)^t \right) \right] \\ &\quad \times \frac{1}{R(x', x'')} - \sum_{\substack{\mu \subseteq \lambda, (\text{v.s.}) \\ \mu_{p+1} = \mu_p + 1}} \sum_{x'+x''} \frac{s_{\mu^{(p)}}(x'/-y^{(n-1)}) s_{\mu_*^{(p)}}(x''/-y^{(n-1)})}{R(x', x'')} (-y_n)^r \quad (\text{with } r = s + t) \end{aligned}$$

The first part in this expression is the result we want. For the second part, one can by induction use (3.10); then it is easy to see that these contributions are zero.  $\square$

We can now give a direct proof of the determinantal formula.

**Theorem 3.5.** *Let  $\lambda \in \mathcal{H}_{m,n}$  and  $k$  be the  $(m, n)$ -index of  $\lambda$ . Then*

$$(3.11) \quad \frac{R(x, y)}{D(x)D(y)} \det \begin{pmatrix} \frac{1}{x-y} & X_\lambda \\ Y_\lambda & 0 \end{pmatrix} = \pm s_\lambda(x/y),$$

where the (rectangular) blocks of the determinant are given by

$$\begin{aligned} \frac{1}{x-y} &= \left( \frac{1}{x_i - y_j} \right)_{1 \leq i \leq m, 1 \leq j \leq n}, \\ X_\lambda &= \left( x_i^{\lambda_j + m - n - j} \right)_{1 \leq i \leq m, 1 \leq j \leq k-1}, \quad Y_\lambda = \left( y_j^{\lambda_i + n - m - i} \right)_{1 \leq i \leq n - m + k - 1, 1 \leq j \leq n}. \end{aligned}$$

*Proof.* Suppose  $|x'| = k-1 \equiv p$  and  $|y'| = n-m+k-1 \equiv q$ . We will indicate the indices of the elements of  $x'$  by  $i_t$  ( $t = 1, \dots, p$ ) and those of  $y'$  by  $j_t$  ( $t = 1, \dots, q$ ). The determinant in (3.11) has a double Laplace expansion, with partitions  $\alpha = (\lambda_1, \dots, \lambda_{k-1}) - (q^p)$  and

$\beta = (\lambda'_1, \dots, \lambda'_q) - (p^q)$  determined by  $X_\lambda$  and  $Y_\lambda$ , so the lhs of (3.11) equals :

$$\begin{aligned} & \frac{R(x, y)}{D(x)D(y)} \sum_{x=x'+x''} \sum_{y=y'+y''} (-1)^P D(x')D(y')s_\alpha(x')s_\beta(y') \det\left(\frac{1}{x''-y''}\right) \\ & \left( \text{with } P = \frac{p(p+1)}{2} + \frac{q(q+1)}{2} + \sum_{t=1}^p i_t + \sum_{t=1}^q j_t \right) \\ & = \frac{R(x, y)}{D(x)D(y)} \sum_{x'+x''} \sum_{y'+y''} (-1)^P D(x')D(y')s_\alpha(x')s_\beta(y')D(x'')D(y'')/R(x'', y'') \\ & = \sum_{x'+x''} \sum_{y'+y''} s_\alpha(x')s_\beta(y') \frac{R(x, y')R(x', y'')}{R(x', x'')R(y', y'')} \\ & = \sum_{y'+y''} \left( \sum_{x'+x''} \frac{s_\alpha(x')R(x', y'')}{R(x', x'')} \right) \frac{s_\beta(y')(-1)^{mq}R(y', x)}{R(y', y'')}. \end{aligned}$$

Now we can apply the special case (1.12) of the Sergeev-Pragacz formula twice. Putting  $\eta = \alpha + ((m-k+1)^p)$  and  $\chi = \beta + (m^q) = (\lambda'_1, \dots, \lambda'_q) + ((m-k+1)^q)$ , there comes

$$\pm \sum_{y'+y''} \left( \sum_{x'+x''} \frac{s_\eta(x' - y'')}{R(x', x'')} \right) \frac{s_\chi(y' - x)}{R(y', y'')} = \pm \sum_{y'+y''} \left( \sum_{x'+x''} \frac{s_\eta(x' - y'')}{R(x', x'')} \right) \frac{s_\chi(-y'/x)}{R(y', y'')}.$$

Finally, we use Lemma 3.4, and duality (1.13); the last expression becomes :

$$\pm \sum_{y'+y''} s_\alpha(x' - y'')s_\chi(-y'/x) \frac{1}{R(y', y'')} = \pm (-1)^{|\alpha|} \sum_{y'+y''} \frac{s_{\alpha'}(-y''/x)s_\chi(-y'/x)}{R(y', y'')}.$$

As  $\alpha' = (\lambda'_{q+1}, \lambda'_{q+2}, \dots)$  and  $\chi = (\lambda'_1, \dots, \lambda'_q) + ((m-k+1)^q)$ , this is equal to  $\pm s_{\chi'}(-y/x) = \pm s_\lambda(x/y)$ , using once more Lemma 3.4 and duality.  $\square$

Thus, the proof uses essentially a double Laplace expansion, twice the application of the factorization case of the Sergeev-Pragacz formula, and twice Lemma 3.4. Finally, observe that the sign in Theorem 3.11 depends on the partition  $\lambda$ , in particular it is equal to

$$(-1)^{\sum_{i=1}^{n-m+k-1} \lambda'_i + \frac{m(m-1)}{2} + \frac{n(n-1)}{2} - \frac{k(k-1)}{2} - 1}$$

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#### REFERENCES

- [1] E.L. Basor and P.J. Forrester, Formulas for the evaluation of Toeplitz determinants with rational generating functions, *Mathematische Nachrichten*, 170 (1994), 5-18.
- [2] N. Bergeron and A. Garsia, Sergeev's formula and the Littlewood-Richardson rule, *Linear and Multilinear Algebra*, 27 (1990), 79-100.
- [3] A. Berele and A. Regev, Hook Young diagrams with applications to combinatorics and to representations of Lie superalgebras, *Advances in Mathematics*, 64 (1987), 118-175.
- [4] V.G. Kac, Lie superalgebras, *Advances in Mathematics*, 26 (1977), 8-96.
- [5] V.G. Kac, Representations of classical Lie superalgebras, *Lecture Notes in Mathematics* 676, pp. 597-626, Springer, Berlin, 1978.

- [6] V.G. Kac and M. Wakimoto, Integrable highest weight modules over affine superalgebras and number theory, *Progress in Mathematics*, 123 (1994), 415-456.
- [7] R.C. King, Supersymmetric functions and the Lie supergroup  $U(m/n)$ , *Ars Combinatoria*, 16 A (1983), 269-287.
- [8] C. Krattenthaler, Advanced determinant calculus, *Séminaire Lotharingien Combinatoire*, 42 (1999), Article B42q, 67 pp.
- [9] A. Lascoux and P. Pragacz, Ribbon Schur functions, *European Journal of Combinatorics*, 9 (1988), 561-574.
- [10] I.G. Macdonald, *Symmetric functions and Hall polynomials*, Oxford University Press, Oxford, 1979.
- [11] I. Penkov and V. Serganova, On irreducible representations of classical Lie superalgebras, *Indagationes Mathematica*, 3 (1992), 419-466.
- [12] P. Pragacz, Algebro-geometric applications of Schur S- and Q-polynomials, *Topics in invariant theory (Paris, 1989/1990)*, *Lecture Notes in Mathematics* 1478, pp. 130-191, Springer, Berlin, 1991.
- [13] P. Pragacz and A. Thorup, On a Jacobi-Trudi identity for supersymmetric polynomials, *Advances in Mathematics*, 95 (1992), 8-17.
- [14] M. Scheunert, *The theory of Lie superalgebras*, Springer, Berlin, 1979.
- [15] A.N. Sergeev, Tensor algebra of the identity representation as a module over the Lie superalgebras  $GL(n, m)$  and  $Q(n)$  (Russian), *Matematicheskii Sbornik*, 123 (1984), 422-430.
- [16] G.E. Shilov, *Linear Algebra*, Dover Publications, New York, 1977.
- [17] J. Stembridge, A characterization of supersymmetric polynomials, *Journal of Algebra*, 95 (1985), 439-444.
- [18] J. Van der Jeugt and V. Fack, The Pragacz identity and a new algorithm for Littlewood-Richardson coefficients, *Computers and Mathematics with Applications*, 21 (1991), 39-47.
- [19] J. Van der Jeugt, J.W.B. Hughes, R.C. King and J. Thierry-Mieg, Character formulas for irreducible modules of the Lie superalgebra  $sl(m/n)$ , *Journal of Mathematical Physics*, 31 (1990), 2278-2304.
- [20] J. Van der Jeugt, J.W.B. Hughes, R.C. King and J. Thierry-Mieg, A character formula for singly atypical modules of the Lie superalgebra  $sl(m/n)$ , *Communications in Algebra*, 18 (1990), 3453-3480.
- [21] J. Van der Jeugt and R.B. Zhang, Characters and composition factor multiplicities for the Lie superalgebra  $gl(m/n)$ , *Letters in Mathematical Physics*, 47 (1999), 49-61.

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