# ON THE AUTOMORPHISM GROUP OF THE SUBMODULE LATTICE OF A MODULE OVER COMPLETE DISCRETE VALUATION RING

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ABSTRACT. The automorphism group $\operatorname{Aut}\mathcal{L}(M)$ of the submodule lattice $\mathcal{L}(M)$ of a finite-length module $M$ over complete discrete valuation ring $\mathfrak{o}$ is studied. Let $\lambda = (\lambda_1, \cdots, \lambda_l)$ be the type of $M$. We show that for those $M$ with $\lambda_1 = \lambda_2$, $\operatorname{Aut}\mathcal{L}(M)$ can be analyzed by computing a certain subgroup of the bijections on a quotient of the scalar ring $\mathfrak{o}$. In particular, when $\mathfrak{o}$ is $\mathbb{Z}_p$ (hence $M$ is a finite abelian $p$-group) or the ring of Witt vectors over the finite field $\mathbb{F}_q$, we can compute $\operatorname{Aut}\mathcal{L}(M)$ more explicitly.

RÉSUMÉ. Étudié est le groupe $\operatorname{Aut}\mathcal{L}(M)$ des automorphismes du réseau $\mathcal{L}(M)$ des sous-modules d'un module $M$ de type fini sur un anneau $\mathfrak{o}$ de valuation discrète complet. Soit $\lambda = (\lambda_1, \cdots, \lambda_l)$ le type de $M$. Nous montrons que, pour tels $M$ que $\lambda_1 = \lambda_2$, le groupe $\operatorname{Aut}\mathcal{L}(M)$ peut être analysé par un calcul d'un certain sous-groupe du groupe des permutations dans un quotient de l'anneau $\mathfrak{o}$ scalaire. En particulier, si $\mathfrak{o}$ est $\mathbb{Z}_p$ (et, par conséquent, $M$ est un $p$-groupe abelien fini), ou l'anneau de vecteurs de Witt sur un corps $\mathbb{F}_q$ fini, nous pouvons calculer le groupe $\operatorname{Aut}\mathcal{L}(M)$ plus explicitement.

## 1. INTRODUCTION

Let $\mathfrak{o}$ be a complete discrete valuation ring with the maximal ideal $\mathfrak{p}$, a prime element $\pi$ and the valuation $v$. Let $M$ be an $\mathfrak{o}$-module of finite length. Then, since $\mathfrak{o}$ is a principal ideal domain, $M$ can be written as a sum of cyclic $\mathfrak{o}$-submodules:

$$M \cong \mathfrak{o}/\mathfrak{p}^{\lambda_1} \oplus \cdots \oplus \mathfrak{o}/\mathfrak{p}^{\lambda_l},$$

with $\lambda = (\lambda_1, \cdots, \lambda_l)$ being some partition. $\lambda$ is called the *type* of $M$. Let $\mathcal{L}(M)$ denote the set of $\mathfrak{o}$-submodules of $M$. $\mathcal{L}(M)$ inherits a lattice structure by inclusion relation. Our main objective is to compute $\operatorname{Aut}\mathcal{L}(M)$, the automorphism group of the lattice $\mathcal{L}(M)$, for $\lambda$ with $\lambda_1 = \lambda_2 \geq \lambda_3$.

When $\mathfrak{o} = \mathbb{Z}_p$, the ring of $p$-adic integers, $M$ becomes nothing but a finite abelian $p$-group and $\mathcal{L}(M)$ the subgroup lattice of $M$. This can be generalized by considering the case $\mathfrak{o} = W[\mathbb{F}_q]$, the ring of Witt vectors over the finite field $\mathbb{F}_q$, for $W[\mathbb{F}_p] \cong \mathbb{Z}_p$.

We call $e = (e_1, \cdots, e_l) \in M^l$ an *ordered basis* for $M$ if $M = \bigoplus_{i=1}^{l} \mathfrak{o}e_i$ and $\mathfrak{o}e_i \cong \mathfrak{o}/\mathfrak{p}^{\lambda_i}$. Let $e$ be fixed, and let us define a subset $\mathcal{F}(e)$ of $\mathcal{L}(M)$ by

$$\mathcal{F}(e) = \{\, \mathfrak{o}e_1, \dots, \mathfrak{o}e_l \,\} \cup \{\, \mathfrak{o}(e_1 + e_2), \dots, \mathfrak{o}(e_1 + e_l) \,\}.$$

Let $R(e)$ denote the element-wise stabilizer of $\mathcal{F}(e)$ in $\operatorname{Aut}\mathcal{L}(M)$. In most cases it boils down to computing $R(e)$ in order to analyze $\operatorname{Aut}\mathcal{L}(M)$, in the sense of the following.

Since an autormophism of $\mathfrak{o}$-module $M$ induces an automorphism of the lattice $\mathcal{L}(M)$, we have the natural group homomorphism

$$\xi : \operatorname{Aut} M \to \operatorname{Aut}\mathcal{L}(M).$$

It can be shown that $\operatorname{Ker}\xi \cong (\mathfrak{o}/\mathfrak{p}^{\lambda_1})^\times$ and that $\operatorname{Aut} M$ can be expressed in matrix form, as mentioned at the end of this section. Naturally $\operatorname{Aut}\mathcal{L}(M)$ contains a subgroup isomorphic to $\operatorname{Aut} M/\operatorname{Ker}\xi$, and we let $\operatorname{PAut} M$ denote this subgroup.

These two subgroups $R(e)$ and $\mathrm{PAut}\, M$ are closely related; the subgroup $R(e)$ is a permutable complement of the subgroup $\mathrm{PAut}\, M$ in $\mathrm{Aut}\,\mathcal{L}(M)$. Namely, we have

$$R(e) \cdot \mathrm{PAut}\, M = \mathrm{Aut}\,\mathcal{L}(M)$$
$$R(e) \cap \mathrm{PAut}\, M = 1.$$

Also, we remark that if $e$ and $e'$ are ordered base for $M$, then it is easily checked that $\varphi R(e) \varphi^{-1} = R(e')$, where $\varphi \in \mathrm{PAut}\, M$ is the lattice automorphism induced by the module automorphism of $M$ defined by $e_i \mapsto e'_i$ $(1 \leq i \leq l)$. Hence the isomorphism type of $R(e)$ does not depend on the choice of $e$. We content ourselves with computing $R(e)$ instead of computing $\mathrm{Aut}\,\mathcal{L}(M)$ for our purpose.

Let us mention the relation with earlier results. We consider the case when the residue field of $\mathfrak{o}$ is the finite field $\mathbb{F}_p$. Let $M = \mathfrak{o}/\mathfrak{p} \oplus \mathfrak{o}/\mathfrak{p} \cong \mathbb{F}_p \oplus \mathbb{F}_p$. Then $\mathrm{Aut}\,\mathcal{L}(M)$ is isomorphic to the symmetric group $\mathfrak{S}_{p+1}$ and $\mathrm{PAut}\, M$ isomorphic to the projective general linear group $PGL(2,p)$ (Note that $|PGL(2,p)| = (p+1)p(p-1)$). In this case, $R(e)$ is a subgroup that fixes three points and isomorphic to $\mathfrak{S}_{p-2}$.

More generally, for $M = \mathbb{Z}_p/p^{\lambda_2}\mathbb{Z}_p \oplus \mathbb{Z}_p/p^{\lambda_2}\mathbb{Z}_p$ ($\mathfrak{o} = \mathbb{Z}_p$ is the ring of $p$-adic integers), Holmes' result [5] states that $\mathrm{Aut}\,\mathcal{L}(M)$ is isormorphic to $\mathfrak{S}_p^{\wr(\lambda_2-1)} \wr \mathfrak{S}_{p+1}$, where $\mathfrak{S}_p^{\wr n}$ means $\mathfrak{S}_p \wr \cdots \wr \mathfrak{S}_p$ ($n$ times). In this case, $\mathrm{PAut}\, M$ is nothing but $PGL_2(\mathbb{Z}_p/p^{\lambda_2}\mathbb{Z}_p)$, and we note that $|PGL_2(\mathbb{Z}_p/p^{\lambda_2}\mathbb{Z}_p)| = (p+1)p(p-1) \cdot (p^{\lambda_2-1})^3$. $R(e)$ is the subgroup that fixes three points $\mathbb{Z}_p(1,0)$, $\mathbb{Z}_p(0,1)$ and $\mathbb{Z}_p(1,1)$; it is indeed isomorphic to

$$(\mathfrak{S}_p^{\wr(\lambda_2-1)} \wr \mathfrak{S}_{p-2}) \times \left\{ \prod_{i=0}^{\lambda_2-2} (\mathfrak{S}_p^{\wr i} \wr \mathfrak{S}_{p-1}) \right\}^3 .$$

Another example is the case when $\lambda_1 = \lambda_2 = \lambda_3$, which is essentially the well-known result of Baer [2]. In this case, $R(e)$ becomes isomorphic to the group $\mathrm{Aut}\,\overline{\mathfrak{o}}$ of ring automorphisms, where $\overline{\mathfrak{o}} = \mathfrak{o}/\mathfrak{p}^{\lambda_2}$. More specifically, we have

$$\mathrm{Aut}\,\overline{\mathfrak{o}} \ltimes \mathrm{PAut}\, M \cong \mathrm{Aut}\,\mathcal{L}(M).$$

In particular, when $\lambda_1 = \cdots = \lambda_l = 1$ $(l \geq 3)$, $M$ becomes a vector space over the residue filed $k$ of $\mathfrak{o}$, and $\mathrm{Aut}\,\mathcal{L}(M)$ is isomorphic to $P\Gamma L(l,k)$, the group of projective semi-linear automorphisms. This result is a variation of so called *the Fundamental Theorem of Finite Projective Geometry.*

There have been works to bridge the gap between Holmes' result and Baer's. Costantini-Holmes-Zacher[3] and Costantini-Zacher[4] treated the case of abelian groups in a rather general framework. Yasuda[11] studied the case of abelian groups for $\lambda_1 > \lambda_2 = \lambda_3$ with explicit computation of $R(e)$ and $\mathrm{Aut}\,\mathcal{L}(M)$. In this work, we shall treat the case $\lambda_1 = \lambda_2 \geq \lambda_3$, in the general setting of finite-length modules over complete valuation ring.

We end this section with the description of the automorphism group $\mathrm{Aut}\, M$ of an $\mathfrak{o}$-module $M$. Let $e$ be fixed. The action of $f \in \mathrm{Aut}\, M$ is then determined by its action on $e = (e_1, \cdots, e_l)$. Write

$$f(e_j) = \sum_{i=1}^{l} a_{ij} e_i$$

and express $f$ as the matrix $(a_{ij})_{i,j=1}^{l}$. Rewriting $\lambda = (\lambda_1, \ldots, \lambda_l) = \langle {d_1}^{m_1}, \ldots, {d_r}^{m_r} \rangle$ $(d_1 > \cdots > d_r)$, $\mathrm{Aut}\, M$ can be expressed in matrix form as

$$\begin{pmatrix} GL_{m_1}(\mathfrak{o}/\mathfrak{p}^{d_1}) & \cdots & \mathrm{Hom}((\mathfrak{o}/\mathfrak{p}^{d_r})^{\oplus m_r}, (\mathfrak{o}/\mathfrak{p}^{d_1})^{\oplus m_1}) \\ \vdots & \ddots & \vdots \\ \mathrm{Hom}((\mathfrak{o}/\mathfrak{p}^{d_1})^{\oplus m_1}, (\mathfrak{o}/\mathfrak{p}^{d_r})^{\oplus m_r}) & \cdots & GL_{m_r}(\mathfrak{o}/\mathfrak{p}^{d_r}) \end{pmatrix},$$

with respect to the ordered basis $e$. Here, the block matrix in the diagonal

$$A \in GL_{m_i}(\mathfrak{o}/\mathfrak{p}^{d_i})$$

is of size $m_i \times m_i$ and has elements of $\mathfrak{o}/\mathfrak{p}^{d_i}$ in its components, satisfying $\pi \nmid \det A$. Also, the block matrix at $(i,j)$-position $(i \neq j)$

$$A \in \mathrm{Hom}((\mathfrak{o}/\mathfrak{p}^{d_j})^{\oplus m_j}, (\mathfrak{o}/\mathfrak{p}^{d_i})^{\oplus m_i})$$

is of size $m_i \times m_j$ and in its components has elements of $\mathfrak{p}^{d_i - \min(d_j, d_i)}(\mathfrak{o}/\mathfrak{p}^{d_i})$, that is, for $i < j$ $(\Longrightarrow\ d_i > d_j)$ elements of $\mathfrak{p}^{d_i - d_j}(\mathfrak{o}/\mathfrak{p}^{d_i})$, and for $i > j$ $(\Longrightarrow\ d_i < d_j)$ elements of $\mathfrak{o}/\mathfrak{p}^{d_i}$.

## 2. Results

### 2.1. $\mathfrak{u}$-crossed automorphisms.

**Lemma 1.** *For $\varphi \in R(e)$, there exists a unique bijective map $\sigma : \overline{\mathfrak{o}} \to \overline{\mathfrak{o}}$ satisfying*

$$\varphi(\mathfrak{o}(e_1 + ae_2)) = \mathfrak{o}(e_1 + \sigma(a)e_2).$$

Note that since $\lambda_1 = \lambda_2$, $(e_2, e_1, e_3, \cdots, e_l)$ is also an ordered basis for $M$. Then with this lemma we also see that there exists a bijection $\tau \in \mathfrak{S}(\overline{\mathfrak{o}})$ such that $\varphi(\mathfrak{o}(ae_1 + e_2)) = \mathfrak{o}(\tau(a)e_1 + e_2)$ for all $a \in \overline{\mathfrak{o}}$. Thus we have obtained the map $R(e) \to \mathfrak{S}(\overline{\mathfrak{o}}) \times \mathfrak{S}(\overline{\mathfrak{o}})$ $(\varphi \mapsto (\tau, \sigma))$.

Put $\overline{\mathfrak{p}} = \mathfrak{p}\overline{\mathfrak{o}} \subset \overline{\mathfrak{o}}$. Let $\mathfrak{u}$ denote the kernel of the natural group homomorphism $\overline{\mathfrak{o}}^{\times} \to (\mathfrak{o}/\mathfrak{p}^{\lambda_3})^{\times}$. In other words,

$$\mathfrak{u} = 1 + \overline{\mathfrak{p}}^{\lambda_3} \subset \overline{\mathfrak{o}}^{\times}.$$

For $a, b \in \overline{\mathfrak{o}}$, we say that $a$ and $b$ are $\mathfrak{u}$-*similar* and write

$$a \overset{\mathfrak{u}}{\sim} b$$

if $a \in \mathfrak{u}b$. It is easily seen that $\overset{\mathfrak{u}}{\sim}$ defines an equivalence relation on $\overline{\mathfrak{o}}$. We call the bijection $\sigma \in \mathfrak{S}(\overline{\mathfrak{o}})$ a $\mathfrak{u}$-*crossed automorphism* of $\overline{\mathfrak{o}}$ if $\sigma$ satisfies the following three conditions:

(1) $\sigma(\overline{\mathfrak{p}}) \subset \overline{\mathfrak{p}}$,
(2) $\sigma(a - b) \overset{\mathfrak{u}}{\sim} \sigma(a) - \sigma(b)$ for all $a, b \in \overline{\mathfrak{o}}$, and
(3) $\sigma(ab) \overset{\mathfrak{u}}{\sim} \sigma(a)\sigma(b)$ for all $a, b \in \overline{\mathfrak{o}}$.

Clearly a ring automorphism $\sigma : \overline{\mathfrak{o}} \to \overline{\mathfrak{o}}$ is a $\mathfrak{u}$-crossed automorphism.

**Lemma 2.** *Let $\varphi \in R(e)$ and $\sigma \in \mathfrak{S}(\overline{\mathfrak{o}})$ the derived map from $\varphi$ as in Lemma 1. Then $\sigma$ is a $\mathfrak{u}$-crossed automorphism.*

We list some basic properties of $\mathfrak{u}$-crossed automorphisms.

**Lemma 3.** *Let $\sigma : \overline{\mathfrak{o}} \to \overline{\mathfrak{o}}$ be a $\mathfrak{u}$-crossed automorphism. Then*

(1) *$\sigma(\overline{\mathfrak{p}}^i) = \overline{\mathfrak{p}}^i$ for all $i \geq 0$. That is, we have $v(\sigma(a)) = v(a)$ for all $a \in \overline{\mathfrak{o}}$.*
(2) *For all $a, b \in \overline{\mathfrak{o}}$ and $i \geq 0$, we have $a \equiv b \mod \overline{\mathfrak{p}}^i$ if and only if $\sigma(a) \equiv \sigma(b) \mod \overline{\mathfrak{p}}^i$. That is, $v(a - b) = v(\sigma(a) - \sigma(b))$ for all $a, b \in \overline{\mathfrak{o}}$.*
(3) *$\sigma(1) \in \mathfrak{u}$. That is, $\sigma(1) \overset{\mathfrak{u}}{\sim} 1$.*
(4) *For all $a, b \in \overline{\mathfrak{o}}$, we have $a \overset{\mathfrak{u}}{\sim} b$ if and only if $\sigma(a) \overset{\mathfrak{u}}{\sim} \sigma(b)$. That is, $\sigma(\mathfrak{u}a) = \mathfrak{u}\sigma(a)$ for all $a \in \overline{\mathfrak{o}}$.*

With this lemma, we see that modulo $\mathfrak{p}^i$ $(i \leq \lambda_2)$ reduction of $\sigma$ induces a $\mathfrak{u}$-crossed automorphism $\mathfrak{o}/\mathfrak{p}^i \to \mathfrak{o}/\mathfrak{p}^i$. In particular, modulo $\mathfrak{p}^{\lambda_3}$ reduction induces a ring automorphism $\mathfrak{o}/\mathfrak{p}^{\lambda_3} \to \mathfrak{o}/\mathfrak{p}^{\lambda_3}$.

Let $\mathrm{Aut}_{\mathfrak{u}}\overline{\mathfrak{o}}$ denote the set of $\mathfrak{u}$-crossed automorphisms of $\overline{\mathfrak{o}}$. Then

**Lemma 4.** $\mathrm{Aut}_{\mathfrak{u}}\,\overline{\mathfrak{o}}$ *forms a subgroup of* $\mathfrak{S}(\overline{\mathfrak{o}})$.

Since a ring automorphism is a $\mathfrak{u}$-crossed automorphism, $\mathrm{Aut}_{\mathfrak{u}}\,\overline{\mathfrak{o}}$ contains the subgroup $\mathrm{Aut}\,\overline{\mathfrak{o}}$ of ring automorphisms. Also, modulo $\mathfrak{p}^{\lambda_3}$ reduction gives us the group homomorphism $\mathrm{Aut}_{\mathfrak{u}}\,\overline{\mathfrak{o}} \to \mathrm{Aut}\,\mathfrak{o}/\mathfrak{p}^{\lambda_3}$.

Thus far we have obtained the group homomorphism $R(e) \to \mathrm{Aut}_{\mathfrak{u}}\,\overline{\mathfrak{o}} \times \mathrm{Aut}_{\mathfrak{u}}\,\overline{\mathfrak{o}}$ ($\varphi \mapsto (\tau, \sigma)$). Next we see that the $\mathfrak{u}$-crossed automorphisms $\tau$ and $\sigma$ derived from $\varphi$ are related as in:

**Lemma 5.** *Let* $\varphi \in R(e)$ *and* $(\tau, \sigma) \in (\mathrm{Aut}_{\mathfrak{u}}\,\overline{\mathfrak{o}})^2$ *be derived from* $\varphi$. *Then*
(1) $\tau(a)^{-1} = \sigma(a^{-1})$ *for all* $a \in \overline{\mathfrak{o}} \setminus \overline{\mathfrak{p}}$,
(2) $\tau(a) \overset{\mathfrak{u}}{\sim} \sigma(a)$ *for all* $a \in \overline{\mathfrak{p}}$.

Now let $\mathrm{Aut}_{\mathfrak{u}}\,\overline{\mathfrak{o}}_1$ denote the stabilizer of 1, i.e., $\mathrm{Aut}_{\mathfrak{u}}\,\overline{\mathfrak{o}}_1 = \{\sigma \in \mathrm{Aut}_{\mathfrak{u}}\,\overline{\mathfrak{o}} \mid \sigma(1) = 1\}$. Also, define $\tilde{\Delta}(\mathrm{Aut}_{\mathfrak{u}}\,\overline{\mathfrak{o}}_1)^2$ to be the set of $(\tau, \sigma) \in (\mathrm{Aut}_{\mathfrak{u}}\,\overline{\mathfrak{o}}_1)^2$ with the properties of Lemma 5. Then $\tilde{\Delta}(\mathrm{Aut}_{\mathfrak{u}}\,\overline{\mathfrak{o}}_1)^2$ forms a subgroup of $(\mathrm{Aut}_{\mathfrak{u}}\,\overline{\mathfrak{o}}_1)^2$. So we have the group homomorphism $R(e) \to \tilde{\Delta}(\mathrm{Aut}_{\mathfrak{u}}\,\overline{\mathfrak{o}}_1)^2$.

Let $\sigma \in \mathrm{Aut}\,\overline{\mathfrak{o}}$. Then $\varphi : \mathcal{L}(M) \to \mathcal{L}(M)$ defined by $\varphi(\mathfrak{o}(\sum_{i=1}^{l} a_i e_i)) = \mathfrak{o}(\sum_{i=1}^{l} \sigma(a_i) e_i)$ is an element of $R(e)$. Thus $R(e)$ contains a subgroup isomorphic to $\mathrm{Aut}\,\overline{\mathfrak{o}}$. Also, $\tilde{\Delta}(\mathrm{Aut}_{\mathfrak{u}}\,\overline{\mathfrak{o}}_1)^2$ contains the subgroup $\Delta(\mathrm{Aut}\,\overline{\mathfrak{o}})^2 = \{(\sigma, \sigma) \in \mathrm{Aut}\,\overline{\mathfrak{o}} \times \mathrm{Aut}\,\overline{\mathfrak{o}}\}$. The diagonal map $\Delta : \mathrm{Aut}\,\overline{\mathfrak{o}} \to \Delta(\mathrm{Aut}\,\overline{\mathfrak{o}})^2$ is compatible with the homomorphism $R(e) \to \tilde{\Delta}(\mathrm{Aut}_{\mathfrak{u}}\,\overline{\mathfrak{o}}_1)^2$.

### 2.2. Main isomorphism theorem.

We now state our main result:

**Theorem 1** (Main isomorphism theorem)**.** *We have the isomorphism of groups*

$$R(e) \cong \tilde{\Delta}(\mathrm{Aut}_{\mathfrak{u}}\,\overline{\mathfrak{o}}_1)^2.$$

One direction of the isomorphism

$$\eta : R(e) \to \tilde{\Delta}(\mathrm{Aut}_{\mathfrak{u}}\,\overline{\mathfrak{o}}_1)^2$$

is already described in the previous subsection. To construct the isomorphism in the other direction

$$\zeta : \tilde{\Delta}(\mathrm{Aut}_{\mathfrak{u}}\,\overline{\mathfrak{o}}_1)^2 \to R(e),$$

we need to divide the map in two stages. Since every bijective map $f \in \mathfrak{S}(M)$ induces a bijective map $\varphi : 2^M \to 2^M$, we can define $\mathfrak{S}_{R(e)}(M)$ to be the set of bijections $f \in \mathfrak{S}(M)$ such that the induced map $\varphi$ satisfies $\varphi(\mathcal{L}(M)) \subset \mathcal{L}(M)$ and the lattice automorphism $\varphi\,|_{\mathcal{L}(M)}: \mathcal{L}(M) \to \mathcal{L}(M)$ is an element of $R(e)$. Then $\mathfrak{S}_{R(e)}(M)$ is a subgroup of $\mathfrak{S}(M)$, and we obtain the group homomorphism

$$\iota : \mathfrak{S}_{R(e)}(M) \to R(e).$$

The other half of the map

$$\zeta' : \tilde{\Delta}(\mathrm{Aut}_{\mathfrak{u}}\,\overline{\mathfrak{o}}_1)^2 \to \mathfrak{S}_{R(e)}(M)$$

is constructed as follows. We partition $M$ into two disjoint sets, by defining

$$M_e^{\leq} = \left\{ \sum_{i=1}^{l} a_i e_i \in M \; ; \; v(a_1) \leq v(a_2) \right\}, \text{ and}$$

$$M_e^{>} = \left\{ \sum_{i=1}^{l} a_i e_i \in M \; ; \; v(a_1) > v(a_2) \right\},$$

so that $M = M_e^{\leq} \sqcup M_e^{>}$. Given $(\tau,\sigma) \in \tilde{\Delta}(\mathrm{Aut}_{\mathfrak{u}}\, \bar{\mathfrak{o}}_1)^2$, we shall define the map $f \in \mathfrak{S}_{R(e)}(M)$ on $M_e^{\leq}$ and on $M_e^{>}$ separately. For $a \in M_e^{\leq}$, write $a = a_1e_1 + a_1\tilde{a}_2e_2 + \sum_{i=3}^{l} a_ie_i$ and define

$$f(a) = \tau(a_1)e_1 + \tau(a_1)\sigma(\tilde{a}_2)e_2 + \sum_{i=3}^{l} \tau(a_i)e_i.$$

Similarly, for $a \in M_e^{>}$, write $a = \tilde{a}_1a_2e_1 + a_2e_2 + \sum_{i=3}^{l} a_ie_i$ and define

$$f(a) = \tau(\tilde{a}_1)\sigma(a_2)e_1 + \sigma(a_2)e_2 + \sum_{i=3}^{l} \sigma(a_i)e_i.$$

**Lemma 6.** *The map $f : M \to M$ above is well-defined and bijective.*

Thus we have the group homomorphism $\tilde{\Delta}(\mathrm{Aut}_{\mathfrak{u}}\, \bar{\mathfrak{o}}_1)^2 \to \mathfrak{S}(M)$. To prove that the map $\varphi : 2^M \to 2^M$ induced by $f$ satisfies $\varphi(\mathcal{L}(M)) \subset \mathcal{L}(M)$ and that $\varphi\mid_{\mathcal{L}(M)}: \mathcal{L}(M) \to \mathcal{L}(M)$ is a lattice automorphism, we make a use of the following lemma, which is a slightly modified variation of a well-known principle in the theory of subgroup lattices (cf. [7]).

**Lemma 7.** *Let $R$ be a unitary commutative ring, $M$ an $R$-module, and $G$ a subgroup of the group $\mathfrak{S}(M)$ of all bijections on $M$. If every $f \in G$ satisfies*

$$f(Ra) \subset Rf(a)$$
$$f(a+b) \in Rf(a) + Rf(b)$$

*for all $a, b \in M$, then every $f \in G$ induces a lattice automorphism on $\mathcal{L}(M)$.*

Also, we have

**Lemma 8.** *The map $f : M \to M$ defined by $(\tau,\sigma) \in \tilde{\Delta}(\mathrm{Aut}_{\mathfrak{u}}\, \bar{\mathfrak{o}}_1)^2$ satisfies the conditions of Lemma 7 and hence induces a lattice automorphism $\varphi : \mathcal{L}(M) \to \mathcal{L}(M)$. Moreover, thus induced $\varphi$ is indeed an element of $R(e)$.*

Hence we have obtained the group homomorphism $\zeta' : \tilde{\Delta}(\mathrm{Aut}_{\mathfrak{u}}\, \bar{\mathfrak{o}}_1)^2 \to \mathfrak{S}_{R(e)}(M)$ and the series of group homomorphisms

$$\iota \circ \zeta' : \tilde{\Delta}(\mathrm{Aut}_{\mathfrak{u}}\, \bar{\mathfrak{o}}_1)^2 \to \mathfrak{S}_{R(e)}(M) \to R(e),$$

which by letting $\zeta = \iota \circ \zeta'$ gives us the desired isomorphism $\zeta : \tilde{\Delta}(\mathrm{Aut}_{\mathfrak{u}}\, \bar{\mathfrak{o}}_1)^2 \to R(e)$.

To show that $\eta$ and $\zeta$ are indeed isomorphisms, we prove that $\eta \circ \zeta = 1$ and $\zeta \circ \eta = 1$. The fact $\eta \circ \zeta = 1$ is almost clear from the constructions. To see $\zeta \circ \eta = 1$, we use the

**Lemma 9.** *Let $\varphi \in R(e)$ and $(\tau,\sigma) \in \tilde{\Delta}(\mathrm{Aut}_{\mathfrak{u}}\, \bar{\mathfrak{o}}_1)^2$ be derived from $\varphi$. Then we have*

(1) $\varphi(\mathfrak{o}(e_1 + a_2e_2 + a_3e_3 + \cdots + a_le_l)) = \mathfrak{o}(e_1 + \sigma(a_2)e_2 + \sigma(a_3)e_3 + \cdots + \sigma(a_l)e_l)$.
(2) $\varphi(\mathfrak{o}(a_2e_2 + a_3e_3 + \cdots + a_le_l)) = \mathfrak{o}(\sigma(a_2)e_2 + \sigma(a_3)e_3 + \cdots + \sigma(a_l)e_l)$.

Using this lemma, we can show that the action of $\zeta \circ \eta(\varphi)$ on $\mathcal{L}(M)$ is actually the same as that of $\varphi$, thus showing $\zeta \circ \eta = 1$.

2.3. **The case of the ring of Witt vectors over $\mathbb{F}_q$.** In this section we apply our main result to the case $\mathfrak{o} = W[\mathbb{F}_q]$, the ring of Witt vectors over the finite field $\mathbb{F}_q$. Let $q = p^r$. Recall that we have the group homomorphism

$$\rho : {\mathrm{Aut}_{\mathfrak{u}}\, W[\mathbb{F}_q]/(p)^{\lambda_2}}_1 \to \mathrm{Aut}\, W[\mathbb{F}_q]/(p)^{\lambda_3}.$$

Since we have $\mathrm{Aut}\, W[\mathbb{F}_q]/(p)^{\lambda_3} \cong \mathbb{Z}/r\mathbb{Z} \cong \mathrm{Aut}\, W[\mathbb{F}_q]/(p)^{\lambda_2}$, the exact sequence

$$1 \to \mathrm{Ker}\, \rho \to {\mathrm{Aut}_{\mathfrak{u}}\, W[\mathbb{F}_q]/(p)^{\lambda_2}}_1 \to \mathrm{Aut}\, W[\mathbb{F}_q]/(p)^{\lambda_3} \to 1$$

splits, and we have
$$\mathrm{Aut_u}\, W[\mathbb{F}_q]/(p)^{\lambda_2}{}_1 \cong \mathbb{Z}/r\mathbb{Z} \ltimes \operatorname{Ker}\rho.$$
So it remains to compute $\operatorname{Ker}\rho$. When $\lambda_1(=\lambda_2) \leq 2\lambda_3 + 1 \geq 3$, $\operatorname{Ker}\rho$ becomes a (finite) abelian $p$-group and indeed can be calculated explicitly. When $\lambda_1 \geq 2\lambda_3$, $\operatorname{Ker}\rho$ is not necessarily abelian, and explicit computation of $\operatorname{Ker}\rho$ (and of $R(e)$) becomes more complicated. The author has obtained two different results in this case, using different approach, and is currently in the stage of determining which one should be the correct result.

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