Chain Lengths in the Dominance Lattice

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Abstract

We find the size of the largest union of two chains in the lattice of partitions of n under dominance order. We also present some partial results and conjectures on chains and antichains in this lattice.

Nous trouvons la taille de la plus grande union de deux chaînes dans le treillis des partitions de n sous l'ordre partiel dominant. Nous présentons aussi des résultats partiels et des conjectures sur les chaînes et antichaînes de ce treillis.

1 Introduction

Let P_n denote the poset of partitions of the positive integer n, ordered by dominance (aka majorization), i.e. $\lambda \leq \mu$ if $\lambda_1+\lambda_2\cdots+\lambda_k \leq \mu_1+\mu_2+\cdots+\mu_k$ for all k. This poset is a lattice, and is self-dual under conjugation. P_n is not graded for $n \geq 7$, since there exist saturated chains from $\{n\}$ to $\{1^n\}$ of all lengths from $2n-3$ to $cn^{3/2}$ [2, 5].

Given any poset P, there exists a partition $\lambda(P)$ such that the sum of the first k parts of λ is the maximal number of elements in a union of k chains in P. In fact, the conjugate of λ has the same property with chains replaced by antichains [1, 3, 4]. Let $\lambda_k(P)$ denote the kth part of this partition.

The length $h(P_n)$ of the longest chain in P_n has been known for some time [5]. If $n = \binom{m+1}{2} + r$, $0 \leq r \leq m$, then $h(P_n) = \frac{m^3-m}{3} + rm$. In other words, $\lambda_1(P_n) = \frac{m^3-m}{3} + rm + 1$. Our main result is the following theorem.

Theorem 1. *For $n > 16$, $\lambda_2(P_n) = \lambda_1(P_n) - 6$.*

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Consider the subposet Q_n of P_n consisting of the partitions that appear in chains of length $h(P_n)$. Clearly Q_n is self-dual under conjugation, since conjugation takes a decreasing chain to an increasing chain of the same length. It seems likely that Q_n is a graded lattice, but for our purposes it will suffice to use a weaker statement, namely: for $\lambda \in Q_n$, define $r(\lambda)$ to be the length of the longest chain from $\{n\}$ to λ; then $\lambda \neq \{1^n\}, \{n\}$ is covered by an element μ such that $r(\mu) = r(\lambda) - 1$ and covers an element ν such that $r(\nu) = r(\lambda) + 1$. In other words, every element of Q_n is on a fixed level. Figure 1 shows an example of a poset Q with this property that is not graded. Note that the top element is level 0, and the levels increase as we move down.

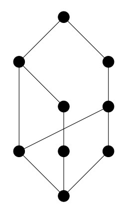

Figure 1: A non-graded poset with well-defined levels.

The covering relation in P_n comes in two flavors. Following the methods of [5], we represent Ferrers diagrams with vertical parts, as illustrated in Figure 2. We say λ covers μ by an H-step if there exists i such that $\mu_i = \lambda_i - 1$, $\mu_{i+1} = \lambda_{i+1} + 1$, and $\mu_k = \lambda_k$ for $k \neq i, i+1$. In terms of Ferrers diagrams, this corresponds to moving a box horizontally one space to the right (and down some distance). The other flavor is a V-step, which is an H-step on the conjugate, and corresponds to moving a box vertically one space down (and right some distance). Chains from $\{n\}$ to $\{1^n\}$ consisting of H-steps followed by V-steps are maximal.

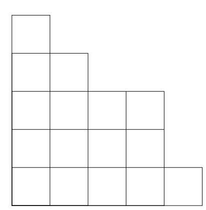

Figure 2: The partition $\{5, 4, 3, 3, 1\}$.

2 Down to work

The cases where $n \leq 16$ will be handled separately, so for now assume $n > 16$.

We will prove Theorem 1 by showing that there exist two disjoint chains in Q_n of lengths $h(P_n)$ and $h(P_n)-6$. Since Q_n is a subposet of P_n, these are also chains in P_n. Since there are six elements of P_n in saturated antichains of size 1, this is clearly the maximum possible number of elements in two chains, thus giving $\lambda_2(P_n)$ exactly.

To that end, we seek two disjoint chains in Q_n from $\{n-2,1,1\}$ and $\{n-3,3\}$ to $\{2,2,2,1^{n-6}\}$ and $\{3,1^{n-3}\}$. Let Q_n^* denote Q_n without the top three and bottom three elements.

Lemma 1. *If Q_n^* has at least two elements on every level, then it has two disjoint chains of maximal length.*

Proof: Clearly we can start two chains with the two elements in the top level, so proceed by induction. The only potential problem is if we reach two elements on level k that both cover only one and the same element on level $k+1$. In that case, take a second element on level $k+1$ and a maximal chain ending at it. This chain has a lowest point of intersection with one of the two old chains, so just replace that old chain with the new one from that point on. See Figure 3. □

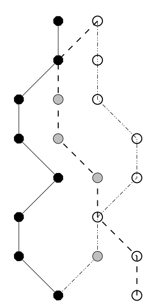

Figure 3: Salvaging a dead end.

Since Q_n^* is self-dual, it will suffice to show that the first half of its levels have at least two elements. We do this by explicitly constructing two disjoint chains to the halfway point. As a first approximation of these chains, take the following construction.

The left chain starts at $\{n-2,1,1\}$. At every step, we take the right-most possible H-step, e.g. the next partition is $\{n-3,2,1\}$. The right chain starts at $\{n-3,3\}$. At every step, we take the left-most possible H-step, e.g. the next partition is $\{n-4,4\}$. The names come from the relative positions of the chains when plotted, as in Figure 4. Both chains will eventually reach $\{m,m-1,\ldots,r+1,r,r,r-1,\ldots,2,1\}$, which is at least the halfway point [5], so the idea is to modify the left chain as little as possible to make it reach the halfway point without intersecting the right chain.

Once we've done that, we can apply Lemma 1 to get two disjoint chains of length $h(P_n) - 6$, then append the top and bottom three elements to one of them two get the desired chains. The following proposition will be used to prove several lemmas concerning the right chain.

Proposition 1. *If $\lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_k\}$ is in the right chain, then $\lambda_i - \lambda_{i+1} \leq 2$ for $i = 1, 2, \ldots, k-2$. In other words, only the last difference can be greater than 2. Moreover, excluding the last difference, λ cannot have more than one difference equal to 2.*

Proof: By construction, we are always doing the left-most possible H-step. At first there is nothing to prove, since $k = 2$ through $\{\frac{n}{2}, \frac{n}{2}\}$ or $\{\frac{n+1}{2}, \frac{n-1}{2}\}$. Think in terms of partition diagrams as in the definition of H-steps. If there are no differences greater than 1 (excluding the last one), then push one box from λ_{k-1} to increase the last part (or from λ_k increase the number of parts). Now move to the left, pushing one box at a time until $\lambda_i - \lambda_{i+1} < 2$ for $i = 1, 2, \ldots, k-2$ again. Clearly we never get a difference greater than 2 or more than one difference of 2 unless we had one before, so the result follows by induction. $\square$

3 Proof of Theorem 1

The proof comes in six cases, depending on r. We begin with general calculations that will be used in multiple cases. If $\lambda = \{\lambda_1, \lambda_2, \lambda_3, \ldots\}$ is reachable from $\{n\}$ by only H-steps, such as the elements of the left and right chains, then $r(\lambda) = \lambda_2 + 2\lambda_3 + 3\lambda_4 + \cdots$, since each box in λ_i had to be moved horizontally $i-1$ times.

Note that any λ in the left chain with $\lambda_1 - \lambda_2 > 2$ is not in the right chain by Proposition 1. This means that the left chain makes it safely to the partition $\{m+r, m-1, m-2, \ldots, 2, 1\}$ at level $\frac{m^3-m}{6}$ for $r \geq 2$. For $r > 2$, we can continue safely to $\{m+2, m-1, m-2, \ldots, r-2, r-2, \ldots, 2, 1\}$ (using both assertions in Proposition 1) for an additional $m(r-2) - \binom{r-2}{2}$ levels. So we're done if $2(m(r-2) - \binom{r-2}{2}) \geq rm$. For $r > 4$, this comes down to $m \geq \frac{r^2-5r+6}{r-4} = r - 1 + \frac{2}{r-4}$. For $r = 5$, this means $m \geq 6$. In fact $m = 5$ also works, since we really just needed $m(r-2) - \binom{r-2}{2} \geq \left\lfloor \frac{rm}{2} \right\rfloor$. For $r > 5$, we just need $m \geq r$ (since m must be an integer), but that's as general as possible since $r \leq m$. Thus we've established Theorem 1 when $r \geq 5$.

If $r = 4$, then the above construction gets us to one level shy of where we need to be, since we only reach $\{m+2, m-1, m-2, \ldots, 3, 2, 2, 1\}$ safely.

Since $h(P_n)$ is always even when r is even, the middle level consists of self-conjugate partitions. Note that not all self-conjugate partitions are in Q_n, but one will be if it is covered by an element of Q_n since by duality it covers the conjugate of that element. Now we simply observe that $\{m+2, m-1, m-2, \ldots, 3, 2, 2, 1\}$ covers the self-conjugate partition $\{m+2, m-1, m-2, \ldots, 3, 2, 1, 1, 1\}$. This partition cannot be in the right chain by Proposition 1 (it is also not H-reachable from $\{n\}$ [5]), so this establishes Theorem 1 when $r = 4$.

If $r = 0$, then we safely reach $\{m+1, m-2, m-2, \ldots, 2, 1\}$, one level shy again. Once again, we simply observe that this covers the self-conjugate partition $\{m+1, m-2, m-2, \ldots, 3, 1, 1, 1\}$, which is not in the right chain by Proposition 1, so this establishes Theorem 1 when $r = 0$.

The remaining cases each require a lemma to get past the shortfall in the above argument.

If $r = 1$, then we safely reach $\{m+2, m-2, m-2, m-3, \ldots, 2, 1\}$, but in fact we can go further along the left chain.

Lemma 2. *The partitions $\{m+1, m-1, m-2, \ldots, 2, 1\}$ and $\{m, m-1, \ldots, k+1, k, k, k-2, \ldots, 2, 1\}$, $5 \leq k \leq m$, do not occur in the right chain.*

Proof: If $\{m+1, m-1, m-2, \ldots, 2, 1\}$ occurred in the right chain, then it would have to be preceded by $\{m+2, m-2, m-2, \ldots, 2, 1\}$ or $\{m+1, m, m-3, \ldots, 2, 1\}$ (otherwise we couldn't have done the left-most H-step), both of which violate Proposition 1.

If $\{m, m-1, \ldots, k+1, k, k, k-2, \ldots, 2, 1\}$ occurred in the right chain, then it would have to be preceded by $\{m, m-1, \ldots, k+1, k, k, k-1, k-4, k-4, \ldots, 2, 1\}$ (note this works even for $k = m$) which violates Proposition 1 unless $k - 4 = 0$, hence the need for $k \geq 5$, or by $\{m, m-1, \ldots, k+1, k+1, k-1, k-2, \ldots, 2, 1\}$. In this case, we can recursively work our way back to $\{m+1, m-1, m-2, \ldots, 2, 1\}$, which is not in the right chain since it would have to be preceded by $\{m+2, m-2, m-2, \ldots, 2, 1\}$ or $\{m+1, m, m-3, \ldots, 2, 1\}$, both of which violate Proposition 1. $\square$

Now apply Lemma 2 to extend the left chain safely to $\{m, m-1, \ldots, 5, 5, 3, 2, 1\}$, which occurs at level $\frac{m^3+5m-24}{6}$. Since $h(P_n) = \frac{m^3+2m}{3}$, it suffices if $m^3+5m-24 \geq m^3+2m$, or $m \geq 8$. $m = 7$ also works since $h(P_{29}) = 119$ and we reach level 59. The case $m = 6$, $n = 22$ can be dealt with individually. The left chain gets to $\{6, 5, 5, 3, 2, 1\}$ at level 37, but intersects the right chain at level 38 with $\{6, 5, 4, 4, 2, 1\}$. However, the right chain reaches $\{6, 5, 4, 4, 3\}$ at level 37, which also covers the self-conjugate partition $\{5, 5, 5, 4, 3\}$, so this establishes Theorem 1 when $r = 1$.

If $r = 2$, then we safely reach $\{m+2, m-1, m-2, m-3, \ldots, 2, 1\}$, but in fact we can go further along the left chain.

Lemma 3. *The partitions* $\{m+1, m, m-2, m-3, \ldots, 2, 1\}$ *and* $\{m+1, m-1, m-2, \ldots, k+1, k, k, k-2, \ldots, 2, 1\}$, $1 \leq k \leq m-1$, *do not occur in the right chain.*

Proof: If $\{m+1, m, m-2, m-3, \ldots, 2, 1\}$ occurred in the right chain, then it would have to be preceded by $\{m+2, m-1, m-2, m-3, \ldots, 2, 1\}$, $\{m+1, m+1, m-3, m-3, \ldots, 2, 1\}$, or $\{m+1, m, m-1, m-4, m-4, \ldots, 2, 1\}$, all of which violate Proposition 1. Note that we are tacitly assuming that $m > 4$, but that's fine since $n > 16$, so $m \geq 5$.

Since $\{m+1, m-1, m-2, \ldots, k+1, k, k, k-2, \ldots, 2, 1\}$ has two differences of size 2 for $k > 2$, Proposition 1 takes care of those cases (note $k = m-1$ means the partition is $\{m+1, m-1, m-1, m-3, \ldots, 2, 1\}$). If $\{m+1, m-1, m-2, \ldots, 3, 2, 2\}$ occurred in the right chain, then it would have to be preceded by $\{m+2, m-2, m-2, \ldots, 3, 2, 2\}$ or $\{m+1, m, m-3, \ldots, 3, 2, 2\}$, both of which violate Proposition 1. $k = 1$ is similar. $\square$

Now apply Lemma 3 to extend the left chain safely to $\{m+1, m-1, m-2, \ldots, 2, 1, 1\}$, which occurs at level $\frac{m^3+5m}{6}$. Since $h(P_n) = \frac{m^3+5m}{3}$, this establishes Theorem 1 when $r = 2$.

Finally, if $r = 3$, we safely reach $\{m+2, m-1, m-2, \ldots, 2, 1, 1\}$. Now we just modify Lemma 3. Note we could also show that the right chain has no elements ending in 1,1 until it's too late, but this method is cleaner.

Lemma 4. *The partitions* $\{m+1, m, m-2, \ldots, 2, 1, 1\}$ *and* $\{m+1, m-1, m-2, \ldots, k+1, k, k, k-2, \ldots, 2, 1, 1\}$, $4 \leq k \leq m-1$ *do not occur in the right chain.*

Proof: Exactly the same as Lemma 3, since the second 1 at the end never comes into play. $\square$

Now apply Lemma 4 to extend the left chain safely to $\{m+1, m-1, m-2, \ldots, 5, 4, 4, 2, 1, 1\}$, which occurs at level $\frac{m^3+11m-18}{6}$. Since $h(P_n) = \frac{m^3+8m}{3}$, it suffices if $m^3 + 11m - 18 \geq m^3 + 8m$, or $m \geq 6$. When $m = 5$, we get to level 27, and $h(P_5) = 55$, so this case is fine as well. This establishes Theorem 1 when $r = 3$, and thus completes the proof. $\square$

4 Smaller cases and related questions

The smaller n for which $\lambda_2(P_n) = \lambda_1(P_n) - 6$ are 10, 13, 14, and 15. Figure 4 shows Q_{16}. Since there are levels of size 1 in the middle, P_{16} cannot possibly have two chains of the desired lengths.

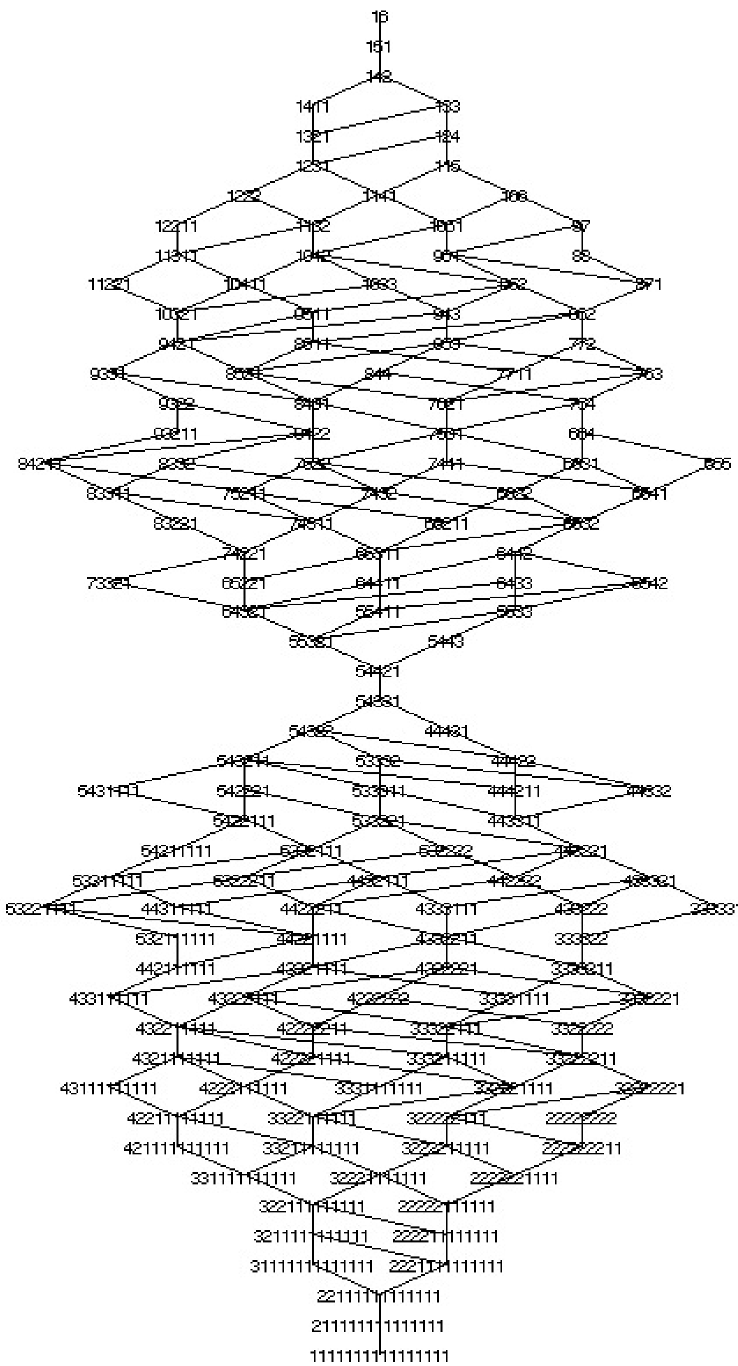

Figure 4: Elements of P_{16} on maximal chains.

More generally, Table 1 shows the partitions of chain lengths for P_n, $1 \leq n \leq 14$. It is interesting to note that in all of these cases, the elements added between $\lambda_{k-1}(P_n)$ and $\lambda_k(P_n)$ form a chain that is added to the previous $k-1$ chains (and similarly for antichains). This is not the case for arbitrary posets, such as Figure 5. The proof of Theorem 1 shows this is the case for every P_n when $k = 2$; it would be interesting to know if it holds for all k.

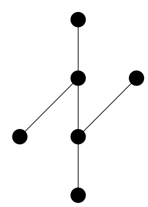

Figure 5: A poset P such that the largest chain is not one of the largest two chains. $\lambda(P) = \{4, 2\}$.

n	$\lambda(P_n)$
1	$\{1\}$
2	$\{2\}$
3	$\{3\}$
4	$\{5\}$
5	$\{7\}$
6	$\{9, 2\}$
7	$\{12, 3\}$
8	$\{15, 7\}$
9	$\{18, 9, 3\}$
10	$\{21, 15, 4, 2\}$
11	$\{25, 18, 10, 3\}$
12	$\{29, 21, 13, 10, 4\}$
13	$\{33, 27, 18, 14, 6, 3\}$
14	$\{37, 31, 24, 19, 15, 6, 3\}$

Table 1: Known values of $\lambda(P_n)$.

While the proof of Theorem 1 is constructive in the cases where $h(P_n)$ is even, so that the middle level consists of self-conjugate partitions, it is not constructive when $h(P_n)$ is odd, since in those cases the proof relies on Lemma 1. It would be interesting to give an explicit construction of two long chains in those cases. Note also that Lemma 1 does not generalize in the most obvious way for finding three chains, due to posets such as the one shown in Figure 6.

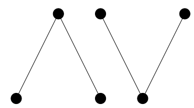

Figure 6: A graded poset with three elements on every level but no three disjoint chains of maximal length.

Conjecture 1. *For large n, $\lambda_i(P_n) - \lambda_{i+1}(P_n)$ depends only on i.*

Note that Proposition 1 holds for the left chain if we exclude the first difference instead of the last, so a partition with a difference of 3 or two differences of 2 in the middle will not be on either chain. We can try to exploit this to construct a third disjoint chain to the middle level, starting at $\{n-5, 4, 1\}$, by keeping the second difference greater than or equal to 3, and similarly for k chains by keeping a different difference large in each one. If r is even, so that we can extend the chains by conjugation, then this will give us k disjoint chains that end on the conjugates of their starting points. By an analogous calculation to the proof of Theorem 1 for $r \geq 5$, this works for getting three chains when $m \geq r - 1 + \frac{12}{r-8}$, proving that $\lambda_2(P_n) - \lambda_3(P_n) = 6$ when $r > 8$ is even and m is suffiently large. The smallest example is $n = 117$, with $m = 14$ and $r = 12$. Note that $\lambda_i(P_n) - \lambda_{i+1}(P_n)$ need not always be 6. It appears that the fourth chain starts just one level further down, so we conjecture that $\lambda_3(P_n) - \lambda_4(P_n) = 2$ for large n.

Let M be the transition matrix from the bases $\{e_\lambda\}$ to $\{m_{\lambda'}\}$ of homogeneous symmetric functions of degree n. Since $M_{\lambda\mu} > 0$ iff $\mu \leq \lambda'$, it is a theorem of Gansner and Saks that a generic matrix with the same 0 entries will have jordan blocks whose sizes are exactly the parts of $\lambda(P_n)$ (see [1]). Using Table 1 and Maple, one can verify that M is sufficiently generic at least for $n \leq 13$.

Another open problem is to find the size $a(n)$ of the largest antichain in P_n. Let $p(n)$ be the number of partitions of n. There is the obvious upper bound $a(n) \leq p(n)$. By Dilworth's theorem, $a(n) \geq p(n)/(h(P_n)+1)$, so we have $\Omega(n^{-5/2}e^{\pi\sqrt{2n/3}}) \leq a(n) \leq O(n^{-1}e^{\pi\sqrt{2n/3}})$. It would be interesting to find a constructive proof that $a(n)$ is at least as large as the lower bound. In addition to the values of $a(n)$ implied by Table 1, we can see that $a(15) = 9$. Moreover, $\lambda_9(P_{15}) = 2$, with the long antichains being $71^8, 6221^5, 541^6, 53221^3, 52^5, 4431^4, 442221, 433311, 3^5$ and their conjugates. One can also verify that $a(16) = 10$, with $\lambda_{10}(P_{16}) = 5$. The sequence of $a(n)$'s is number A076269 in [6].

One construction that shows $a(n)$ has a lower bound of the form $e^{c\sqrt{n}}$ is as follows. Begin with the antichain 7321^4, 722221, 651^5, 642211, 63322, 553111, 55222, 54421, 4444 in P_{16}. Let $\nu + 7n$ denote a partition ν from the list with $7n$ added to each part. Consider ν to have 7 parts, so some of them might be 0. Then $\{\nu + 7n, \nu + 7(n-1), \ldots, \nu + 7, \nu\}$ is a partition of $N = 16(n+1) + 49\frac{n^2+n}{2} = \frac{49}{2}n^2 + O(n)$. There are 9^{n+1} choices for the ν's, yielding an antichain of size 9^{n+1} in P_N. This yields a lower bound for $a(n)$ of $e^{c\sqrt{n}}$ where $c = \ln 9\sqrt{2/49} = 0.4439\ldots$. By starting with a 28-element antichain in P_{27} where each ν has at most 9 parts, and largest part at most 8, one can similarly get $c = \frac{\ln 28}{6} = 0.555\ldots$. This is still a long way from $\pi\sqrt{2/3} = 2.565\ldots$, but at least it's constructive.

5 Acknowledgements

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