HALL-LITTLEWOOD ANALOGS IN THE Q-FUNCTION ALGEBRA

GEANINA TUDOSE AND MICHAEL ZABROCKI

ABSTRACT. We present a family of analogs of the Hall-Littlewood symmetric functions in the Q-function algebra. The change of basis coefficients between this family and Schur's Q-functions are q-analogs of numbers of marked shifted tableaux. These coefficients exhibit many parallel properties to the Kostka-Foulkes polynomials.

RÉSUMÉ. Nous présentons une famille des analogues des fonctions symétriques de Hall-Littlewood dans l'algébre des Q-fonctions. Le changement des coefficients de base sont les q-analogues des nombres de tableaux deplacés et marqués. Ces coefficients présentent beaucoup de propriétés parallèles aux polynômes de Kostka-Foulkes.

1. INTRODUCTION

The symmetric functions form a fundamental algebra Λ associated to the representation theory and the combinatorics of the symmetric group. The Q-functions form a sub-algebra $\Gamma \subset \Lambda$ of the symmetric functions and are associated to the projective representations of the symmetric group. The fundamental bases for these algebras are (respectively) the Schur functions, $s_\lambda[X]$, indexed by partitions, and Schur's Q-functions, $Q_\lambda[X]$, indexed by strict partitions.

Another important basis of Λ is the Hall-Littlewood symmetric functions. They were introduced by Hall [3] as an algebra whose structure coefficients count chains of submodules of a certain type. In addition, they have remarkable combinatorial and algebraic properties and interpolate several other bases with a parameter q. They may also be seen in other contexts such as a Demazure character formula for the Hecke algebra, or as the Frobenius series of certain symmetric group or $GL(n)$-modules.

The change of basis coefficients between the Hall-Littlewood symmetric functions and the Schur basis are known as the Kostka-Foulkes polynomials. They can be seen as a q-generating function for the set of column strict tableaux of fixed shape and content. The combinatorial tools of the RSK-algorithm, jeu de taquin and the plactic monoid [13] were developed in part to help answer the question of a combinatorial interpretation for these coefficients. Lascoux and Schützenberger [12] produced a solution by the introduction of the statistic *charge* on column strict tableaux.

The aim of this paper is to introduce the analogs of the Hall-Littlewood symmetric functions and the Kostka-Foulkes polynomials for the Q-function algebra. These analogs do not exist in the literature and are certainly missing elements in the combinatorial theory of the Q-function algebra. The combinatorics of the shifted tableaux developed in [2] [15] [19] [21] correspond to the Q-functions in the same way that the column strict tableaux correspond to the symmetric functions. The functions that we introduce here share many of the same properties as the Hall-Littlewood symmetric functions. The analogs of Kostka-Foulkes polynomials are q-generating functions of the set of marked shifted tableaux of fixed shape and content. Some of the most important properties of these functions are still conjecture and they suggest a yet undiscovered structure in the combinatorics of shifted tableaux including a poset structure associated to a charge-like statistic.

This abstract is divided into two sections. In the first section we present some well known theory related to the symmetric functions and in particular the Hall-Littlewood functions. For a more detailed account of this theory we refer the reader to [14]. The second section is a development of the functions $G_\lambda[X;q]$ (the functions that are analogs of the Hall-Littlewoods in Γ) and the change of basis coefficients $L_{\lambda\mu}(q)$ (analogs of the Kostka-Foulkes polynomials). We present several formulas for computing these coefficients and list some properties and conjectures. In an appendix we also list a table of transition coefficients $L_{\lambda\mu}(q)$ and draw as an example a conjectured poset for the marked shifted tableaux of content $(4,3,1)$.

Finally, we briefly consider the *parabolic* version of $G_\lambda[X;q]$ which are analogs of the functions introduced in [16, 17]. The definition follows the generalization of Jing's Hall-Littlewood vertex operator to a more general class of operators, as was considered in [18]. The coefficients that appear in this generalization can be viewed as q-analogs of the structure coefficients of Schur's Q-functions.

2. Notation and Definitions

2.1. **Symmetric functions, partitions, tableaux.** Define the ring of symmetric functions as the polynomial ring $\Lambda = \mathbb{C}[p_1, p_2, p_3, \ldots]$ with $deg(p_k) = k$. A typical monomial of degree n in this ring will be $p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_\ell} := p_\lambda$, where $\sum_i \lambda_i = n$ and a basis will indexed by the sequences λ such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 0$.

The sequence λ is a partition of n (denoted by $\lambda \vdash n$) if the entries are non-negative integers and are is weakly decreasing. The size of λ is given by $|\lambda| := \sum_i \lambda_i = n$. The entries of λ are called the parts of the partition. The number of parts that are of size i in λ will be represented by $m_i(\lambda)$ and the total number of non-zero parts is represented by $\ell(\lambda) = \sum_i m_i(\lambda)$. A common statistic on partitions λ is $n(\lambda) := \sum_i (i-1)\lambda_i$.

The dominance order, $\lambda \leq \mu$ if and only if $\sum_{i=1}^k \lambda_i \leq \sum_{i=1}^k \mu_i$ for all $1 \leq k \leq \ell(\lambda)$, is a partial order on partitions. Using this partial order, the operators

$$R_{ij}\lambda = (\lambda_1, \ldots, \lambda_i + 1, \ldots, \lambda_j - 1, \ldots, \lambda_{\ell(\lambda)})$$

for $1 \leq i \leq j \leq \ell(\lambda)$ have the property that $R_{ij}\lambda \geq \lambda$ if $R_{ij}\lambda$ is a partition.

We will consider three fundamental bases of Λ here. Following the notation of [14], we define the homogeneous (complete) symmetric functions are $h_\lambda := h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_{\ell(\lambda)}}$ where $h_n = \sum_{\lambda \vdash n} p_\lambda / z_\lambda$ and $z_\lambda = \prod_{i=1}^{\ell(\lambda)} i^{m_i(\lambda)} m_i(\lambda)!$. The elementary symmetric functions are $e_\lambda := e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_{\ell(\lambda)}}$ where $e_n = \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} p_\lambda / z_\lambda$. By convention we set $p_0 = h_0 = e_0 = 1$ and $p_{-k} = h_{-k} = e_{-k} = 0$ for $k > 0$. The Schur functions are given by $s_\lambda = det\,|h_{\lambda_i + i - j}|_{1 \leq i,j \leq \ell(\lambda)}$. The sets $\{p_\lambda\}_{\lambda \vdash n}$, $\{h_\lambda\}_{\lambda \vdash n}$, $\{e_\lambda\}_{\lambda \vdash n}$ and $\{s_\lambda\}_{\lambda \vdash n}$ all form bases for the symmetric functions of degree n.

The fundamental theorem of symmetric functions says that the subring $\mathbb{C}[p_1, p_2, \ldots, p_n]$ is isomorphic to the ring of symmetric polynomials $\Lambda^{X_n} = \mathbb{C}[x_1, x_2, \ldots, x_n]^{S_n}$ (the polynomials in n variables which are invariant under the action $\sigma(x_i) = x_{\sigma(i)}$ for any $\sigma \in S_n$) using the map that sends $p_k \to x_1^k + x_2^k + \cdots + x_n^k$. The space Λ^X of symmetric series in an infinite number of variables $x_1, x_2, x_3, \ldots$ of finite degree is isomorphic to Λ under the map that sends $p_k \to x_1^k + x_2^k + x_3^k + \cdots$.

Much of our notation for the symmetric functions thus far has reflected that of [14], but we will concentrate on operations involving the Hopf algebra structure of the symmetric functions and specialization of variables. To this end we extend the notation for these maps in a natural manner and represent a set of variables as a sum $X = x_1 + x_2 + x_3 + \ldots$ and act on this sum with elements of Λ. We define $p_k[X] = x_1^k + x_2^k + x_3^k + \cdots$ and for any $P \in \Lambda$ we set $P[X]$ equal to P with p_k replaced by $p_k[X]$. That is for $P = \sum_\lambda c_\lambda p_\lambda$,

(1)
$$P[X] = \sum_\lambda c_\lambda p_{\lambda_1}[X] p_{\lambda_2}[X] \cdots p_{\lambda_{\ell(\lambda)}}[X].$$

It is clearly true for two sets of variables X and $Y = y_1 + y_2 + y_3 + \cdots$ that $p_k[X + Y] = p_k[X] + p_k[Y]$ and to extend this linearly we set $p_k[X - Y] = p_k[X] - p_k[Y]$ and $p_k[XY] = p_k[X]p_k[Y]$. We will also consider the Cauchy element

(2)
$$\Omega = \sum_{n\geq 0} \sum_{\lambda \vdash n} p_\lambda / z_\lambda = \sum_{n \geq 0} h_n$$

in the completion of Λ. This special element has the property that $\Omega[X + Y] = \Omega[X]\Omega[Y]$, $\Omega[X - Y] = \Omega[X]/\Omega[Y]$ and $\Omega[X] = \prod_i (1 - x_i)^{-1}$.

Notice that for an arbitrary element $c \in \mathbb{C}$, we have $p_k[cX] = cp_k[X]$. This implies that cX does not represent $cx_1 + cx_2 + cx_3 + \cdots$, instead it represents c 'copies of' the variables X. We introduce a special parameter q or t that interacts with the variable set in that $p_k[qX] = q^k p_k[X]$. Sometimes this element will be an arbitrary parameter and other times we will specialize it to values in the base field $\mathbb{C}$. To obtain operations such as replacing x_i by cx_i in a symmetric function we use our special parameter q and at the end of our calculations we specialize this parameter to c. In particular, the operation of replacing x_i by $-x_i$ is useful and we will represent it with the notation

(3)
$$P[\epsilon X] = P[qX]\Big|_{q=-1}.$$

We also have the relations $p_k[\epsilon X] = (-1)^k p_k[X]$, $\Omega[\epsilon X] = \prod_i (1 + x_i)^{-1}$ and $h_n[X] = e_n[-\epsilon X]$. Of course if the symmetric function P or the set of variables X already has a parameter q, the one that is set to -1 is unique and does not interfere with parameters in P or X.

It follows from the definition of the Schur function and the expansion of the Vandermonde determinant $det|x_i^{j-1}|_{1\leq i,j \leq n} = \prod_{1 \leq i < j \leq n}(x_i - x_j)$ that $s_\lambda[X] = \prod_{1\leq i<j\leq n}(1 - R_{ij})h_\lambda[X]$, where $R_{ij}h_\lambda[X] = h_{R_{ij}\lambda}[X]$. Since the coefficient of z^λ in $\Omega[Z_n X]$ is $h_\lambda[X]$ and $(z_j/z_i)^{-1} z_\lambda = z^{R_{ij}\lambda}$, then the Schur function is equal to

(4)
$$s_\lambda[X] = \Omega[Z_n X] \prod_{1 \leq i < j \leq n} (1 - z_j/z_i)\Big|_{z^\lambda}.$$

Remark: We follow [14] in the use of R_{ij} acing on symmetric functions, however one should note that these operators are not associative. This issue can be resolved however and is dealt with in more detail in [1] or [8].

Now for any symmetric function $P \in \Lambda$ define $\mathbf{S}(z)P[X] := P\left[X - \frac{1}{z}\right]\Omega[zX]$. Since we have that $\mathbf{S}(z_1)\mathbf{S}(z_2)\cdots\mathbf{S}(z_n)1 = \Omega[Z_n X]\prod_{1\leq i<j\leq n}(1 - z_j/z_i)$, then the operator $\mathbf{S}_m P[X] = \mathbf{S}(z)P[X]\Big|_{z^m}$ raises the degree of a symmetric function by m and has the property that $\mathbf{S}_m(s_\lambda[X]) = s_{(m,\lambda)}[X]$ as long as $m \geq \lambda_1$. The $\mathbf{S}_m$ operators also have the commutation relations $\mathbf{S}_m\mathbf{S}_{m+1} = 0$ and $\mathbf{S}_m\mathbf{S}_n = -\mathbf{S}_{n-1}\mathbf{S}_{m+1}$.

A Young diagram for a partition will be a collection of cells of the integer grid lying in the first quadrant. For a partition λ, $Y(\lambda) = \{(i, j) : 0 \leq j < \ell(\lambda) \text{ and } 0 \leq i \leq \lambda_j\}$. The reason why we consider empty cells rather than say points is because we wish to consider fillings of these cells. A tableau is a map from the set $Y(\lambda)$ to $\mathbb{N}$, this may be represented on a Young diagram by writing integers within the cells of a graphical representation of a

Young diagram (see figure 1). The shape of the tableau is the partition λ. We say that a tableau T is column strict if $T(i,j) \leq T(i+1,j)$ and $T(i,j) < T(i,j+1)$ whenever the points $(i+1,j)$ or $(i,j+1)$ are in $Y(\lambda)$. Let $m_k(T)$ represent the number of points p in $Y(\lambda)$ such that $T(p) = k$. The vector $(m_1(T), m_2(T), \ldots)$ is the content of the tableau T.

The Pieri rule describes a combinatorial method for computing the product of $h_m[X]$ and $s_\mu[X]$ expanded in the Schur basis. We will use the notation $\lambda/\mu \in \mathcal{H}_m$ to represent that $|\lambda| - |\mu| = m$ and for $1 \leq i \leq \ell(\lambda)$, $\mu_i \leq \lambda_i$ and $\mu_i \geq \lambda_{i+1}$. It may be easily shown that

$$h_m[X]s_\mu[X] = \sum_{\lambda/\mu \in \mathcal{H}_m} s_\lambda[X]. \tag{5}$$

This gives a method for computing the expansion of the $h_\mu[X]$ basis in terms of the Schur functions. Consider the coefficients $K_{\lambda\mu}$ defined by the expression

$$h_\mu[X] = \sum_{\lambda \vdash |\mu|} K_{\lambda\mu} s_\lambda[X]. \tag{6}$$

$K_{\lambda\mu}$ are called the Kostka numbers and are equal to the number of column strict tableaux of shape λ and content μ. Now define a q analog of the $\{h_\lambda\}$ basis by setting

$$H_\lambda[X;q] = \prod_{i<j} \frac{1 - R_{ij}}{1 - qR_{ij}} h_\lambda[X] = \prod_{i<j} (1 + (q-1)R_{ij} + (q^2 - q)R_{ij}^2 + \cdots) h_\lambda[X]. \tag{7}$$

Since the coefficient of z^λ in $\Omega[Z_k X]$ is $h_\lambda[X]$, it is clear that we have the formula

$$H_\lambda[X;q] = \Omega[Z_k X] \prod_{1 \leq i < j \leq k} \frac{1 - z_j/z_i}{1 - qz_j/z_i}\Big|_{z^\lambda}. \tag{8}$$

This leads us to a 'vertex operator' definition for these functions. If we define the operation $\mathbf{H}(z)P[X] = P\left[X - \frac{1-q}{z}\right] \Omega[zX]$, then

$$\mathbf{H}(z_1)\mathbf{H}(z_2) \cdots \mathbf{H}(z_k) 1 = \Omega[Z_k X] \prod_{1 \leq i < j \leq k} \frac{1 - z_j/z_i}{1 - qz_j/z_i}, \tag{9}$$

and therefore defining the operator $\mathbf{H}_m$ that raises the degree of a symmetric function by m as $\mathbf{H}_m P[X] := \mathbf{H}(z)P[X]\Big|_{z^m}$, has the property that $\mathbf{H}_m H_\lambda[X;q] = H_{(m,\lambda)}[X;q]$ as long as $m \geq \lambda_1$. The vertex operator also satisfies the relations $\mathbf{H}_{m-1}\mathbf{H}_m = q\mathbf{H}_m\mathbf{H}_{m-1}$ and $\mathbf{H}_{m-1}\mathbf{H}_n - q\mathbf{H}_m\mathbf{H}_{n-1} = q\mathbf{H}_n\mathbf{H}_{m-1} - \mathbf{H}_{n-1}\mathbf{H}_m$.

The functions $H_\lambda[X;q]$ interpolate between the functions $s_\lambda[X] = H_\lambda[X;0]$ and $h_\lambda[X] = H_\lambda[X;1]$. The Kostka-Foulkes polynomials are defined as the q-polynomial coefficient of $s_\lambda[X]$ in $H_\mu[X;q]$ and hence we have the expansion analogous to (6).

$$H_\mu[X;q] = \sum_{\lambda \vdash |\mu|} K_{\lambda\mu}(q) s_\lambda[X]. \tag{10}$$

The coefficients $K_{\lambda\mu}(q)$ are clearly polynomials in q, but it is surprising to find that the coefficients of the polynomials are non-negative integers. A defining recurrence can be derived $K_{\lambda\mu}(q)$ in terms of the Kostka-Foulkes polynomials indexed by partitions of size $|\mu| - \mu_1$ using the formula for $\mathbf{H}_m$. This recurrence is often referred to as the 'Morris recurrence' for the Kostka-Foulkes polynomials.

The Kostka-Foulkes polynomials and the generating functions $H_\mu[X;q]$ have the following important properties which we simply list here so that we may draw a connection to analogous formulae. For a more detailed reference of these sorts of properties we refer the interested reader to the excellent survey article [1].

FIGURE 1. The diagram on the left represents a column strict tableau of shape $(6,5,3,3)$ and content $(4,3,3,2,2,2,1)$. The diagram on the right represents a shifted marked tableau of shape $(7,5,4,1)$ and content $(2,5,5,3,2)$. This tableau has labels which are marked on the diagonal.

(1) $K_{\lambda\mu}(q)$ has non-negative integer coefficients.
(2) $K_{\lambda\mu}(q) = \sum_T q^{c(T)}$, where the sum is over all column strict tableaux of shape λ and content μ and $c(T)$ denotes the charge of a tableau T (see [12]). In addition there is a combinatorial interpretation for these coefficients in terms of objects called rigged configurations (see [10]).
(3) The degree in q of $K_{\lambda\mu}(q)$ is $n(\mu) - n(\lambda)$.
(4) $K_{\lambda\mu}(0) = \delta_{\lambda\mu}$ which implies $H_\mu[X;0] = s_\mu[X]$, $K_{\lambda\mu}(1) = K_{\lambda\mu}$, so that $H_\mu[X;1] = h_\mu[X]$, $K_{\lambda\lambda}(q) = 1$ and $K_{(|\mu|)\mu}(q) = q^{n(\mu)}$. We also have that $K_{\lambda\mu}(q) = 0$ if $\lambda < \mu$.
(5) $H_{(1^n)}[X;q] = e_n\left[\frac{X}{1-q}\right](q;q)_n$ where $(q;q)_n = \prod_{i=1}^n (1-q^i)$.
(6) If ζ is k^{th} root of unity, $H_\mu[X;\zeta]$ factors into a product of symmetric functions.
(7) Set $K'_{\mu\lambda}(q) := q^{n(\lambda)-n(\mu)} K_{\mu\lambda}(1/q)$, then $K'_{\mu\lambda}(q) \geq K'_{\mu\nu}(q)$ for $\lambda \leq \nu$.
(8) $K_{\lambda+(a),\mu+(a)}(q) \geq K_{\lambda,\mu}(q)$, where $\lambda + (a)$ represents the partition λ with a part of size a inserted into it.
(9) $K_{\lambda\mu}(q) = \sum_{w\in S_n} sign(w)\mathcal{P}_q(w(\lambda+\rho) - (\mu+\rho))$ where $\mathcal{P}_q(\alpha)$ is the coefficient of x^α in $\prod_{1\leq i<j\leq n}(1-qx_i/x_j)^{-1}$, a q analog of the Kostant partition function and $\rho = (\ell(\mu)-1, \ell(\mu)-2, \ldots, 1, 0)$.
(10) $H_\mu[X;q]H_\lambda[X;q] = \sum_\gamma d^\nu_{\lambda\mu}(q) H_\nu[X;q]$, for some coefficients $d^\nu_{\lambda\mu}(q)$ with the property that if the Littlewood-Richardson coefficient $c^\nu_{\lambda\mu} = 0$ then $d^\nu_{\lambda\mu}(q) = 0$. These coefficients are a transformation of the Hall algebra structure coefficients.
(11) For the scalar product $\langle s_\lambda[X], s_\mu[X]\rangle = \delta_{\lambda\mu}$, we have that $\langle H_\lambda[X;q], H_\mu[X(1-q);q]\rangle = 0$ if $\lambda \neq \mu$.

2.2. **Schur's Q-functions, strict partitions, and marked shifted tableaux.** The Q-function algebra is a sub-algebra of the symmetric functions $\Gamma = \mathbb{C}[p_1, p_3, p_5, \ldots]$. A typical monomial in this algebra will be p_λ, where λ is a partition and λ_i is odd. A partition λ is strict if $\lambda_i > \lambda_{i+1}$ for all $1 \leq i \leq \ell(\lambda)-1$ and a partition λ is odd if λ_i is odd for $1 \leq i \leq \ell(\lambda)$. We will use the notation $\lambda \vdash_s n$ (respectively $\lambda \vdash_o n$) to denote that λ is a partition of size n that is strict (respectively odd). Note that the number of strict partitions of size n and the number of odd partitions of size n is the same (proof: write out a generating function for each sequence).

The analog of the homogeneous and elementary symmetric functions in Γ are the functions $q_\lambda := q_{\lambda_1} q_{\lambda_2} \cdots q_{\lambda_{\ell(\lambda)}}$, where $q_n = \sum_{\lambda\vdash_o n} 2^{\ell(\lambda)} p_\lambda / z_\lambda$. Define an algebra morphism $\theta : \Lambda \to \Gamma$ by the action on the p_n generators as $\theta(p_n) = (1-(-1)^n)p_n$. That is $\theta(p_n) = 2p_n$ if n is odd and $\theta(p_n) = 0$ for n even. θ has the property that $\theta(h_n) = \theta(e_n) = q_n$ and may be represented in our notation as $\theta(p_n[X]) = p_n[(1-\epsilon)X]$. Under this morphism, our Cauchy element may also be considered a generating function for the q_n elements since

$$\Omega[(1-\epsilon)X] = \sum_{n\geq 0} q_n[X] = \prod_i \frac{1+x_i}{1-x_i}. \tag{11}$$

It follows that $\{p_\lambda\}_{\lambda\vdash_o n}$, $\{q_\lambda\}_{\lambda\vdash_o n}$, $\{q_\lambda\}_{\lambda\vdash_s n}$ are all bases for the subspace of Q-functions of degree n. Another fundamental basis for this space are the Schur's Q-functions $Q_\lambda[X] = \theta(H_\lambda[X;-1])$. These functions hold a similar place in the Q-function algebra that the Schur functions hold in Λ. In particular, $\{Q_\lambda[X]\}_{\lambda\vdash_s n}$ is a basis for the Q-functions of degree n.

In analogy with the Schur functions, $Q_\lambda[X]$ may also be defined with a raising operator formula by setting $q = -1$ and applying the θ homomorphism to equation (7). We arrive at the formula:

$$Q_\lambda[X] = \prod_{i<j} \frac{1-R_{ij}}{1+R_{ij}} q_\lambda[X] = \prod_{i<j}(1 - 2R_{ij} + 2R_{ij}^2 - \cdots) q_\lambda[X], \tag{12}$$

where the operators now act as $R_{ij} q_\lambda[X] = q_{R_{ij}\lambda}[X]$. Furthermore, they have a formula as the coefficient in a generating function:

$$Q_\lambda[X] = \Omega[(1-\epsilon)Z_n X] \prod_{1\le i<j\le n} \frac{1-z_j/z_i}{1+z_j/z_i}\Big|_{z^\lambda}. \tag{13}$$

As with Schur functions and the Hall-Littlewood functions, the raising operator formula leads us to a vertex operator definition. By setting $\mathbf{Q}(z)P[X] = P\left[X - \frac{1}{z}\right]\Omega[(1-\epsilon)zX]$, it is easily shown that $\mathbf{Q}(z_1)\mathbf{Q}(z_2)\cdots\mathbf{Q}(z_n)1 = \Omega[(1-\epsilon)Z_n X]\prod_{1\le i<j\le n}\frac{1-z_j/z_i}{1+z_j/z_i}$, and hence if we set $\mathbf{Q}_m P[X] = \mathbf{Q}(z)P[X]\Big|_{z^m}$ then $\mathbf{Q}_m(Q_\lambda[X]) = Q_{(m,\lambda)}[X]$ as long as $m > \lambda_1$. The commutation relations for the $\mathbf{Q}_m$ are

$$\mathbf{Q}_m\mathbf{Q}_n = -\mathbf{Q}_n\mathbf{Q}_m \text{ for } m \neq -n, \tag{14}$$

$$\mathbf{Q}_m\mathbf{Q}_{-m} = 2(-1)^m - \mathbf{Q}_{-m}\mathbf{Q}_m \text{ if } m \neq 0, \tag{15}$$

$$\mathbf{Q}_m^2 = 0 \text{ if } m \neq 0 \text{ and } \mathbf{Q}_0^2 = 1. \tag{16}$$

These formulas allow us to straighten the $Q_\mu[X]$ functions when they are not indexed by a strict partition.

A shifted Young diagram for a partition will again be a collection of cells lying in the first quadrant. For a strict partition λ, let $YS(\lambda) = \{(i,j) : 0 \le j \le \ell(\lambda) \text{ and } j-1 \le i \le \lambda_j + j - 1\}$. A marked shifted tableau T of shape λ is a map from $YS(\lambda)$ to the set of marked integers $\{1' < 1 < 2' < 2 < \ldots\}$ that satisfy the following conditions

- $T(i,j) \le T(i+1,j)$ and $T(i,j) \le T(i,j+1)$
- If $T(i,j) = k$ for some integer k (i.e. has an unmarked label) then $T(i,j+1) \neq k$
- If $T(i,j) = k'$ for some marked label k' then $T(i+1,j) \neq k'$.

We may represent these objects graphically with a diagram representing λ and the cells filled with the marked integer alphabet. If T is a marked shifted tableau, then we will set $m_i(T)$ as the number of occurrences of i and i' in T. The sequence $(m_1(T), m_2(T), m_3(T), \ldots)$ is the content of T.

The combinatorial definition of the marked shifted tableaux is defined so that it reflects the change of basis coefficients between the q_λ and Q_μ basis. The rule for computing the product of $q_m[X]$ and $Q_\mu[X]$ when expanded in the Schur Q-functions is the analog of the Pieri rule for the Γ space. If $\lambda/\mu \in \mathcal{H}_m$ then $a(\lambda/\mu)$ will represent 1+ the number of $1 < j \le \ell(\lambda)$ such that $\lambda_j > \mu_j$ and $\mu_{j-1} > \lambda_j$. We may show that

$$q_m[X]Q_\mu[X] = \sum_{\lambda/\mu\in\mathcal{H}_m} 2^{a(\lambda/\mu)-\ell(\lambda)+\ell(\mu)} Q_\lambda[X]. \tag{17}$$

Denote by $L_{\lambda\mu}$ the number of marked shifted tableaux T of shape λ and content μ (where λ is a strict partition) such that $T(i,i)$ is not a marked integer. We may expand the function $q_\mu[X]$ in terms of the Q-functions using (17) to show

$$q_\mu[X] = \sum_{\lambda \vdash |\mu|} L_{\lambda\mu} Q_\lambda[X]. \tag{18}$$

3. The Q-Hall-Littlewood basis $G_\lambda(x;q)$ for the algebra Γ

Note: From here, unless otherwise stated, all partitions are considered strict.

3.1. **Raising operator formula.** We define the following analog of the Hall-Littlewood functions in the subalgebra Γ

$$G_\lambda[X;q] := \prod_{1\le i<j\le n} \left(\frac{1+qR_{ij}}{1-qR_{ij}}\right)\left(\frac{1-R_{ij}}{1+R_{ij}}\right) q_\lambda[X] = \prod_{1\le i<j\le n} \left(\frac{1+qR_{ij}}{1-qR_{ij}}\right) Q_\lambda[X]. \tag{19}$$

We call the functions $G_\lambda \in \Gamma \otimes_{\mathbb{C}} \mathbb{C}(q)$ the *Q-Hall-Littlewood functions.*

In $\Gamma \otimes \mathbb{C}(q)$ this family can be expressed in the basis of Q-functions as

$$G_\mu[X;q] = \sum_\lambda L_{\lambda\mu}(q) Q_\lambda[X], \tag{20}$$

which can be viewed as a q-analog of (18). We call the coefficients $L_{\lambda\mu}(q)$ the *Q-Kostka polynomials.* We shall see that this family of polynomials shares many of the same properties with the classical Kostka-Foulkes polynomials. Tables of these coefficients are given in an Appendix. It follows from (19) that $L_{\lambda\mu}(q)$ have integer coefficients and $L_{\lambda\mu}(q) = 0$ if $\lambda < \mu$. This shows

Proposition 1. *The G_λ, λ strict, form a $\mathbb{Z}$-basis for $\Gamma \otimes_{\mathbb{Z}} \mathbb{Z}(q)$.*

The basis G_λ interpolates between the Schur's Q-functions and the functions q_μ because $G_\lambda[X;0] = Q_\lambda[X]$ and $G_\lambda[X;1] = q_\lambda[X]$ as is clear from (19).

Since the coefficient of z^λ in $\Omega[(1-\epsilon)Z_nX]$ is $q_\lambda[X]$ equation (19) implies

$$G_\lambda[X;q] = \prod_{1\le i<j\le n} \left(\frac{1-z_j/z_i}{1+z_j/z_i}\right)\left(\frac{1+qz_j/z_i}{1-qz_j/z_i}\right) \Omega[(1-\epsilon)Z_nX]\Big|_{z^\lambda}. \tag{21}$$

By defining $\mathbf{G}(z)P[X] = P[X - \frac{1-q}{z}]\Omega[(1-\epsilon)zX]$, we may show that

$$\mathbf{G}(z_1)\mathbf{G}(z_2)\cdots\mathbf{G}(z_n)1 = \prod_{1\le i<j\le n} \left(\frac{1-z_j/z_i}{1+z_j/z_i}\right)\left(\frac{1+qz_j/z_i}{1-qz_j/z_i}\right) \Omega[(1-\epsilon)Z_nX]. \tag{22}$$

This implies that if we define the operator

$$\mathbf{G}_m P[X] = P\left[X - \frac{1-q}{z}\right] \Omega[(1-\epsilon)zX]\Big|_{z^m}, \tag{23}$$

then

$$G_\lambda[X;q] = \mathbf{G}_{\lambda_1}\dots\mathbf{G}_{\ell(\lambda)}(1).$$

The operator $\mathbf{G}_m$ satisfies the following commutation relation.

Proposition 2. *For all $r,s \in \mathbb{Z}$ we have*

$$(1-q^2)(\mathbf{G}_r\mathbf{G}_s+\mathbf{G}_s\mathbf{G}_r)+q(\mathbf{G}_{r-1}\mathbf{G}_{s+1}-\mathbf{G}_{s+1}\mathbf{G}_{r-1}+\mathbf{G}_{s-1}\mathbf{G}_{r+1}-\mathbf{G}_{r+1}\mathbf{G}_{s-1}) = 2(-1)^r(1-q)^2\delta_{r,-s}.$$

For $q = 0$ in the equation above we recover the commutation relations of the operator $\mathbf{Q}$ given in equations (14), (15) and (16).

We can use formula (23) to derive the action of this operator on the basis of Schur's Q-functions.

Proposition 3. *For $m > 0$,*

$$\mathbf{G}_m(Q_\lambda[X]) = \sum_{i\geq 0} q^i \sum_{\mu:\lambda/\mu\in\mathcal{H}_i} 2^{a(\lambda/\mu)}(-1)^{\epsilon(m+i,\mu)} Q_{\mu+(m+i)}[X], \tag{24}$$

where $\mu + (k)$ denotes the partition formed by adding a part of size k to the partition μ, and $\epsilon(k,\mu) + 1$ represents which part k becomes in $\mu + (k)$. For $m \leq 0$ a similar statement can be made using the commutation relations (14), (15) and (16).

Proof From (23) the action of $\mathbf{G}_m$ on a function $P[X] \in \Gamma$ can be written as

$$\begin{aligned}
\mathbf{G}_m P[X] &= P[X - (1-q)/z]\Omega[(1-\epsilon)zX]\Big|_{z^m} \\
&= \sum_{i\geq 0} q^i (q_i^\perp P)[X - 1/z]\Omega[(1-\epsilon)zX]\Big|_{z^m} \\
&= \sum_{i\geq 0} q^i \mathbf{Q}_{m+i} q_i^\perp P[X]
\end{aligned}$$

where $q_i^\perp$ is

$$\mathbf{Q}[X+z]\Big|_{z^i} = q_i^\perp Q_\lambda[X] = \sum_{\mu:\lambda/\mu\in\mathcal{H}_i} 2^{a(\lambda/\mu)} Q_\mu[X],$$

and thus equation (24) follows from (14) and (15). □

Example 1. *We compute $G_{(3,2,1)}[X;q]$ using the Proposition above. We have*

$$G_{(3,2,1)}[X;q] = \mathbf{G}_3(\mathbf{G}_2(Q_{(1)}[X])) = \mathbf{G}_3\left(\sum_{i\geq 0}\sum_{(1)/\mu\in\mathcal{H}_i} 2^{a((1)/\mu)}(-1)^{\epsilon(2+i,\mu)} Q_{\mu+(2+i)}[X]\right)$$

$$= \mathbf{G}_3(Q_{(2,1)}) + 2q\mathbf{G}_3(Q_{(3)}) = \sum_{i\geq 0}\sum_{(2,1)/\mu\in\mathcal{H}_i} 2^{a((2,1)/\mu)}(-1)^{\epsilon(3+i,\mu)} Q_{\mu+(3+i)}[X] +$$

$$+2q\left(\sum_{i\geq 0}\sum_{(3)/\nu\in\mathcal{H}_i} 2^{a((3)/\nu)}(-1)^{\epsilon(3+i,\nu)} Q_{\nu+(3+i)}[X]\right)$$

$$= (q^0 2^0 Q_{(3,2,1)} + q^1 2^1 Q_{(4,2)} + q^2 2^1 Q_{(5,1)}) + 2q(q^1 2^1 Q_{(4,2)} + q^2 2^1 Q_{(5,1)} + q^3 2^1 Q_{(6)})$$

$$= Q_{(3,2,1)} + (2q + 4q^2) Q_{(4,2)} + (2q^2 + 4q^3) Q_{(5,1)} + 4q^4 Q_{(6)}.$$

3.2. **Properties of the polynomials $L_{\lambda\mu}(q)$.** The Q-Kostka polynomials introduced here have a number of remarkable properties that are very similar to those of Kostka Foulkes polynomials listed in the previous section. We have already seen the analog of Property 4 holds for Q-Kostka polynomials. In what follows we will consider the other remaining properties.

An important consequence of equation (24) is a Morris-like recurrence which expresses the Q-Kostka polynomials $L_{\lambda\mu}(q)$ in terms of smaller ones.

Proposition 4. *We have the following recurrence*

$$L_{\alpha,(n,\mu)}(q) = \sum_{s=1}^{t:\alpha_t\geq n} (-1)^{s-1} q^{\alpha_s - n} \sum_{\lambda:\lambda/\alpha^{(s)}\in\mathcal{H}_{(\alpha_s-n)}} 2^{a(\lambda/\alpha^{(s)})} L_{\lambda\mu}(q), \tag{25}$$

where $n > \mu_1$ and $\alpha^{(s)}$ is α with part α_s removed.

Proof If $n > \mu_1$ we have that

$$\mathbf{G}_n G_\mu[X;q] = G_{(n,\mu)}[X;q] = \sum_\alpha L_{\alpha,(n,\mu)}(q) Q_\alpha[X]. \tag{26}$$

On the other hand $G_\mu[X;q] = \sum_\lambda L_{\lambda\mu}(q) Q_\lambda[X]$ and so

$$\mathbf{G}_n \left(\sum_\lambda L_{\lambda\mu}(q) Q_\lambda[X] \right) = \sum_\mu L_{\lambda\mu}(q) \mathbf{G}_n(Q_\lambda[X]).$$

Using the action in (24) we have

$$\mathbf{G}_n G_\mu[X;q] = \sum_\lambda L_{\lambda\mu}(q) \sum_{i\geq 0} q^i \sum_{\nu:\lambda/\nu\in\mathcal{H}_i} 2^{a(\lambda/\nu)} (-1)^{\epsilon(n+i,\nu)} Q_{\nu+(n+i)}[X]. \tag{27}$$

For $\alpha = \nu + (n+i)$, equating the coefficients of Q_α in (26) and (27) we get

$$L_{\alpha,(n,\mu)}(q) = \sum_\lambda \sum_{i\geq 0} q^i 2^{a(\lambda/\alpha-(n+i))} (-1)^{\epsilon(n+i,\alpha-(n+i))} L_{\lambda\mu}(q).$$

By reindexing $i := \alpha_s - n$ for $\alpha_s - n \geq 0$ we obtain the desired recurrence (25). □

Example 2. *Let $n = 5$ and $L_{(6,2),(5,2,1)}(q) = 2q + 4q^2$. Using the recurrence we have one s such that $\alpha_s \geq 5$, i.e. $\alpha_1 = 6$. So*

$$L_{(6,2),(5,2,1)}(q) = q^{6-5} \sum_{\lambda/(2)\in\mathcal{H}_1} 2^{a(\lambda/(2))} L_{\lambda(2,1)}(q)$$

$$= q\big(2L_{(21),(21)}(q) + 2L_{(3),(21)}(q)\big) = q(2 + 2\cdot 2q) = 2q + 4q^2.$$

As a consequence of the Morris-like recurrence we have the following

Corollary 5. *Let $\mu \leq \lambda$ in dominance order.*

1. *If $n > \lambda_1$ then $L_{(n,\lambda),(n,\mu)}(q) = L_{\lambda\mu}(q)$.*
2. *$L_{\lambda\lambda}(q) = 1$ and $L_{(|\lambda|)\lambda}(q) = 2^{\ell(\lambda)-1} q^{n(\lambda)}$.*
3. *$2^{\ell(\mu)-\ell(\lambda)}$ divides $L_{\lambda\mu}(q)$.*

Proof 1. There is only one term in the recurrence (25) in this case which is exactly $L_{\lambda\mu}(q)$.
2. The first is a consequence of (1). For the second, we have that the only term on the right hand side is $q^{|\lambda|-\lambda_1} 2L_{(|\lambda|-\lambda_1)(\lambda_2,\ldots)}(q)$ which by induction is $q^{|\lambda|-\lambda_1+n((\lambda_2,\ldots))} 2\cdot 2^{\ell(\lambda)-2} = 2^{\ell(\lambda)-1} q^{n(\lambda)}$. This is the analog of Property 4 for the Kostka-Foulkes polynomials.
3. This property can be easily derived by induction from the recurrence. □

Using the Morris-like recurrence one can obtain a formula for the degree of $L_{\lambda\mu}(q)$ similar to Property 3 for Kostka-Foulkes.

Proposition 6. *If $\mu \leq \lambda$ in dominance order, we have*

$$deg_q L_{\lambda\mu}(q) = n(\mu) - n(\lambda).$$

The property that is most suggestive that these polynomials are analogs of the Kostka-Foulkes polynomials is

Conjecture 7. *The Q-Kostka polynomials $L_{\lambda\mu}(q)$ have non-negative coefficients.*

We can prove this conjecture for some particular cases. In general we believe that there should exist a similar combinatorial interpretation as for the Kostka-Foulkes polynomials. More precisely there should exist a statistic function d on the set of marked shifted tableaux, similar to the charge function on column strict tableaux, such that

$$L_{\lambda\mu}(q) = \sum_T q^{d(T)}$$

summed over marked shifted tableaux of shifted shape λ and content μ with diagonal entries unmarked.

In addition, we conjecture that this function must have the property that if T and S are two marked shifted tableaux such that by erasing the marks the two resulting tableaux coincide, then $d(T) = d(S)$.

For some of the polynomials $L_{\lambda\mu}(q)$, this observation determines completely the statistic on the tableaux. For instance there are two marked shifted tableaux classes of shape $(5,3)$ and content $(4,3,1)$ and $L_{(5,3),(4,3,1)}(q) = 2q + 4q^2$. Clearly the tableau with a 3 in the first row must have statistic 1 and with 3 in the second row has statistic 2. On the other hand, $L_{(6,2),(4,3,1)}(q) = 4q^2 + 4q^3$. This polynomial does not uniquely determine which of the two tableaux have statistic 2 and 3. We have used the function $G_{(4,3,1)}[X;q]$ to draw a conjectured tableau poset (similar to the case of column strict tableau) for the marked shifted tableaux with unmarked diagonals of content $(4,3,1)$ in an appendix.

We also note that monotonicity properties, similar to Property 7 and 8, hold for the Q-Kostka polynomials.

Conjecture 8. *Let $L'_{\lambda\mu}(q) := q^{n(\mu)-n(\lambda)} L_{\lambda\mu}(q^{-1})$. We have*

$$L'_{\lambda\mu}(q) \geq 2^{\ell(\nu)-\ell(\mu)} L'_{\lambda\nu}(q), \qquad \text{for } \mu \leq \nu \text{ in dominance order.}$$

We can prove this fact by using induction and the recurrence (25) for the case $\mu_1 = \nu_1$.

Example 3. *Let $\lambda = (6,2)$, $\mu = (4,3,1)$, $\nu = (5,2,1)$. We have $n(\lambda) = 2$, $n(\mu) = 5$, and $n(\nu) = 4$. The L' polynomials are*

$$L'_{\lambda\mu} = q^{5-2}(4/q^2 + 4/q^3) = 4 + 4q, \qquad L'_{\lambda\nu} = q^{4-2}(2/q + 4/q^2) = 4 + 2q,$$

and thus $L'_{\lambda\mu}(q) \geq 2^{3-3} L'_{\lambda\nu}(q)$.

Another property of the Kostka-Foulkes polynomials case that seems to hold in our case refers to the growth of the polynomials L. For the Kostka-Foulkes polynomials the conjecture is due to Gupta (see [1] and references therein).

Conjecture 9. *If r is an integer that is not a part in either partitions λ or μ, then*

$$L_{\lambda+(r),\mu+(r)}(q) \geq L_{\lambda\mu}(q).$$

The case where $r > \lambda_1$ (which also ensures that $r > \mu_1$) is obviously true since $L_{(r,\lambda),(r,\mu)}(q) = L_{\lambda\mu}(q)$ (see Corollary 5).

Example 4. *Let $\lambda = (5,3), \mu = (4,3,1)$ and $r = 2$. We have*

$$L_{(5,3,2),(4,3,2,1)}(q) - L_{(5,3),(4,3,1)}(q) = 2q + 4q^2 + 8q^3 - (2q + 4q^2) = 8q^3.$$

The polynomials $L_{\lambda\mu}(q)$ have a similar interpretation to property 9 using an analog of the q-Kostant partition function. Using the formal inversion from [1], equation (12) may be written as

$$q_\lambda[X] = \prod_{i<j} \left(\frac{1-R_{ij}}{1+R_{ij}}\right)^{-1} Q_\lambda[X]. \tag{28}$$

In fact if we let $\zeta_n := \prod_{i<j} \left(\frac{1-x_i/x_j}{1+x_i/x_j}\right)^{-1}$, we have that $\zeta_n = \sum_{\alpha\in\mathbb{Z}^n} \mathcal{R}(\alpha)e^\alpha$ where $\mathcal{R}(\alpha) = \sum_t a_t 2^t$ and a_t counts the number of ways the vector α can be written as a sum of positive roots of type A_{n-1}, t of which are distinct. The positive roots in the root lattice of A_{n-1} are $\{e_i - e_j\}_{1\le i<j\le n}$, where $e_i = (0,\dots,1,\dots 0)$ is the canonical basis of $\mathbb{Z}^n$.
The q-analog of ζ_n is defined to be

$$\zeta_n(q) := \prod_{i<j} \left(\frac{1-qx_i/x_j}{1+qx_i/x_j}\right)^{-1},$$

and thus $\zeta_n(q) = \sum_{\alpha\in\mathbb{Z}^n} \mathcal{R}_q(\alpha)e^\alpha$ where $\mathcal{R}_q(\alpha) = \sum_{t,k} a_{t,k}2^t q^k$ and $a_{t,k}$ counts the number of ways the vector α can be written as a sum of k positive roots, t of which are distinct.

We can express the Q-Kostka polynomials in terms of $\mathcal{R}_q(\alpha)$ as

$$L_{\lambda\mu}(q) = \sum_{\alpha\colon Q_{\alpha+\mu}=\pm 2^t Q\lambda} \pm 2^t \mathcal{R}_q(\alpha).$$

It is possible to express the equation above using the action of the symmetric group on Schur's Q-functions, yielding an alternating sum similar to Property 9. Unfortunately the action of the symmetric group on Schur's Q-functions indexed by a general integer vector is not as elegant as for Schur functions (due to relation (15)).
Remark: Most of the properties of the Q-Kostka polynomials $L_{\lambda\mu}(q)$ are analogous to the Kostka-Foulkes. A few properties for the Kostka-Foulkes polynomials do not have a corresponding property for the Q-Kostka polynomials.

(1) The analog of Property 6 does not seem to hold since computations of $G_\lambda[X;q]$ where q is set to a root of unity do not factor.
(2) There does not seem to exist an elegant relationship between $G_\lambda[X;q]$ and its dual basis (Property 11).
(3) A property similar to that of Property 10 does not seem to hold. We do not know if there is a relationship between $G_\lambda[X;q]$ and a Hall-like algebra.
(4) The symmetries of the Macdonald symmetric function in Λ cannot hold in Γ and do not suggest what a two parameter analog of what these functions must be.

3.3. **Generalized (parabolic) Q-Kostka polynomials.** Shimozono and Weyman [17], defined a generalization of the Kostka-Foulkes polynomials that are a q-analog of the Littlewood-Richardson coefficients. They were originally defined as the coefficient of a Schur function in a symmetrized rational series, however it became clear in later work [18] that they can be defined as coefficients in families of symmetric functions using formulas similar to those presented here.

This construction exists in complete analogy within the Q-function algebra. We will create a family of functions in Γ which are indexed by a sequence of strict partitions. Let $\mu^* = (\mu^{(1)}, \mu^{(2)}, \dots, \mu^{(k)})$ where $\mu^{(i)}$ is a strict partition and set $\eta = (\ell(\mu^{(1)}), \ell(\mu^{(2)}), \dots, \ell(\mu^{(k)}))$. Define $Roots_\eta = \{(i,j) : 1 \le i \le \eta_1 + \cdots + \eta_r < j \le n \text{ for some } r\}$ and then define the function

$$G_{\mu^*}[X;q] = \prod_{(i,j)\in Roots_\eta} \frac{1+qR_{ij}}{1-qR_{ij}} Q_{\bar{\mu^*}}[X] \tag{29}$$

A generating function, vertex operator, and a Morris-like recurrence analogous to equations (21), (23) and (25) may be derived from this definition.

If we set $\bar{\mu^*}$ equal to the concatenation of the partitions in μ^*, then $G_{\mu^*}[X;0] = Q_{\bar{\mu^*}}[X]$ and $G_{\mu^*}[X;1] = Q_{\mu^{(1)}}[X]Q_{\mu^{(2)}}[X]\cdots Q_{\mu^{(k)}}[X]$. Define the polynomials $L_{\lambda\mu^*}(q)$ by the expansion

$$G_{\mu^*}[X;q] = \sum_\lambda L_{\lambda\mu^*}(q)Q_\lambda[X]. \tag{30}$$

Computing these coefficients suggests the following remarkable conjecture and indicates that these coefficients are an important q-analog of the structure coefficients of the $Q_\lambda[X]$ functions in the same way that the parabolic Kostka coefficients are q-analogs of the Littlewood-Richardson coefficients.

Conjecture 10. *For a sequence of partitions μ^*, if $\bar{\mu^*}$ is a partition then $L_{\lambda\mu^*}(q)$ is a polynomial in q with non-negative integer coefficients.*

4. Appendix: Tables of $2^{\ell(\lambda)-\ell(\mu)}L_{\lambda\mu}(q)$ for $n = 4,5,6,7,8,9$

$$\begin{bmatrix} (3,1) & (4) \\ 1 & q \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} (3,2) & (4,1) & (5) \\ 1 & 2q & q^2 \\ 0 & 1 & q \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} (3,2,1) & (4,2) & (5,1) & (6) \\ 1 & 2q^2+q & 2q^3+q^2 & q^4 \\ 0 & 1 & 2q & q^2 \\ 0 & 0 & 1 & q \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} (4,2,1) & (4,3) & (5,2) & (6,1) & (7) \\ 1 & q & 2q^2+q & 2q^3+q^2 & q^4 \\ 0 & 1 & 2q & 2q^2 & q^3 \\ 0 & 0 & 1 & 2q & q^2 \\ 0 & 0 & 0 & 1 & q \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} (4,3,1) & (5,2,1) & (5,3) & (6,2) & (7,1) & (8) \\ 1 & 2\,q & 2\,q^2+q & 2\,q^2+2\,q^3 & q^3+2\,q^4 & q^5 \\ 0 & 1 & q & 2\,q^2+q & 2\,q^3+q^2 & q^4 \\ 0 & 0 & 1 & 2\,q & 2\,q^2 & q^3 \\ 0 & 0 & 0 & 1 & 2\,q & q^2 \\ 0 & 0 & 0 & 0 & 1 & q \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} (4,3,2) & (5,3,1) & (5,4) & (6,2,1) & (6,3) & (7,2) & (8,1) & (9) \\ 1 & 2\,q+4\,q^2 & 2\,q^3+q^2 & 2\,q^2+4\,q^3 & q^2+2\,q^4+4\,q^3 & 4\,q^4+q^3+2\,q^5 & 2\,q^6+2\,q^5 & q^7 \\ 0 & 1 & q & 2\,q & 2\,q^2+q & 2\,q^2+2\,q^3 & q^3+2\,q^4 & q^5 \\ 0 & 0 & 1 & 0 & 2\,q & 2\,q^2 & 2\,q^3 & q^4 \\ 0 & 0 & 0 & 1 & q & 2\,q^2+q & 2\,q^3+q^2 & q^4 \\ 0 & 0 & 0 & 0 & 1 & 2\,q & 2\,q^2 & q^3 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2\,q & q^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & q \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

5. Appendix: example of conjectured tableaux poset of content $(4,3,1)$

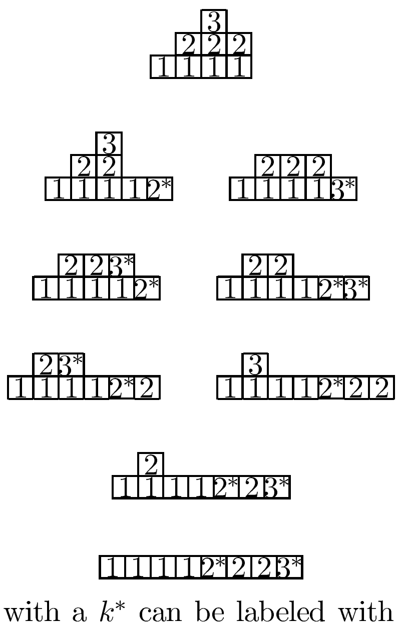

Figure 2. The cells marked with a k^* can be labeled with either k or k', we conjecture that the statistic is independent of these markings. The value of $G_{(4,3,1)}[X;q]$ determines the position of each of the shifted tableaux here except for the two of shape $(6,2)$. The covering relation is unknown, but the rank function indicates that it is not the same as the charge statistic.

Acknowledgement: Thank you to Nantel Bergeron for many helpful suggestions on this research.

References

[1] Désarménien J., Leclerc B., and Thibon J.-Y., "Hall-Littlewood Functions and Kostka-Foulkes Polynomials in Representation Theory." *Sèminaire Lotharingien de Combinatoire*, **B32c**(1994), 38 pp.

[2] Haiman M., "On Mixed Insertion, Symmetry, and Shifted Young Tableaux." *J. Comb Theory (A)*, **50**(1989) 196–225.

[3] P. Hall, "The algebra of partitions." *Proc. 4th Canadian Math. Congress, Banff*, (1959).

[4] Jing, N., "Vertex Operators, Symmetric Functions and the Spin Group Γ_n." *J. Algebra* **138**(1991), 340–398.

[5] Jing N., "Vertex Operators and Hall-Littlewood Symmetric Functions." *Advances in Math.* **87**(1991), 226–248.

[6] Foulkes H. O., "A survey of some combinatorial aspects of symmetric functions." in *Permutations.* (1974) Gauthier-Villars, Paris.

[7] Fulton W., *Young Tableaux.* Cambridge University Press, Cambridge, 1997.

[8] Garsia, A. M., "Orthogonality of Milne's polynomials and raising operators." *Discrete Math*, **99**, 247–64.

[9] Kirillov A., "Ubiquity of Kostka polynomials." *Phys. and Comb.*, *Proceedings of the Nagoya 1999 International Workshop on Physics and Combinatorics, Nagoya University*, August 23-27, (1999), 85–200.

[10] A. N. Kirillov and N. Yu. Reshetikhin, *The Bethe Ansatz and the combinatorics of Young tableaux*, J. Soviet Math. **41** (1988), 925–955.

[11] Lascoux A., Pragacz P., "Orthogonal Divided Differences and Schubert Polynomials, $\tilde{P}$-Functions and Vertex Operators." *Mich. Math. J.* **48**(2000), 417–441.

[12] Lascoux A., Schützenberger M. P., "Sur une conjecture de H. O. Foulkes." *C. R. Acad. Sci. Paris,* **286A**(1978) 323–324.

[13] Lascoux A. and Schützenberger M. P., "Le Monoid plaxique." *Noncommutative structures in Algebra and Geometric Combinatorics*, CNR, Rome. **109**(1981), 129–156.

[14] Macdonald, I. G., *Symmetric Functions and Hall Polynomials.* Cambridge University Press, Cambridge, 1995.

[15] Sagan, B. E., "Shifted Tableaux, Schur Q-functions and a Conjecture of R. Stanley." *J. Comb. Theory (A)*, **45**(1987), 62–103.

[16] Schilling A. and Warnaar O., *Inhomogeneous lattice paths, generalized Kostka polynomials and A_{n-1} supernomials*, Comm. in Math. Phys. 202, (1999), 359–401.

[17] Shimozono M., Weyman J., "Graded characters of modules supported in the closure of a nilpotent conjugacy class." *European J. Combin.* **21** no. 2(2000) 257–288.

[18] Shimozono M., Zabrocki M., "Hall-Littlewood Vertex Operators and Generalized Kostka Polynomials." *Adv. Math.* **158**(2001), 66–85.

[19] Stembridge, J. R., "Shifted Tableaux and the Projective Representations of Symmetric Groups." *Advances in Math.* **74**(1989), 87–134.

[20] Stembridge, J. R., "On Schur's Q-functions and the Primitive Idempotents of a Commutative Hecke Algebra." *J. Alg. Combinatorics* **1**(1992), 71–95.

[21] Worley, D. R., "A Theory of Shifted Young Tableaux." Thesis, MIT (1984).

School of Mathematics, University of Minnesota, Minneapolis, MN, 55455 USA
E-mail address: `tudose@math.umn.edu`

Department of Mathematics and Statistics, York University, Toronto, Ontario, M3J 1P3 CANADA
E-mail address: `zabrocki@mathstat.yorku.ca`