# CLUSTER MANIFOLDS, GENERIC SYMPLECTIC LEAVES, AND CONNECTED COMPONENTS

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Abstract. We assign to any cluster algebra a geometric object equipped with a Poisson structure and called a cluster manifold. The cluster manifold is foliated into symplectic leaves of this Poisson structure. Generic leaves of this foliation form the nonsingular locus of a certain toric action on the cluster manifold. We suggest a general approach to finding the number of connected components in the union of all generic symplectic leaves, which extends our previous calculations of the number of connected components of double Bruhat cells for semisimple Lie groups. As a new application of these techniques we find the number of connected components of refined Bruhat cells in real Grassmannians.

**1. Cluster algebras of rational functions on $\mathbb{A}^n$.** Let $A$ be an arbitrary matrix, $I=\{i_1,\dots,i_m\}$, $J=\{j_1,\dots,j_n\}$ be two ordered multi-indices. We denote by $A(I;J)$ the $m\times n$ submatrix of $A$ whose entries lie in the rows $i_1,\dots,i_m$ and columns $j_1,\dots,j_n$. Instead of $A([1,m];[1,n])$ we write just $A[m;n]$ (here and in what follows we use the notation $[i,j]$ for a contiguous index set $\{i,\dots,j\}$). Given a diagonal matrix $D$ with positive integer diagonal entries $d_1,\dots,d_m$, let $\mathcal{Z}_{mn}^D$ be the set of all $m\times n$ integer matrices $Z$ such that $m\leqslant n$ and $Z[m;m]$ is $D$-skew-symmetrizable (that is, $DZ[m;m]$ is skew-symmetric); clearly, $\mathcal{Z}_{mn}^D=\mathcal{Z}_{mn}^{\lambda D}$ for any positive integer $\lambda$. According to [FZ], any $Z=(z_{ij})\in\mathcal{Z}_{mn}^D$ defines a cluster algebra of geometric type in the following way. Let us fix a set of $m$ cluster variables $f_1,\dots,f_m$, and a set of $n-m$ tropic variables $f_{m+1},\dots,f_n$. For each $i\in[1,m]$ we introduce a transformation $T_i$ of cluster variables by

$$T_i(f_j)=\bar f_j=\begin{cases}\dfrac{1}{f_i}\left(\prod\limits_{z_{ik}>0}f_k^{z_{ik}}+\prod\limits_{z_{ik}<0}f_k^{-z_{ik}}\right) & \text{for } j=i\\ f_j & \text{for } j\neq i,\end{cases}\tag{1.1}$$

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX and the corresponding matrix transformation $\bar Z = T_i(Z)$, called mutation, by

$$\bar z_{kl} = \begin{cases} -z_{kl} & \text{for } (k-i)(l-i) = 0 \\ z_{kl} + \dfrac{|z_{ki}|z_{il} + z_{ki}|z_{il}|}{2} & \text{for } (k-i)(l-i) \neq 0. \end{cases} \tag{1.2}$$

Observe that the tropic variables are not affected by $T_i$, and that $\bar Z$ belongs to $\mathcal{Z}^D_{mn}$. Thus, one can apply transformations $T_i$ to the new set of cluster variables (using the new matrix), etc. The cluster algebra (of geometric type) is the subalgebra of the field of rational functions in cluster variables $f_1, \dots, f_m$ generated by the union of all clusters; its ground ring is the ring of integer polynomials over tropic variables. We denote this algebra by $\mathcal{A}(Z)$.

One can represent $\mathcal{A}(Z)$ with the help of an $m$-regular tree $\mathbb{T}_m$ whose edges are labeled by the numbers $1, \dots, m$ so that the $m$ edges incident to each vertex receive different labels. To each vertex $v$ of $\mathbb{T}_m$ we assign a set of $m$ cluster variables $f_{v,1}, \dots, f_{v,m}$ and a set of $n-m$ tropic variables $f_{m+1}, \dots, f_n$. For an edge $(v, \bar v)$ of $\mathbb{T}_m$ that is labeled by $i \in [1, m]$, the variables $f = f_v$ and $\bar f = f_{\bar v}$ are related by the transformation $T_i$ given by (1.1). The first monomial in the right hand side of (1.1) is sometimes denoted by $M^i = M^i_v$. Transformations (1.2) then guarantee that the second monomial is $\bar M^i = M^i_{\bar v}$.

Let us say that cluster and tropic variables together form an extended cluster. Assume that the entries of the initial extended cluster are coordinate functions on $\mathbb{A}^n$. We thus get a realization of a cluster algebra of geometric type as a cluster algebra of rational functions on $\mathbb{A}^n$. It is easy to see that in this situation entries of any extended cluster are functionally independent.

*Remark.* If all entries of $Z$ belong to the set $\{0, \pm 1\}$, it is sometimes convenient to represent $Z$ by a directed graph $\mathcal{E}$ with vertices corresponding to the variables (both cluster and tropic) and with edges $i \to j$ for every pair of vertices $i, j$, such that $Z_{ij} = 1$ (in particular, there are no edges between vertices corresponding to tropic variables). If we assume, in addition, that the resulting graph has no nonoriented 3-cycles, then the graph that corresponds to $\bar Z$ differs from the one that corresponds to $Z$ as follows. All edges through $i$ change directions. Furthermore, for every two vertices $j, k$ such that edges $j \to i$ and $i \to k$ belong to the graph of $Z$, the graph of $\bar Z$ contains an edge $j \to k$ if and only if the graph of $Z$ does not contain an edge $k \to j$.

**2. The cluster manifold.** In this section we construct an algebraic variety $\mathcal{X}$ (which we call the *cluster manifold*) related to the cluster algebra $\mathcal{A}(Z)$. Our approach is suggested by considering coordinate rings of double Bruhat cells, which provide main examples of cluster algebras.

We will describe $\mathcal{X}$ by means of charts and transition functions. Assume that $\mathcal{A}(Z)$ is given by an $m$-regular tree $\mathbb{T}_m$ (see Section 1). For each vertex $v$ of $\mathbb{T}_m$ we define the chart, that is, an open subset $X = X_v \subset \mathcal{X}$ by

$$X_v = \operatorname{Spec}(\mathbb{F}[f_{v,1}, f_{v,1}^{-1}, \dots, f_{v,m}, f_{v,m}^{-1}, f_{m+1}, \dots, f_n])$$

(as before, $\mathbb{F}$ is a field of characteristic 0). An edge $(v, \bar v)$ of $\mathbb{T}_m$ labeled by a number $i \in [1, m]$ defines a transition function $X_v \to X_{\bar v} = \bar X$ by the equations $\bar f_j = f_j$ if $j \neq i$, and the *three-term relation* $\bar f_i f_i = M^i_v + M^i_{\bar v}$, where $M^i_u$, $u \in \{v, \bar v\}$, are some monomials in $f_1, \dots, f_n$.

Note that the tree $\mathbb{T}_m$ is connected, hence any pair of its vertices is connected by a unique path. Therefore, the transition map between the charts corresponding to two arbitrary vertices can be computed as the composition of the transitions along this path. Finally, put

$$\mathcal{X} = \cup_{v\in\mathbb{T}_m} X_v.$$

It follows immediately from the definition that $\mathcal{X} \subset \operatorname{Spec}(\mathcal{A}(Z))$. However, $\mathcal{X}$ contains only such points $x \in \operatorname{Spec}(\mathcal{A}(Z))$ for which there exists a vertex $v$ of $\mathbb{T}_m$ whose cluster variables form a coordinate system in some neighbourhood $\mathcal{X}(x) \subset \mathcal{X}$ of this point.

*Example.* Consider a cluster algebra $\mathcal{A}_1$ over $\mathbb{C}$ given by two clusters $\{f_1, f_2, f_3\}$ and $\{\bar f_1, f_2, f_3\}$ subject to one relation: $f_1\bar f_1 = f_2^2 + f_3^2$.

In this case $\operatorname{Spec}(\mathcal{A}_1) = \operatorname{Spec}(\mathbb{C}[x,y,u,v]/\{xy - u^2 - v^2 = 0\})$ is a singular affine hypersurface $H \subset \mathbb{C}^4$ given by the equation $xy = u^2 + v^2$ and containing a singular point $x = y = u = v = 0$. On the other hand, $\mathcal{X} = H \setminus \{x = y = u^2 + v^2 = 0\}$ is nonsingular.

Let $\omega$ be a Poisson bracket on an $n$-dimensional manifold $M$. We say that functions $g_1, \dots, g_n$ are *log-canonical* with respect to $\omega$ if $\omega(g_i, g_j) = \omega_{ij} g_i g_j$, where $\omega_{ij}$ are integer constants. The matrix $\Omega = (\omega_{ij})$ is called the *coefficient matrix* of $\omega$ (in the basis $g$); evidently, $\Omega \in so_n(\mathbb{Z})$.

In the general case, the following proposition is true.

**Theorem 2.1** ([GSV2, Lemma 2.1]). *The cluster manifold $\mathcal{X}$ is nonsingular and possesses a Poisson bracket such that for each vertex $v$ of $\mathbb{T}_m$ the corresponding extended cluster is log-canonical w.r.t. this bracket.*

Let $\omega$ be one of these Poisson brackets. Recall that a *Casimir element* corresponding to $\omega$ is a function that is in involution with all the other functions on $\mathcal{X}$. All rational Casimir functions form a subfield $\mathbb{F}_C$ in the field of rational functions $\mathbb{F}(\mathcal{X})$. The following proposition provides a complete description of $\mathbb{F}_C$.

**Theorem 2.2** ([GSV2, Lemma 2.2]). *$\mathbb{F}_C = \mathbb{F}(\mu_1, \dots, \mu_s)$, where $\mu_j$ has a monomial form*

$$\mu_j = \prod_{i=m+1}^{n} f_i^{\alpha_{ji}}$$

*for some integer $\alpha_{ji}$, and $s = \operatorname{corank}\omega$.*

**3. Toric action on the cluster algebra.** Assume that an integer weight $w_v = (w_{v,1}, \dots, w_{v,n})$ is given at any vertex $v$ of the tree $\mathbb{T}_m$. Then we define a *local toric action* on the cluster at $v$ as the $\mathbb{F}^\star$-action given by the formula $\{f_{v,1}, \dots, f_{v,n}\} \mapsto \{f_{v,1} \cdot t^{w_{v,1}}, \dots, f_{v,n} \cdot t^{w_{v,n}}\}$. We say that local toric actions are compatible if for any two extended clusters $C_v$ and $C_u$ the following diagram is commutative:

$$\begin{tikzcd} C_v \arrow[r] \arrow[d, "t^{w_v}"'] & C_u \arrow[d, "t^{w_u}"'] \\ C_v \arrow[r] & C_u \end{tikzcd}$$

In this case, local toric actions together define a global toric action on $\mathcal{A}(Z)$. This toric action is said to be an *extension* of the above local actions.

A toric action on the cluster algebra gives rise to a well-defined $\mathbb{F}^\star$-action on $\mathcal{X}$. The corresponding flow is called a *toric flow.*

**Lemma 3.1.** *Let $Z$ denote the transformation matrix at a vertex $v$ of $\mathbb{T}_m$, and let $w$ be an arbitrary integer weight. The local toric action at $v$ defined by $\{f_1,\dots,f_n\}\mapsto\{f_1t^{w_1},\dots,f_nt^{w_n}\}$ can be extended to a toric action on $\mathcal{A}(Z)$ if and only if $w^T$ belongs to the right nullspace $N_r(Z)$. Moreover, if such an extension exists, then it is unique.*

*Proof.* Given a monomial $g=\prod_j f_j^{p_j}$, we define its (weighted) degree by $\deg g=\sum_j p_jw_j$. It is easy to see that local toric actions are compatible if and only if all the monomials in each relation defining the transition $X\to\bar X$ have the same degree.

Consider an edge $(v,\bar v)$ of $\mathbb{T}_m$ labeled by $i$. Among the relations defining the transition $X\to\bar X$ there is a three-term relation $\bar f_i f_i=M^i+\bar M^i$. Hence, the compatibility condition implies $\deg M^i=\deg\bar M^i$, which by (1.1) is equivalent to

$$\sum_{z_{ik}>0} z_{ik}w_k=\sum_{z_{ik}<0}(-z_{ik})w_k.$$

The latter condition written for all the $m$ edges incident to $v$ gives $Zw^T=0$. Therefore, condition $w^T\in N_r(Z)$ is necessary for the existence of global extension.

Let us find the weight $\bar w$ that defines the local toric action at $\bar v$ compatible with the initial local toric action at $v$. First, identities $\bar f_j=f_j$ for $j\ne i$ immediately give $\bar w_j=w_j$ for $j\ne i$. Next, the three-term relation gives

$$\bar w_i=\sum_{z_{ik}>0} z_{ik}w_k-w_i,$$

so the weight at $\bar v$ is defined uniquely. It remains to prove that $w^T\in N_r(Z)$ implies $\bar w^T\in N_r(\bar Z)$.

Let $k\ne i$, then the $k$th entry of $\bar Z\bar w^T$ is equal to

$$\begin{aligned}\sum_{j=1}^n \bar z_{kj}\bar w_j&=\sum_{j\ne i} z_{kj}w_j+\frac12\sum_{j\ne i}\left(|z_{ki}|z_{ij}+z_{ki}|z_{ij}|\right)w_j-z_{ki}\left(\sum_{z_{il}>0} z_{il}w_l-w_i\right)\\&=\sum_{j=1}^n z_{kj}w_j=0.\end{aligned}$$

The $i$th entry of $\bar Z\bar w^T$ is equal to

$$\sum_{j=1}^n \bar z_{ij}\bar w_j=-\sum_{j\ne i} z_{ij}w_j=0,$$

since $\bar z_{ii}=0$. Hence, $\bar w^T\in N_r(\bar Z)$. $\square$

**4. Symplectic leaves.** Evidently, $\mathcal{X}$ is foliated into a disjoint union of symplectic leaves of the Poisson brackets $\omega$. We are interested only in generic leaves, which means the following.

Fix some generators $q_1,\dots,q_s$ of the field of rational Casimir functions $\mathbb{F}_C$. They define a map $Q\colon\mathcal{X}\to\mathbb{F}^s$, $Q(x)=(q_1(x),\dots,q_s(x))$. Let $\mathcal{L}$ be a symplectic leaf, and let $z = (z_1, \dots, z_s) = Q(\mathcal{L}) \in \mathbb{F}^s$. We say that $\mathcal{L}$ is *generic* if there exist $s$ vector fields $u_i$ in a neighborhood of $\mathcal{L}$ such that

a) at every point $x \in \mathcal{L}$, the vector $u_i(x)$ is transversal to the surface $q_i(x) = z_i$, which means that $\nabla_{u_i} q_i(x) \neq 0$;

b) the translation along $u_i$ for a sufficiently small time $t$ gives a diffeomorphism between $\mathcal{L}$ and a close symplectic leaf $\mathcal{L}_t$.

Let us denote by $\mathcal{X}^0$ the open part of $\mathcal{X}$ given by the conditions $f_i \neq 0$ for $i \in [m+1, n]$. It is easy to see that $\mathcal{X}^0 = \cup_{v \in \mathbb{T}_m} X_v^0$, where $X_v^0$ is the part of $X_v$ given by the same nonvanishing conditions.

**Lemma 4.1.** *$\mathcal{X}^0$ is foliated into a disjoint union of generic symplectic leaves of the Poisson bracket $\omega$.*

*Proof.* Consider first the special case when the Poisson structure on $\mathcal{X}$ is nondegenerate at the generic point, i.e., its rank equals to the dimension of the manifold. Then we show that every point of $\mathcal{X}^0$ is generic, i.e., the rank of the Poisson bracket is maximal at each point. Note that for every point $x \in \mathcal{X}^0$ there exists a cluster chart $X_v$ such that on $X_v$ one has $f_{v,j} \neq 0$ for $j \in [1, n]$. Therefore the coordinates $\log f_{v,j}$ form a local coordinate system on $X_v$, and the Poisson structure written in these coordinates becomes a constant Poisson structure. If a constant Poisson structure is nondegenerate, it is nondegenerate at each point, which proves the statement. Moreover, note that the complement $\mathcal{X} \setminus \mathcal{X}^0$ consists of degenerate symplectic leaves of smaller dimension. Hence, if the Poisson structure is symplectic, i.e., nondegenerate at a generic point, then $\mathcal{X}^0$ is a union of generic symplectic leaves.

Assume now that the rank of the Poisson structure is $r < n$. There exist $s = n - r$ Casimir functions that generate the field $\mathbb{F}_C$. By Theorem 2.2, one can build these Casimir functions by choosing $s$ independent integer vectors $u$ in the left nullspace $N_{\mathrm{l}}(\Omega^\tau)$ and by constructing the corresponding monomials $\mu^u$. Observe that if $u = (\, u^1 \quad u^2 \,) \in N_{\mathrm{l}}(\Omega^\tau)$ and $u' = (\, D^{-1}u^1 \quad u^2 \,)$, then $(u')^T \in N_{\mathrm{r}}(Z)$. Therefore, by Lemma 3.1, such a $u'$ defines a toric flow on $\mathcal{X}$. To accomplish the proof it is enough to show that the toric flow corresponding to the vector $u'$ is transversal to the level surface $\{y \in \mathcal{X}^0 : \mu^u(y) = \mu^u(x)\}$, and that a small translation along the trajectory of this toric flow transforms one symplectic leaf into another one.

We will first show that if $x(t)$ is a trajectory of the toric flow corresponding to $u'$ with the initial value $x(1) = x$ and the initial velocity vector $\nu = dx(t)/dt|_{t=1} = (u'_1 f_1, \dots, u'_n f_n)$ then $d\mu^u(\nu)/dt \neq 0$. Indeed, by Theorem 2.2,

$$d\mu^u(\nu)/dt = \sum_{i=m+1}^{n} \alpha_i \frac{\prod_{j=m+1}^{n} f_j^{\alpha_j}}{f_i} \cdot u_i f_i = \mu^u \alpha^2 (u^2)^T.$$

Since $x \in \mathcal{X}^0$, one has $\mu^u(x) \neq 0$. To find $\alpha^2(u^2)^T$ recall that by Theorem 2.2, $\alpha^2 = u^1 Z_2 + u^2 Z_4 + u^2 K'$, where $K'$ is the submatrix of $K$ whose entries lie in the last $n - m$ rows and columns. Clearly, $u^2 Z_4 (u^2)^T = 0$, since $Z_4$ is skew-symmetric. Next, $u^1 Z_2 (u^2)^T = u^1 \Lambda^{-1} \Omega_2 (u^2)^T = u^1 \Lambda^{-1} \Omega_1^T (u^1)^T = 0$, since $u^2 \Omega_2^T = u^1 \Omega_1$ and $\Lambda^{-1} \Omega_1^T$ is skew-symmetric. Thus, $\alpha^2 (u^2)^T = u^2 K' (u^2)^T \neq 0$, since $K'$ can be chosen to be a diagonal matrix with positive elements on the diagonal.

Consider now another basis vector $\bar{u} \in N_{\mathrm{l}}(\Omega^\tau)$ and the corresponding Casimir function $\mu^{\bar{u}}$. It is easy to see that $d\mu^{\bar{u}}(\nu)/dt = \mu^{\bar{u}} \bar{\alpha}^2 (u^2)^T$. Note that the latter expression does not depend on the point $x$, but only on the value $\mu^{\bar{u}}(x)$ and on the vectors $\bar{\alpha}$ and $u$. Therefore the value of the derivative $d\mu^{\bar{u}}/dt$ is the same for all points $x$ lying on the same symplectic leaf, and the toric flow transforms one symplectic leaf into another one. □

In general, it not true that $\mathcal{X}^0$ coincides with the union of all "generic" symplectic leaves. A simple counterexample is provided by the cluster algebra given by two clusters $\{f_1, f_2, f_3\}$ and $\{\bar{f}_1, f_2, f_3\}$ subject to one relation: $\bar{f}_1 f_1 = f_2^2 f_3^2 + 1$. One can choose the Poisson bracket on $\mathcal{X}$ as follows: $\{f_1, f_2\} = f_1 f_2$, $\{f_1, f_3\} = f_1 f_3$, $\{f_2, f_3\} = 0$. Equivalently, $\{\bar{f}_1, f_2\} = -\bar{f}_1 f_2$, $\{\bar{f}_1, f_3\} = -\bar{f}_1 f_3$, $\{f_2, f_3\} = 0$. Generic symplectic leaves of this Poisson structure are described by the equation $Af_2 + Bf_3 = 0$ where $(A : B)$ is a homogeneous coordinate on $P^1$. All generic symplectic leaves form $P^1$. In particular, two leaves $(1 : 0)$ and $(0 : 1)$ (correspondingly, subsets $f_2 = 0, f_3 \neq 0, f_1 \bar{f}_1 = 1$ and $f_3 = 0, f_2 \neq 0, f_1 \bar{f}_1 = 1$) are generic symplectic leaves in $\mathcal{X}$. According to the definition of $\mathcal{X}^0$ these leaves are not contained in $\mathcal{X}^0$.

We can describe $\mathcal{X}^0$ as the nonsingular locus of the toric action. The main source of examples of cluster algebras are coordinate rings of homogeneous manifolds. Toric actions on such cluster algebras are induced by the natural toric actions on these manifolds.

All this suggests that $\mathcal{X}^0$ is a natural geometrical object in the cluster algebra theory, intrinsically related to the corresponding Poisson structure.

**5. Connected components of $\mathcal{X}^0$.** In what follows we assume that $\mathbb{F} = \mathbb{R}$. In this case, the first natural question concerning the topology of $\mathcal{X}^0$ is to find the number $\#(\mathcal{X}^0)$ of connected components of $\mathcal{X}^0$. To answer this question we follow the approach developed in [SSV1, SSV2, Z].

Given a vertex $v$ of $\mathbb{T}_m$, we define an open subset $S(X^0) = S(X_v^0) \subset \mathcal{X}^0$ by

$$S(X_v^0) = X_v^0 \cup \bigcup_{(v,\bar{v}) \in \mathbb{T}_m} X_{\bar{v}}^0,$$

where $(v, \bar{v}) \in \mathbb{T}_m$ means that $(v, \bar{v})$ is an edge of $\mathbb{T}_m$.

Recall that $X^0 \simeq (\mathbb{R} \setminus 0)^n$. We can decompose $X^0$ as follows. Let $\Sigma$ be the set of all possible sequences $(\sigma(1), \dots, \sigma(n))$ of $n$ signs $\sigma(i) = \pm 1$. For $\sigma \in \Sigma$ we define $X^0(\sigma)$ as the octant $\sigma(j) f_j > 0$ for all $j \in [1, n]$. Two octants $X^0(\sigma_1)$ and $X^0(\sigma_2)$ are called *essentially connected* if the following two conditions are fulfilled:

1) there exists $i \in [1, n]$ such that $\sigma_1(i) \neq \sigma_2(i)$ and $\sigma_1(j) = \sigma_2(j)$ for $j \neq i$;

2) there exists $x^* \in S(X^0)$ that belongs to the intersection of the closures of $X^0(\sigma_1)$ and $X^0(\sigma_2)$.

The second condition can be restated as follows:

2') there exists $x^* \in S(X^0)$ such that $f_i(x^*) = 0$, $\hat{f}_i(x^*) \neq 0$, and $f_j(x^*) \neq 0$ for $j \neq i$, where $\hat{f} = f_{\hat{v}}$ and $(v, \hat{v})$ is the edge of $\mathbb{T}_m$ labeled by $i$.

**Lemma 5.1.** *Let $(v, \bar{v})$ be an edge of $\mathbb{T}_m$, $\sigma_1, \sigma_2 \in \Sigma$, and let the octants $X^0(\sigma_1)$ and $X^0(\sigma_2)$ be essentially connected. Then $\bar{X}^0(\sigma_1)$ and $\bar{X}^0(\sigma_2)$ are essentially connected as well.*

*Proof.* Assume that the edge $(v, \bar{v})$ is labeled by $j$, and consider first the case $\sigma_1(j) \neq \sigma_2(j)$. Then $\hat{v}$ in condition 2' coincides with $\bar{v}$, and hence $f_j(x^*)\bar{f}_j(x^*) = M^j(x^*) + \bar{M}^j(x^*) = 0$. Since $M^j$ and $\bar{M}^j$ both do not contain $f_j$, any point $x$ such that $f_l(x) = f_l(x^*)$ for $l \neq j$, $f_j(x) \neq 0$, $\bar{f}_j(x) = 0$ belongs to $S(\bar{X}^0)$, and hence $\bar{X}^0(\sigma_1)$ and $\bar{X}^0(\sigma_2)$ are essentially connected.

Assume now that $\sigma_1(i) \neq \sigma_2(i)$ for some $i \neq j$. As before, we get $M^i(x) + \widehat{M}^i(x) = 0$ for any $x$ such that $f_l(x) = f_l(x^*)$ for $l \neq i$. Consider the edge $(\bar{v}, \bar{\bar{v}})$ of $\mathbb{T}_m$ labeled by $i$; by the above assumption, $\bar{\bar{v}} \neq v$. Without loss of generality assume that $\bar{f}_j$ does not enter $\bar{\bar{M}}^i$. Then by [FZ] one has

$$M^i + \widehat{M}^i = \widehat{M}^i \left( \frac{M^i}{\widehat{M}^i} + 1 \right) = \widehat{M}^i \left( \frac{\bar{M}^i}{\bar{\bar{M}}^i} + 1 \right) \Bigg|_{\bar{f}_j \leftarrow \frac{M_0}{f_j}},$$

where $M_0 = M^j + \bar{M}^j|_{f_i=0}$. Therefore, for $x^{**}$ such that $\bar{f}_l(x^{**}) = f_l(x^*)$ if $l \neq i, j$, $\bar{f}_j(x^{**}) = M_0(x^*)/f_j(x^*)$, $\bar{f}_i(x^{**}) = 0$, one has $\bar{M}^i(x^{**}) + \bar{\bar{M}}^i(x^{**}) = M^i(x^*) + \widehat{M}^i(x^*) = 0$, and hence one can choose $\bar{\bar{f}}_i(x^{**}) \neq 0$. $\square$

**Corollary 5.2.** *If $X_v^0(\sigma_1)$ and $X_v^0(\sigma_2)$ are essentially connected, then $X_{v'}^0(\sigma_1)$ and $X_{v'}^0(\sigma_2)$ are essentially connected for any vertex $v'$ of $\mathbb{T}_m$.*

*Proof.* Since the tree $\mathbb{T}_m$ is connected, one can pick up the path connecting $v$ and $v'$. Then the corollary follows immediately from Lemma 5.1. $\square$

Let $\#_v$ denote the number of connected components in $S(X_v^0)$.

**Theorem 5.3.** *The number $\#_v$ does not depend on $v$ and is equal to $\#(\mathcal{X}^0)$.*

*Proof.* Indeed, since $S(X_v^0)$ is dense in $\mathcal{X}^0$, one has $\#(\mathcal{X}^0) \leqslant \#_v$. Conversely, assume that there are points $x_1, x_2 \in \mathcal{X}^0$ that are connected by a path in $\mathcal{X}^0$. Therefore their small neighborhoods are also connected in $\mathcal{X}^0$, since $\mathcal{X}^0$ is a topological manifold. Since $X_v^0$ is dense in $\mathcal{X}^0$, one can pick $\sigma, \sigma' \in \Sigma$ such that the intersection of the first of the above neighborhoods with $X_v^0(\sigma)$ and of the second one with $X_v^0(\sigma')$ are nonempty. Thus, $X_v^0(\sigma)$ and $X_v^0(\sigma')$ are connected in $\mathcal{X}^0$. Hence, there exist a loop $\gamma$ in $\mathbb{T}_m$ with the initial point $v$, a subset $v_1, \dots, v_p$ of vertices of this loop, and a sequence $\sigma_1 = \sigma, \sigma_2, \dots, \sigma_{p+1} = \sigma' \in \Sigma$ such that $X_{v_l}^0(\sigma_l)$ is essentially connected with $X_{v_l}^0(\sigma_{l+1})$ for all $l \in [1, p]$. Then by Corollary 5.2 $X_v^0(\sigma_l)$ and $X_v^0(\sigma_{l+1})$ are essentially connected. Hence all $X_v^0(\sigma_l)$ are connected with each other in $S(X_v^0)$. In particular, $X_v^0(\sigma)$ and $X_v^0(\sigma')$ are connected in $S(X_v^0)$. This proves the assertion. $\square$

By virtue of Theorem 5.3, we write $\#$ instead of $\#_v$.

Let $\mathbb{F}_2^n$ be an $n$-dimensional vector space over $\mathbb{F}_2$ with a fixed basis $\{e_i\}$. Consider an $n \times n$ diagonal matrix $\widehat{D} = \operatorname{diag}(d_1, \dots, d_m, 1, \dots, 1)$ and denote by $\mathcal{Z}_{nn}^{\widehat{D}}(Z)$ the set of all $n \times n$ integer matrices $Z' \in \mathcal{Z}_{nn}^{\widehat{D}}$ such that $Z'[m; n] = Z$. Fix an arbitrary matrix $Z' \in \mathcal{Z}_{nn}^{\widehat{D}}(Z)$ and let $\omega = \omega_v$ be a (skew-)symmetric bilinear form on $\mathbb{F}_2^n$, such that $\omega(e_i, e_j) = z'_{ij}$. Define a linear operator $\mathfrak{t}_i \colon \mathbb{F}_2^n \to \mathbb{F}_2^n$ by the formula $\mathfrak{t}_i(\xi) = w - \omega(\xi, e_i)e_i$, and let $\Gamma = \Gamma_v$ be the group generated by $\mathfrak{t}_i$, $i \in [1, m]$.

The following lemma is a minor modification of the result presented in [Z].

**Lemma 5.4.** *The number of connected components $\#_v$ equals to the number of $\Gamma_v$-orbits in $\mathbb{F}_2^n$.*

For reader's convenience we will repeat the proof of this statement here.

Let us identify $\mathbb{F}_2^n$ with $\Sigma$ by the following rule: a vector $\xi \in \mathbb{F}_2^n$ corresponds to $\sigma \in \Sigma$ such that $\sigma(i) = (-1)^{\xi_i}$. Abusing notation we will also write $X^0(\xi)$ instead of $X^0(\sigma)$.

**Lemma 5.5.** *Let $\xi$ and $\xi'$ be two distinct vectors in $\mathbb{F}_2^n$. Then the closures of $X^0(\xi)$ and $X^0(\xi')$ intersect in $S(X^0)$ if and only if $\xi' = \mathfrak{t}_i(\xi)$ for some $i \in [1,m]$.*

*Proof.* Suppose $x \in S(X^0)$ belongs to the intersection of the closures of $X^0(\xi)$ and $X^0(\xi')$. Then $f_l(x) = 0$ whenever $\xi_l \neq \xi'_l$. From the definition of $S(X^0)$ we see that there is a unique $i$ such that $\xi_i \neq \xi'_i$; evidently, $i \in [1,m]$. Furthermore, if $(v,\bar v)$ is an edge of $\mathbb{T}_m$ labeled by $i$, then $\bar f_i(x) \neq 0$. Since any neighborhood of $x$ intersects both $X^0(\xi)$ and $X^0(\xi')$, it follows that monomials $M^i$ and $\bar M^i$ on the right hand side of the 3-term relation $f_i \bar f_i = M^i + \bar M^i$ must have opposite signs at $x$. Therefore

$$\xi_i - \xi'_i = 1 = \sum_{j=1}^n z_{ij}\xi_j.$$

Comparing this with the definition of the transvection $\mathfrak{t}_i$, we conclude that $\xi' = \mathfrak{t}_i(\xi)$, as claimed.

Conversely, suppose $\xi' = \mathfrak{t}_i(\xi) \neq \xi$, then $\sum_{j=1}^n z_{ij}\xi_j = 1$. Therefore, there exists a point $x \in S(X^0)$ such that $(-1)^{\xi_l} f_l(x) > 0$ for all $l \neq i$, and the right hand side of three 3-term relation vanishes at $x$. Then any neighborhood of $x$ contains points where the signs of all $f_l$ for $l \neq i$ remain the same, while the right hand side of the 3-term relation is positive (or negative). Thus, $x$ belongs to the intersection of the closures of $X^0(\xi)$ and $X^0(\xi')$, and we are done. □

Now we are ready to complete the proof of Lemma 5.4. Let $\Xi$ be a $\Gamma$-orbit in $\mathbb{F}_2^n$, and let $X^0(\Xi) \subset S(X^0)$ be the union of the closures of $X^0(\xi)$ over all $\xi \in \Xi$. Each $X^0(\xi)$ is a copy of $\mathbb{R}^n_{>0}$, and is thus connected. Using the "if" part of Lemma 5.5, we conclude that $X^0(\Xi)$ is connected (since the closure of a connected set and the union of two non-disjoint connected sets are connected as well). On the other hand, by the "only if" part of the same lemma, all the sets $X^0(\Xi)$ are pairwise disjoint. Thus, they are the connected components of $S(X^0)$, and we are done. □

Theorem 5.3 and Lemma 5.4 imply the following theorem.

**Theorem 5.6.** *The number of connected components $\#(\mathcal{X}^0)$ equals the number of $\Gamma$-orbits in $\mathbb{F}_2^n$.*

**6. Poisson and cluster algebra structures on Grassmannians.** It is proved in [GSV2] that one can assign a cluster algebra to any algebraic Poisson manifold equipped with a system of rational log-canonical coordinates. A rich collection of non-trivial examples of this sort is provided by the theory of Poisson-Lie groups and Poisson homogeneous spaces. This collection includes real Grassmannians, which will serve as our main example.

Let $\mathcal{P}$ be a Lie subgroup of a Poisson-Lie group $(G, \omega_G)$. A Poisson structure on the homogeneous space $\mathcal{P}\backslash G$ is called *Poisson homogeneous* if the action map $\mathcal{P}\backslash G \times G \to \mathcal{P}\backslash G$ is Poisson. Conditions on $\mathcal{P}$ for the standard Sklyanin bracket to descend to a Poisson homogeneous structure on $\mathcal{P}\backslash G$ are conveniently formulated it terms of the Manin triple that corresponds to $(G, \omega_G)$ and can be found, e.g., in [ReST]. In particular, these conditions are satisfied for $G = SL_n$ and

$$\mathcal{P} = \mathcal{P}_k = \left\{ \begin{pmatrix} A & 0 \\ B & C \end{pmatrix} : A \in SL_k, C \in SL_{n-k} \right\}.$$

The resulting homogeneous space is the Grassmannian $G(k,n)$.

The Sklyanin bracket can be extended from $SL_n$ to the associative algebra $\mathrm{Mat}_n$ of $n \times n$ matrices. In terms of matrix elements $x_{ij}$, $i,j \in [1,n]$, of a matrix $X \in \mathrm{Mat}_n$, it is given there by

$$\omega(x_{ij}, x_{\alpha\beta}) = (\mathrm{sign}(\alpha - i) + \mathrm{sign}(\beta - j))x_{i\beta}x_{\alpha j}.$$

If $X \in SL_n$ admits a factorization into block-triangular matrices

$$X = \begin{pmatrix} X_1 & 0 \\ Y' & X_2 \end{pmatrix} \begin{pmatrix} \mathbf{1}_k & Y \\ 0 & \mathbf{1}_{n-k} \end{pmatrix} = VU,$$

then $Y$ represents an element of the cell $G_0(k,n)$ in $G(k,n)$ characterized by non-vanishing of the Plücker coordinate $\pi_{[1,k]}$.

Relations between the Plücker coordinates $\pi_I$, $I = \{i_1, \dots, i_k \colon 1 \leqslant i_1 < \dots < i_k \leqslant n\}$, and minors $Y^{\beta_1 \dots \beta_l}_{\alpha_1 \dots \alpha_l} = \det(y_{\alpha_i, \beta_j})^l_{i,j=1}$ of $Y$ are given by

$$Y^{\beta_1 \dots \beta_l}_{\alpha_1 \dots \alpha_l} = (-1)^{kl - l(l-1)/2 - (\alpha_1 + \dots + \alpha_l)} \frac{\pi_{([1,k] \setminus \{\alpha_1 \dots \alpha_l\}) \cup \{\beta_1 + k \dots \beta_l + k\}}}{\pi_{[1,k]}}.$$

Note that, if the row index set $\{\alpha_1 \dots \alpha_l\}$ in the above formula is contiguous then the sign in the right hand side can be expressed as $(-1)^{(k-\alpha_l)l}$.

It is not hard to see that in terms of matrix elements $y_{ij}$, the formula for the Sklyanin brackets on $G(k,n)$ looks as follows:

$$\omega(y_{ij}, y_{\alpha\beta}) = (\mathrm{sign}(\alpha - i) - \mathrm{sign}(\beta - j))y_{i\beta}y_{\alpha j}. \tag{6.1}$$

Next, we will introduce new coordinates on $G(k,n)$, log-canonical w. r. t. $\omega$. This will require some preparation.

For every $(i,j)$-entry of $Y$ define $l(i,j) = \min(i-1, n-k-j)$ and $F_{ij} = Y^{j, \dots, j+l(i,j)}_{i-l(i,j), \dots, i}$. It is easy to see that the change of coordinates $(y_{ij}) \mapsto (F_{ij})$ is a birational transformation.

**Lemma 6.1** ([GSV2, Prop. 3.4]). *Put $t_{ij} = F_{ij}/F_{i-1,j+1}$, then*

$$\omega_{G(n,k)}(\ln t_{ij}, \ln t_{\alpha\beta}) = \mathrm{sign}(j - \beta)\delta_{i\alpha} - \mathrm{sign}(i - \alpha)\delta_{j\beta}. \tag{6.2}$$

We now proceed to compute a maximal dimension of a symplectic leaf of the bracket (6.1). Denote $n - k$ by $m$ and put

$$J_1 = F_{11}, \dots, J_k = F_{k1}, J_{k+1} = F_{k2}, \dots, J_n = F_{km}. \tag{6.3}$$

**Theorem 6.2** ([GSV2, Th. 3.5]). *Let $l = \gcd(k,n)$. The co-dimension of a maximal symplectic leaf of $G(k,n)$ is equal to 0 if $\frac{k}{l}$ is even or $\frac{n-k}{l}$ is even, and is equal to $l$ otherwise. In the latter case, a symplectic leaf via a point in general position is parametrized by values of Casimir functions*

$$I_{V(i)} = \prod_{\alpha=0}^{\frac{n}{l}-1} J_{i+\alpha l}^{(-1)^\alpha}.$$

Our next goal is to build a cluster algebra $\mathcal{A}_{G(k,n)}$ associated with the Poisson bracket (6.1). The initial cluster consists of functions

$$f_{ij} = (-1)^{(k-i)(l(i,j)-1)} F_{ij} = \frac{\pi_{([1,k]\setminus[i-l(i,j),\ i])\cup[j+k,j+l(i,j)+k]}}{\pi_{[1,k]}}, \qquad i \in [1,k], \quad j \in [1,n-k]).$$

We designate functions $f_{11}, f_{21}, \dots, f_{k1}, f_{k2}, \dots, f_{km}$ (cf. (6.3)) to serve as tropical coordinates. This choice is motivated by the last statement of Theorem 6.2 and by the following observation: functions (6.3) have log-canonical brackets with all coordinate functions $y_{\alpha\beta}$.

Now we need to define the matrix $Z$ that gives rise to cluster transformations compatible with the Poisson structure. We want to choose $Z$ in such a way that the submatrix of $Z$ corresponding to cluster coordinates will be skew-symmetric and irreducible. According to our choice of tropic coordinates, this means that $Z$ must satisfy

$$Z\Omega^F = const \cdot (\, \mathrm{diag}(P, \dots, P) \quad 0\,)$$

where $\mathcal{P} = \sum_{i=1}^{m-1} e_{i,i+1}$ is a $(m-1)\times m$ matrix and $\Omega^F$ is the coefficient matrix of Poisson brackets $\omega$ in the basis $F_{ij}$.

This condition can be translated to the following: for $x = (x_{11}, \dots, x_{1,m}, \dots, x_{k1}, \dots, x_{k,m})$ one has

$$(Zx)_{ij} = x_{i+1,j} + x_{i,j-1} + x_{i-1,j+1} - x_{i+1,j-1} - x_{i,j+1} - x_{i-1,j}.$$

It is easy to see that the submatrix of $Z$ corresponding to the non-tropic coordinates is indeed skew-symmetric and irreducible.

The matrix $Z$ thus obtained can be conveniently represented by a directed graph with vertices forming a rectangular $k \times (n-k)$ array and labeled by pairs of integers $(i,j)$, and edges $(i,j) \to (i,j+1)$, $(i+1,j) \to (i,j)$ and $(i,j) \to (i+1,j-1)$ (cf. Fig. 1).

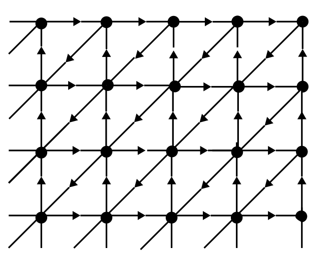

FIG. 1. GRAPH THAT CORRESPONDS TO $G(k,n)$

**7. The number of connected components of refined Bruhat cells in real Grassmannians.** Consider the union of regular $\mathbb{R}^*$-orbits in $\mathcal{X}^0_{G(k,n)}$ corresponding to the described above cluster algebra $\mathcal{A}_{G(k,n)}$ compatible with the Sklyanin Poisson bracket in $G(k,n)$. Recall that by construction functions $f_{11}, \dots, f_{k1}, f_{k2}, \dots, f_{km}$

serve as tropical coordinates. Moreover, any matrix element in the standard representation of the maximal Bruhat cell in the Grassmannian enters as a cluster coordinate for some cluster in $\mathcal{A}_{G(k,n)}$. Therefore, $\mathcal{X}^0_{G(k,n)}$ is naturally embedded into $G(k,n)$ and we can consider it as the subset in $G(k,n)$ determined by the conditions that all tropical coordinates $f_{ik}$ and $f_{kl}$ do not vanish. Tropical coordinates are all "cyclically dense" minors among all the Plücker coordinates, i.e., minors containing all columns with indices $i, i+1, \dots, i+k$ or $i, i+1, \dots, i+l = n, 1, 2, \dots, k+i+1-n$. We call $\mathcal{X}^0_{G(k,n)}$ a *refined open Bruhat cell* in $G(k,n)$, since it is an intersection of $n$ usual open Bruhat cells in $G(k,n)$ in general position.

A method to compute the number of connected components of $\mathcal{X}^0_{G(k,n)}$ was discussed in Section 5 of this paper. Let us recall certain notions and results from [SSVZ].

We denote by $\mathcal{E}$ the graph corresponding to the matrix $Z$ (see the remark at the end of Section 1) and by $\mathbb{F}_2^{\mathcal{E}}$ the vector space generated by the characteristic vectors of the vertices of $\mathcal{E}$. Similarly, $\mathbb{F}_2^C$ denotes a subspace of $\mathbb{F}_2^{\mathcal{E}}$ generated by the vertices corresponding to cluster variables. A finite (undirected) graph is said to be *$E_6$-compatible* if it is connected and contains an induced subgraph with 6 vertices isomorphic to the Dynkin graph $E_6$. A directed graph is said to be $E_6$-compatible if the corresponding undirected graph obtained by replacing each directed edge by an undirected one is $E_6$-compatible.

**Theorem 7.1** ([SSVZ, Th. 3.11]). *Suppose that the induced subgraph of $\mathcal{E}$ spanned by the vertices corresponding to cluster variables is $E_6$-compatible. Then the number of $\Gamma$-orbits in $\mathbb{F}_2^{\mathcal{E}}$ is equal to*

$$2^t \cdot (2 + 2^{\dim(\mathbb{F}_2^C \cap \ker\ \overline{\Omega})}),$$

*where $\overline{\Omega}$ denotes the $\mathbb{F}_2$-valued bilinear form on $\mathbb{F}_2^{\mathcal{E}}$ obtained by reduction modulo 2 from the form $\Omega$ and $t$ is the number of tropic variables.*

Combining this theorem with Theorem 5.6 we get the following corollary.

**Corollary 7.2.** *The number of connected components of a refined open Bruhat cell $\mathcal{X}^0_{G(k,n)}$ is equal to $3 \cdot 2^{n-1}$ if $k > 3$, $n > 7$.*

*Proof.* Indeed, by Theorem 5.6 we know that the number of connected components of $\mathcal{X}^0_{G(k,n)}$ equals the number of orbits of $\Gamma$-orbits in $\mathbb{F}_2^{\mathcal{E}}$, where the graph $\mathcal{E}$ is shown on Fig. 1, and the subset $C$ is formed by all the vertices except for the first column and the last row. Since in the case $k \geqslant 4$, $n \geqslant 8$ the subgraph spanned by $C$ is evidently $E_6$-compatible, Theorem 7.1 implies that to prove the statement it is enough to show that $\mathbb{F}_2^C \cap \ker \overline{\Omega} = 0$; in other words, that there is no nontrivial vector in $\ker \overline{\Omega}$ with vanishing tropical components.

Indeed, let us denote such a vector by $h \in \mathbb{F}_2^C$, and let $\delta_{ij}$ be the $i,j$-th basis vector in $\mathbb{F}_2^{\mathcal{E}}$. Note that the condition $\overline{\Omega}(h, \delta_{k,n-k}) = 0$ implies that $h_{k-1,n-k} = 0$. Further, assuming $h_{k-1,n-k} = 0$ we see that the condition $\overline{\Omega}(h, \delta_{k,n-k-1}) = 0$ implies $h_{k-1,n-k-1} = 0$ and so on. Since $\overline{\Omega}(h, \delta_{kj}) = 0$ for any $j \in [1,m]$, we conclude that $h_{k-1,j} = 0$ for any $j \in [1, n-k]$. Proceeding by induction we prove that $h_{ij} = 0$ for any $i \in [1,k]$ and any $j \in [1, n-k]$. Hence $h = 0$. Note that $t = n$, which accomplishes the proof of the statement. □

It is easy to notice that the number of connected components for $\mathcal{X}^0_{G(k,n)}$ equals the number of connected components for $\mathcal{X}^0_{G(n-k,n)}$. Therefore, taking in account

Corollary 7.2, in order to find the number of connected components for refined open Bruhat cells for all Grassmannians we need to consider only two remaining cases: $G(2,n)$ and $G(3,n)$.

**Proposition 7.3.** *The number of connected components of a refined open Bruhat cell equals to* $(n-1)\cdot 2^{n-2}$ *for* $G(2,n)$, $n \geqslant 3$, *and to* $3\cdot 2^{n-1}$ *for* $G(3,n)$, $n \geqslant 6$.

*Proof.* The proof follows immediately from Lemmas 1,2 and corollary of Lemma 3 of [GSV1]. Following notations of [GSV1], let us denote by $U$ the subgraph of the graph $\mathcal{E}$ (for the Grassmannian $G(k,n)$) consisting of the first $k-1$ rows, and by $L$ the subgraph containing only the last row. The corresponding vector subspaces of $\mathbb{F}_2^{\mathcal{E}}$ are denoted by $\mathbb{F}_2^U$ and $\mathbb{F}_2^L$; the corresponding subgroups of $\Gamma$ are denoted by $\Gamma^U$ and $\Gamma^L$. Lemma 1 of [GSV1] states that for any $G(k,n)$ we have $2^{n-k}\#(\Gamma^U)$ orbits of $\Gamma$-action, where $2^{n-k}$ is the number of vectors in $\mathbb{F}_2^L$, and $\#(\Gamma^U)$ is the number of $\Gamma^U$-orbits in $\mathbb{F}_2^U$. Lemma 2 counts $\#(\Gamma^U) = n-1$ for $G(2,n)$, $n \geqslant 3$. Finally, the corollary of Lemma 3 calculates $\#(\Gamma^U) = 12$ for $G(3,n)$, $n \geqslant 6$. Substituting these values of $\#(\Gamma^U)$ into the above mentioned formula from Lemma 1 proves the proposition. □

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