Alternating Sign Matrices, Weighted Enumerations, and Symplectic Shifted Tableaux

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Abstract

Alternating sign matrices with a U–turn boundary (UASM) are a recent generalization of ordinary alternating sign matrices. Here we show that these matrices are in bijective correspondence with symplectic shifted tableaux that were recently introduced in the context of a symplectic version of Tokuyama's deformation of Weyl's denominator formula. This bijection yields a formula for a weighted enumeration of UASM.

Une matrice à signes alternants avec "U–turn" (UASM) est une généralisation d'une matrice a signes alternants. Nous présentons ici une bijection entre les UASM et les tableaux décalés qui sont associes à une déformation pour $sp(2n)$ de la formule du dénominateur de Weyl. Cette bijection produit divers types d'énumérations des UASM.

1 Introduction

Alternating sign matrices with a U–turn boundary (UASM) first appeared in a paper by Tsuchiya [T98] but have been given a wider audience by Kuperberg [K02] and Propp [P01] (who calls them half alternating sign matrices). In this paper we introduce a generalization of UASM called μ–UASM that combine the UASM with the μ–ASM concept of Okada [O93]. We show the natural relationship between μ–UASM and the symplectic shifted tableaux of Hamel and King [HK02]. Through this bijection we demonstrate a general formula from which we derive weighted enumeration formulae for μ–UASM and UASM. The most basic corollary of our result is $\sum_{A\in\mathcal{UA}} 2^{neg(A)} = 2^{n^2}$, where $neg(A)$ is the number of -1's in A and UA is the

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set of UASM. This result was conjectured by Propp [P01] and proved by Eisenkölbl [E02] and independently by Chapman. It is also derivable from Kuperberg [K02]. Our two main results are:

The Bijection Result: Let $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$ be a partition of length $\ell(\mu) = n$ whose parts are all distinct. Then there is a bijection between $sp(2n)$–standard shifted tableaux of shifted shape μ and the set $\mathcal{UA}^\mu$ of $2n \times m$ μ–alternating sign matrices with a U–turn boundary and $m = \mu_1$.

The Weighted Enumeration Result: Let A be an alternating sign matrix with a U–turn boundary. Then the following formula holds:

$$(1+t)^{n^2} = \sum_{A \in \mathcal{UA}} t^{ssi(A)+2bar(A)}(1+t)^{neg(A)}$$

where $ssi(A)$ and $bar(A)$ are parameters associated to UASM defined below and UA is the set of UASM.

2 Alternating Sign Matrices

Alternating sign matrices (ASM) are $n \times n$ matrices containing 0's, 1's, and -1's such that the first and last nonzero entries of each row and column are 1's and the nonzero entries within a row or column alternate in sign. See, for example, the following ASM A in equation 2.1. Here and elsewhere we use $\overline{1}$ to denote -1.

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & \overline{1} & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}. \tag{2.1}$$

The number, $A(n)$, of ASM of size n is described by the famous formula $A(n) = \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}$. The first proof of this formula was given by Zeilberger [Z96]. A second proof is due to Kuperberg [K96], and a complete history is to be found in Bressoud [B99].

Okada [O93] generalized ASM by defining a set of μ–alternating sign matrices associated with a partition μ with distinct parts. These have properties similar to ordinary ASM but have column sums 0 in columns indexed by some $q \neq \mu_j$ for any j and column sums 1 in columns indexed by $q = \mu_j$ for some j. More formally, for each partition μ, all of whose parts are distinct and for which $\ell(\mu) = n$ and $\mu_1 \leq m$, an $n \times m$ matrix $A = (a_{iq})$ belongs to the set $\mathcal{A}^\mu(2n)$ of $n \times m$ μ-alternating sign matrices if the following conditions are satisfied:

$$\begin{array}{lll} \text{(O1)} & a_{iq} \in \{-1,0,1\} & \text{for } 1 \leq i \leq n,\ 1 \leq q \leq m; \\ \text{(O2)} & \sum_{q=p}^{m} a_{iq} \in \{0,1\} & \text{for } 1 \leq i \leq n,\ 1 \leq p \leq m; \\ \text{(O3)} & \sum_{i=j}^{n} a_{iq} \in \{0,1\} & \text{for } 1 \leq j \leq n,\ 1 \leq q \leq m; \\ \text{(O4)} & \sum_{q=1}^{m} a_{iq} = 1 & \text{for } 1 \leq i \leq n; \\ \text{(O5)} & \sum_{i=1}^{n} a_{iq} = & \left\{ \begin{array}{ll} 1 & \text{if } q = \mu_k \text{ for some } k \\ 0 & \text{otherwise} \end{array} \right. \\ & & \text{for } 1 \leq q \leq m,\ 1 \leq k \leq n. \end{array} \tag{2.2}$$

The alternating sign matrices with a U-turn boundary, UASM, are a variation on ordinary ASM developed by Kuperberg [K02] after a paper of Tsuchiya [T98]. UASM have an even number of rows. Each column of a UASM is of the same form as that of an ordinary ASM. Each successive pair of rows of a UASM reads first left-right and then right-left like a row of an ASM. The number, $A_U(2n)$, of UASM of size $2n$ is

$$A_U(2n) = 2^n(-3)^{n^2} \prod_{\substack{1\leq i\leq 2n+1 \\ 1\leq k\leq n}} \frac{1+6k-3i}{2n+1+2k-i}. \tag{2.3}$$

Alternatively, thanks to their connection with vertically symmetric ASM's (VSASM's) or flip symmetric ASM's (FSASM's), and a recurrence relation for the number of the latter due to Robbins[R00], we have

$$A_U(2n) = A_U(2n-2)\binom{6n-2}{2n}\Big/\binom{4n-2}{2n}. \tag{2.4}$$

with $A_U(2) = 2$. In either case we obtain:

$$\begin{array}{cccccccc} n & 1 & 2 & 3 & 4 & 5 & 6 & \cdots \\ A_U(2n) & 2 & 2^2\cdot 3 & 2^3\cdot 26 & 2^4\cdot 646 & 2^5\cdot 45885 & 2^6\cdot 9304650 & \cdots \end{array} \tag{2.5}$$

Here we extend this idea to the case of μ-alternating sign matrices with a U-turn boundary. These were first defined in Hamel and King [HK02] in the context of deformations of Weyl's denominator formula for characters of the symplectic group and were called *sp(2n) generalised alternating sign matrices.*

Definition 2.1 *Let μ be a partition of length $\ell(\mu) = n$, all of whose parts are distinct, and for which $\mu_1 \leq m$. Then the matrix $A = (a_{iq})$ is said to belong to the set $\mathcal{UA}^\mu(2n)$ of μ-alternating sign matrices with a U-turn boundary if it is a $2n \times m$ matrix whose elements a_{iq} satisfy the conditions:*

$$\begin{array}{lll} \text{(A1)} & a_{iq} \in \{-1,0,1\} & \textit{for } 1\leq i\leq 2n,\ 1\leq q\leq m; \\ \text{(A2)} & \sum_{q=p}^{m} a_{iq} \in \{0,1\} & \textit{for } 1\leq i\leq 2n,\ 1\leq p\leq m; \\ \text{(A3)} & \sum_{i=j}^{2n} a_{iq} \in \{0,1\} & \textit{for } 1\leq j\leq 2n,\ 1\leq q\leq m. \\ \text{(A4)} & \sum_{q=1}^{m}(a_{2i-1,q}+a_{2i,q}) = 1 & \textit{for } 1\leq i\leq n; \\ \text{(A5)} & \sum_{i=1}^{2n} a_{iq} = & \begin{cases} 1 & \textit{if } q=\mu_k \textit{ for some } k \\ 0 & \textit{otherwise} \end{cases} \\ & & \textit{for } 1\leq q\leq m,\ 1\leq k\leq n. \end{array} \tag{2.6}$$

In the case for which $\mu = \delta = (n, n-1, \ldots, 1)$ and $m = n$, for which (A5) becomes $\sum_{i=1}^{2n} a_{iq} = 1$ for $1 \leq q \leq n$, this definition is such that the set $\mathcal{UA}^\delta(2n)$ coincides with the set of U-turn alternating sign matrices, UASM, defined by Kuperberg [K02]. The more general case is exemplified in the case of the partition $\mu = (9,7,6,2,1)$ and $n = 5$ the example in equation (2.7).

In the proof of the bijection between UASM and symplectic shifted tableaux in Section 5 it will be useful to refine the matrix UA. Any UASM UA contains two types of zeros: zeros for which there is a nearest non-zero element to the right in the same row taking the value 1 (positive zeros), and all other zeros (negative zeros). We can then define a map ϕ from the matrix UA to a signature matrix $\phi(UA)$, replacing positive zeros and positive ones with plus signs, and negative zeros and negative ones with minus signs. It should be noted that there is no ambiguity in determining which zeros are positive and which negative, so that for each UASM UA the signature matrix $\phi(UA)$ is unique. Moreover, to recover UA from $\phi(UA)$ by means of the inverse map ϕ^{-1} it is only necessary in each row to replace each rightmost $+$ in a continuous sequence of $+$'s by 1 and all others $+$'s by 0, and the rightmost $-$ of any continuous sequence of $-$'s by -1, provided that its immediate right hand neighbour is $+$, and all other $-$'s by 0. This is illustrated in the case of our example (2.7) by

$$UA = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bar{1} & 1 \\ \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{0} & 0 & 0 & 0 & \mathbf{0} & \mathbf{1} & 0 \\ \bar{1} & \mathbf{1} & 0 & \bar{1} & \mathbf{0} & 0 & \mathbf{1} & 0 & 0 \\ 1 & 0 & \bar{1} & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{1} & \mathbf{0} & 1 & 0 & 0 & 0 \\ 0 & \bar{1} & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 \\ \mathbf{0} & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 \\ \bar{1} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \Rightarrow \quad \phi(UA) = \begin{bmatrix} - & - & - & - & - & - & - & - & + \\ + & - & - & - & - & - & - & - & - \\ + & + & + & + & + & + & + & + & - \\ - & + & - & - & + & + & + & - & - \\ + & - & - & + & - & - & - & - & - \\ - & - & - & - & + & + & - & - & - \\ - & - & + & + & - & - & - & - & - \\ + & + & + & - & - & - & - & - & - \\ - & + & - & - & - & - & - & - & - \\ + & - & - & - & - & - & - & - & - \end{bmatrix} \tag{2.7}$$

We define three additional UASM parameters, also from Hamel and King [HK02]. The first is the signed sum of sites of special interest, or ssi. A *site of special interest*, (i,q), occurs in row i and column q for $q \neq 1$ if it satisfies the following properties. First, the nearest nonzero neighbours to the right and below $(i, q-1)$ are both equal to 1. And second, the entry at site $(i, q-1)$ is 0 if i is odd and 0 or -1 if i is even. If $q = 1$ then we redefine these so that i is even, *and*, either the entry at site $(i,1)$ is 1 *or* it is 0 with nearest nonzero neighbour to its right equal to 1. Then the parameter $ssi(A)$ is the signed sum of these elements:

$$ssi(A) = \#\{(2n+1-2k, q) \in S(A)\} - \#\{(2n+2-2k, q) \in S(A)\}. \tag{2.8}$$

The sites of special interest are indicated in boldface in equation (2.7).

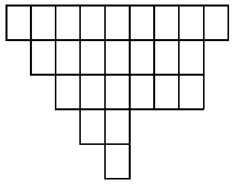
The second parameter is $bar(A)$. First we will define $m_k(A) = \sum_{q=1}^{m} q\, a_{2n+1-2k,q}$ and $m_{\bar{k}}(A) = \sum_{q=1}^{m} q\, a_{2n+2-2k,q}$. Then we define $bar(A) = \sum_{k=1}^{n} m_{\bar{k}}(A)$. Note also we can use the m's defined above to define an x–weighting for a μ–UASM; to wit, $x^{\mathrm{wgt}(A)} = x_1^{m_1(A)-m_{\bar{1}}(A)}$ $x_2^{m_2(A)-m_{\bar{2}}(A)} \cdots x_n^{m_n(A)-m_{\bar{n}}(A)}$. Recall also that $neg(A)$ is the number of -1's in A. In our running example the various parameters are: $ssi(A) = 0-1+3-3+0-1+1-2+0-1 = -4$, $bar(A) = 1+3+2+4+1 = 11$, $wgt(A) = x_1^0 x_2^{-1} x_3^0 x_4^4 x_5^0 = x_2^{-1} x_4^4$, $neg(A) = 7$.

3 Symplectic Shifted Tableaux

Symplectic shifted tableaux are variations on ordinary tableaux and were first introduced in [HK02] in the context of a symplectic version of Tokuyama's formula [T88]. A partition $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$ is a weakly decreasing sequence of non-negative integers. The weight, $|\mu|$, of the partition μ is the sum of its parts, and its length, $\ell(\mu) \leq n$, is the number of its non-zero parts. Now suppose all of the parts of μ are distinct. Define a shifted Young diagram SF^{μ} to be a set of $|\mu|$ boxes arranged in $\ell(\mu)$ rows of lengths μ_i that are left adjusted to a diagonal line. More formally, $SF^{\mu} = \{(i,j) \mid 1 \leq i \leq \ell(\mu), i \leq j \leq \mu_i + i - 1\}$.

For example, for $\mu = (9,7,6,2,1)$ we have

$$SF^\mu = \tag{3.1}$$



It should be noted that the parts of the partition $\mu' = (\mu'_1, \mu'_2, \ldots, \mu'_m)$, with $m = \mu_1$, which is conjugate to μ specify the lengths of successive diagonals of SF^μ. In the above example, $\mu' = (5,4,3,3,3,3,2,1,1)$. Quite generally, if all the parts of μ are distinct, it follows that successive parts of μ' differ by at most 1. In fact, in such a case we have

$$\mu'_{q+1} = \begin{cases} \mu'_q - 1 & \text{if } q = \mu_k \text{ for some } k \\ \mu'_q & \text{otherwise} \end{cases} \tag{3.2}$$

The symplectic shifted tableaux, $\mathcal{ST}^\mu(n,\overline{n})$, result from filling the boxes of SF^μ with integers from 1 to n and $\overline{1}$ to $\overline{n}$, ordered $\overline{1} < 1 < \overline{2} < 2 < \ldots < \overline{n} < n$, subject to a number of restrictions. We require a few more definitions. The *profile* of a shifted tableau is the sequence of entries on the main diagonal of the shifted tableau. Let A be a totally ordered set and let A^r be the set of all sequences $a = (a_1, a_2, \ldots, a_r)$ of elements of A of length r. Then the general set $\mathcal{ST}^\mu(A;a)$ is defined to be the set of all standard shifted tableaux, S, with respect to A, of profile a and shape μ, formed by placing an entry from A in each of the boxes of SF^μ in such that the following five properties hold:

$$\begin{array}{lll} \text{(S1)} & \eta_{ij} \in A & \text{for all } (i,j) \in SF^\mu; \\ \text{(S2)} & \eta_{ii} = a_i \in A & \text{for all } (i,i) \in SF^\mu; \\ \text{(S3)} & \eta_{ij} \leq \eta_{i,j+1} & \text{for all } (i,j),(i,j+1) \in SF^\mu; \\ \text{(S4)} & \eta_{ij} \leq \eta_{i+1,j} & \text{for all } (i,j),(i+1,j) \in SF^\mu; \\ \text{(S5)} & \eta_{ij} < \eta_{i+1,j+1} & \text{for all } (i,j),(i+1,j+1) \in SF^\mu. \end{array} \tag{3.3}$$

Informally we may describe these tableaux as having shifted shape and as being filled with entries from A with profile a such that the entries are weakly increasing from left to right across each row and from top to bottom down each column, and strictly increasing from top-left to bottom-right along each diagonal. Continuing the example above for $\mu = (9,7,6,2,1)$ we have typically a tableaux as in equation (3.5).

The set $\mathcal{ST}^\mu(n,\overline{n})$ of symplectic shifted tableaux is a specific instance of $\mathcal{ST}^\mu(A;a)$ defined:

Definition 3.1 *Let $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$ be a partition of length $\ell(\mu) = n$, all of whose parts are distinct. Then the set of all $sp(2n)$-standard shifted tableaux of shape μ is defined by:*

$$\mathcal{ST}^\mu(n,\overline{n}) = \{S \in \mathcal{ST}^\mu(A;a) \mid A = [n,\overline{n}],\ a \in [n,\overline{n}]^n \text{ with } a_i \in \{i,\overline{i}\} \text{ for } i = 1,2,\ldots,n\}, \tag{3.4}$$

where the entries η_{ij} of each $sp(2n)$-standard shifted tableau S satisfy the conditions (S1)-(S5) of (3.3), with the elements of $A = [n,\overline{n}] = \{1,2,\ldots,n\} \cup \{\overline{1},\overline{2},\ldots,\overline{n}\}$ subject to the order relations $\overline{1} < 1 < \overline{2} < 2 < \ldots < \overline{n} < n$.

Within a symplectic shifted tableau we can identify a further construct, namely, a ribbon strip [HK02].

Definition 3.2 *The ribbon strip,* $\mathrm{str}_k(S)$ *(respectively* $\mathrm{str}_{\overline{k}}(S)$*) consists of all boxes in the symplectic shifted tableau containing* k *(resp.* $\overline{k}$*) with no two such boxes on the same diagonal. Each ribbon strip may consist of one or more continuously connected parts.*

Each symplectic shifted tableaux is nothing other than a collection of ribbon strips wrapped around one another so as to produce a diagram of standard shifted shape. It follows that each $S \in \mathcal{ST}^{\mu}(n, \overline{n})$ may be encoded by means of a map ψ from S to a $2n \times m$ matrix $\psi(S)$, with $m = \mu_1$, in which the rows of $\psi(S)$, specified by k and $\overline{k}$ taken in reverse order from n at the top to $\overline{1}$ at the bottom, consist of a sequence of symbols $+$ or $-$ in the qth column of $\psi(S)$, counted from 1 on the left to m on the right, indicating whether or not $\mathrm{str}_k(S)$ and $\mathrm{str}_{\overline{k}}(S)$, as appropriate, intersects the qth diagonal of S, where diagonals are counted in the north-east direction starting from the main, first diagonal to which the rows of S are left-adjusted. Typically, applying ψ to our example (3.5) for S gives $\psi(S)$ as shown:

$$S = \begin{array}{ccccccccc} \overline{1} & 1 & \overline{2} & 2 & \overline{3} & \overline{3} & \overline{4} & 4 & 5 \\ & \overline{2} & \overline{2} & 2 & 3 & \overline{4} & \overline{4} & 4 & \\ & & 3 & \overline{4} & 4 & 4 & 4 & 4 & \\ & & & 4 & 4 & & & & \\ & & & & \overline{5} & & & & \end{array} \quad \Rightarrow \quad \psi(S) = \begin{array}{c} \begin{bmatrix} - & - & - & - & - & - & - & - & + \\ + & - & - & - & - & - & - & - & - \\ + & + & + & + & + & + & + & + & - \\ - & + & - & - & + & + & + & - & - \\ + & - & - & + & - & - & - & - & - \\ - & - & - & - & + & + & - & - & - \\ - & - & + & + & - & - & - & - & - \\ + & + & + & - & - & - & - & - & - \\ - & + & - & - & - & - & - & - & - \\ + & - & - & - & - & - & - & - & - \end{bmatrix} \end{array} \begin{array}{c} 5 \\ \overline{5} \\ 4 \\ \overline{4} \\ 3 \\ \overline{3} \\ 2 \\ \overline{2} \\ 1 \\ \overline{1} \end{array} \tag{3.5}$$

Clearly $\psi(S)$ is uniquely determined by S and *vice versa.* The inverse map ψ^{-1} from $\psi(S)$ back to S is accomplished by noting that the elements $+$ in each column of $\psi(S)$ simply signify by virtue of their row label, k or $\overline{k}$, those entries that appear in the corresponding diagonal of S, arranged in strictly increasing order.

The strips $\mathrm{str}_k(S)$ and $\mathrm{str}_{\overline{k}}(S)$, whose connected components are well represented by sequences of consecutive $+$'s in $\psi(S)$, play a key role in establishing the bijection between symplectic shifted tableaux and alternating sign matrices with a U–turn boundary.

A related element is the ordinary symplectic tableau [K76, KEl-S83, S89]. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ be a partition of length r. Then the set of all $sp(n)$-standard tableaux of shape λ is defined by

$$\begin{aligned} \mathcal{T}^{\lambda}(sp(2n)) \;&=\; \{T \in \mathcal{T}^{\lambda}(A; a) \mid A = [n, \overline{n}], a \in [n, \overline{n}]^r \\ &\qquad \text{with } a_i \geq i \text{ for } i = 1, 2, \ldots, r\}, \end{aligned} \tag{3.6}$$

where the entries η_{ij} of each $sp(2n)$-standard tableau T with $A = [n, \overline{n}] = \{1, 2, \ldots, n\} \cup \{\overline{1}, \overline{2}, \ldots, \overline{n}\}$, and the elements of $[n, \overline{n}]$ subject to the order relations $\overline{1} < 1 < \overline{2} < 2 < \ldots < \overline{n} < n$ satisfy the following conditions:

$$\begin{array}{lll} \text{(T1)} & \eta_{ij} \in A & \text{for all } (i,j) \in T; \\ \text{(T2)} & \eta_{i1} = a_i \in A & \text{for all } (i,1) \in T; \\ \text{(T3)} & \eta_{ij} \leq \eta_{i,j+1} & \text{for all } (i,j), (i,j+1) \in T; \\ \text{(T4)} & \eta_{ij} < \eta_{i+1,j} & \text{for all } (i,j), (i+1,j) \in T. \end{array} \tag{3.7}$$

Then the symplectic Schur function [K76, KEl-S83, S89] can be defined as

$$sp_\lambda(x) = sp_\lambda(x_1, x_2, \ldots, x_n) = \sum_{T \in \mathcal{T}^\lambda(sp(2n))} x^{\mathrm{wgt}(T)},$$

where the sum is now over all $sp(2n)$-standard tableaux T of shape λ and $x^{\mathrm{wgt}(T)} = x_1^{m_1(T)-m_{\overline{1}}(T)}\ x_2^{m_2(T)-m_{\overline{2}}(T)}\ \cdots\ x_n^{m_n(T)-m_{\overline{n}}(T)}$ with $m_k(T)$ and $m_{\overline{k}}(T)$ equal to the number of entries k and $\overline{k}$, respectively, in T. We have found it useful to introduce a factor of t^2 to yield $sp_\lambda(x;t) = \sum_{T \in \mathcal{T}^\lambda(sp(2n))} t^{2\mathrm{bar}(T)}\ x^{\mathrm{wgt}(T)}$, where $\mathrm{bar}(T)$ is the number of barred entries in T; that is $\mathrm{bar}(T) = \sum_{k=1}^n m_{\overline{k}}(T)$.

4 Square Ice

Square ice is a physics model that has proved to be an invaluable tool in the area of alternating sign matrices. Square ice is a two dimensional grid that models the orientation of hydrogen and oxygen molecules in frozen water (ref. Leib [L67], Bressoud [B99], Lascoux [L99]). The square ice graph is an $(n+2) \times (n+2)$ array of vertices such that each internal vertex has four directed edges and each noncorner boundary vertex has one directed edge. The corner vertices have no edges. Each internal vertex has one of six possible degree patterns given in Figure 1. Each left or right noncorner boundary vertex has an edge pointing towards the adjacent internal vertex. Each top or bottom noncorner boundary vertex has an edge pointing away from the adjacent internal vertex. These vertices are easily associated with entries in an ASM, as seen in Figure 1. Using this association Kuperberg employed known results in square ice to provide a second proof of the alternating sign matrix conjecture [K96].

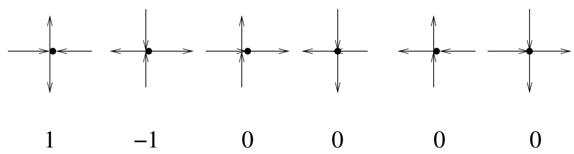


Figure 1: Six possible square ice configurations

The concept of square ice can be generalized slightly to account for U–turns and zero sum columns of μ–UASM. A zero sum in column p corrresponds to square ice models with incoming rather than outgoing edges at the top boundary in column p. A U–turn corresponds to either an outgoing left boundary at row $2i-1$ and an incoming left boundary at row $2i$, or an incoming left boundary at row $2i-1$ and an outgoing left boundary at row $2i$ (see Figure 2). These can also be represented as U–turns [K02]. Then with these changes in convention we can map the six types of vertices to 1's, -1's, and 0's exactly as before and produce μ–UASM. It is also worth noting that with the additional refinement of square ice we are able to distinguish immediately our positive and negative zeros. The positive zeros correspond to square ice vertices in which the horizontal arrows point right; the negative zeros correspond to square ice vertices in which the horizontal arrows point left. If we also include the effect of the vertical arrows, we can distinguish northeast (NE) and southeast (SE) positive zeros whose vertical arrows point up (north), and northwest (NW) and southwest (SW) negative zeros whose vertical arrows point down (south).

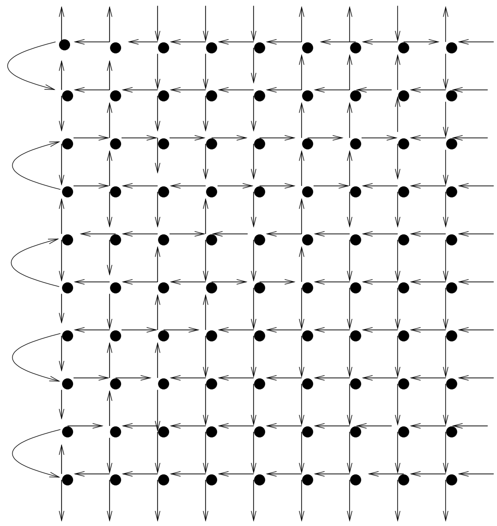


Figure 2: Square ice diagram for μ–UASM in equation 2.7. Only the internal vertices are shown.

5 The Bijection

In Hamel and King [HK02], we derived a relationship between UASM and symplectic shifted tableaux by first going through monotone triangles. Here we prove the relationship directly. We will find it useful to use the refinement of the UASM defined by ϕ.

Since the image $\psi(S)$ of ψ acting on each symplectic shifted tableaux S is a matrix of $\pm$'s, the inverse ϕ^{-1} may be applied to $\psi(S)$ to give a matrix of 1's, $\overline{1}$'s and 0's, which may or may not be a U–turn alternating sign matrix, UA. In fact the resulting matrix $\phi^{-1}\circ\psi(S)$ is always a U–turn alternating sign matrix, and it is shown below in Theorem 5.1 that the map $\Psi = \phi^{-1}\circ\psi$ is a bijective mapping from $\mathcal{ST}^{\mu}(n,\overline{n})$ to $\mathcal{UA}^{\mu}(2n)$.

In the case of our example, the outcome of this procedure mapping from S to $\psi(S)$, identifying $\psi(S)$ with $\phi(UA)$, and then recovering $UA = \phi^{-1}\circ\psi(S) = \Psi(S)$ is illustrated by:

$$
\begin{array}{ccccccccc}
\boxed{\overline{1}} & \boxed{1} & \boxed{\overline{2}} & \boxed{2} & \boxed{\overline{3}} & \boxed{\overline{3}} & \boxed{\overline{4}} & \boxed{4} & \boxed{5} \\
 & \boxed{\overline{2}} & \boxed{\overline{2}} & \boxed{2} & \boxed{3} & \boxed{\overline{4}} & \boxed{\overline{4}} & \boxed{4} & \\
 & & \boxed{3} & \boxed{\overline{4}} & \boxed{4} & \boxed{4} & \boxed{4} & \boxed{4} & \\
 & & & \boxed{4} & \boxed{4} & & & & \\
 & & & & \boxed{\overline{5}} & & & &
\end{array}
\Longrightarrow
\begin{bmatrix}
- & - & - & - & - & - & - & - & + \\
+ & - & - & - & - & - & - & - & - \\
+ & + & + & + & + & + & + & + & - \\
- & + & - & - & + & + & + & - & - \\
+ & - & - & + & - & - & - & - & - \\
- & - & - & - & + & + & - & - & - \\
- & - & + & + & - & - & - & - & - \\
+ & + & + & - & - & - & - & - & - \\
- & + & - & - & - & - & - & - & - \\
+ & - & - & - & - & - & - & - & -
\end{bmatrix}
\Longrightarrow
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & \overline{1} & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
\overline{1} & 1 & 0 & \overline{1} & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & \overline{1} & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \overline{1} & 0 & 1 & 0 & 0 & 0 \\
0 & \overline{1} & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\overline{1} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\tag{5.2}
$$

where the rows of the matrices are labelled from top to bottom $n = 5, \overline{5}, 4, \overline{4}, 3, \overline{3}, 2, \overline{2}, 1, \overline{1}$, and the columns from left to right $1, 2, \ldots, 9 = m = \mu_1$.

Theorem 5.1 *Let $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$ be a partition of length $\ell(\mu) = n$ whose parts are all distinct. Then the mapping $\Psi = \phi^{-1}\circ\psi$ defines a bijection between the set $\mathcal{ST}^{\mu}(n,\overline{n})$ of $sp(2n)$-*

standard shifted tableaux S of shape SF^μ, and the set $\mathcal{UA}^\mu(2n)$ of $2n \times m$ μ-alternating sign matrices UA with a U-turn boundary and $m = \mu_1$.

Proof: The Definition 3.1 of $\mathcal{ST}^\mu(n,\overline{n})$ ensures that each $sp(2n)$-standard shifted tableau S satisfies the properties (S1)-(S5). We need to show, in accordance with the Definition 2.1 of $\mathcal{UA}^\mu(2n)$, that the properties (A1)-(A5) hold for the matrix $UA = \Psi(S)$ obtained from S by means of the map Ψ.

First, it is obvious from the description of the mappings involved that the only possible matrix elements of UA are 1, -1, and 0. Thus (A1) holds.

Conditions (S3)-(S5) imply that each diagonal S contains no repeated entries, leading to the observation that S consists of a union of ribbon strips as described in Definition 3.2. The map from S to the matrix $\psi(S)$ is then such that reading across each row of the matrix $\phi(UA)$ gives sequences of $+$'s corresponding to each connected component of the relevant ribbon strip. The matrix $\psi(S)$ is now to be identified with $\phi(UA)$ for some UA. The fact that the rightmost $+$ of each sequence of consecutive $+$'s in $\phi(UA)$ is mapped to an element 1 in UA, and that the rightmost $-$ of each sequence of consecutive $-$'s is mapped to an element -1, provided that such a $-$ is followed by a $+$, means that across each row of the resulting matrix UA we have non-zero entries 1 and -1 that alternate in sign, with the rightmost non-zero entry always 1. This implies the validity of (A2).

To establish the U–turn nature of UA it is necessary to invoke condition (S2) and the fact that S is standard only if the entry $\eta_{ii} = a_i$ in the ith box of the leading diagonal of S is either i or $\overline{i}$. The map from S to $\psi(S)$ is then such that the elements in the first column of the ith and $\overline{i}$th rows are different, one is always $+$ and the other always $-$. Identifying $\psi(S)$ with $\phi(UA)$, the first non-zero entries, if they exist, in the corresponding ith and $\overline{i}$th rows of UA must also differ, one being 1 and the other -1. This is sufficient to show that the U–turn sequence obtained by reading across the ith row from right to left and then back along the $\overline{i}$th row from left to right is an alternating sequence of 1's and -1's. The fact that in both rows the rightmost non-zero element must be 1 then ensures the validity of (A4) since this U–turn alternating sign sequence begins and ends with 1. If on the other hand either i or $\overline{i}$ is not present in S, then the corresponding row of $\psi(S)$ will consist wholly of $-$'s, and identifying $\psi(S)$ with $\phi(UA)$ leads to the conclusion that the corresponding row of UA consists solely of 0's, containing no non-zero elements and making no contribution to the U-turn sequence. However, the other row of the pair i and $\overline{i}$ in $\psi(S)$ must start with a $+$ thereby ensuring that the first non-zero entry in the corresponding row of UA must be 1. Since the last non-zero element is also 1, the row sum is 1 and the U–turn condition (A4) holds yet again.

To deal with (A3), we consider the diagonals of S. To this end the following schematic diagrams of various portions of the qth and $(q+1)$th diagonals of S will prove to be helpful.

$$D_1 = \begin{array}{|c|c|cccc} \hline i & b & & & & \\ \hline & a & b & & & \\ & & a & b & & \\ & & & a & b & \\ & & & & a & b \\ & & & & & j \\ \end{array} \qquad D_2 = \begin{array}{cccccc} i & & & & & \\ a & b & & & & \\ & a & b & & & \\ & & a & b & & \\ & & & a & b & \\ & & & & a & j \\ \end{array} \qquad D_3 = \begin{array}{ccccc} \hline a & b & & & \\ & a & b & & \\ & & a & b & \\ & & & a & j \\ \end{array} \qquad (5.3)$$

In these diagrams the labels i and j are the actual entries in the corresponding boxes of S, which may of course be barred or unbarred, while the rules (S3)-(S5) of (3.3) are such that the actual entries of S in the boxes labeled by a are all distinct, as are those in the boxes labeled by b. Moreover, in each case that we will consider each such entry k will necessarily be such that $i < k < j$. We use the notation n_a and n_b to indicate the number of entries a and b, respectively.

All elements 1 in the qth column of the matrix UA constructed from S by means of the map Ψ correspond to connected components of ribbon strips of S terminating in the qth diagonal, by virtue of their connection with rightmost $+$'s in continuous sequences of $+$'s in the rows of $\psi(S) = \phi(UA)$. Similarly all elements -1 in the qth column of the matrix UA correspond to connected components of ribbon strips starting in the $q+1$th diagonal, by virtue of their connection with the rightmost $-$'s immediately preceding a $+$ in the rows of $\psi(S) = \phi(UA)$. To see that these non-zero elements in the qth column of UA necessarily alternate in sign, consider two consecutive 1's and the corresponding boxes on the qth diagonal of S. In the schematic diagram D_1 above, these have been labeled by their entries i and j (which could be barred or unbarred entries). They correspond to the termination of connected components of the strips $\mathrm{str}_i(S)$ and $\mathrm{str}_j(S)$ in the qth diagonal of S. All n_a boxes on the qth diagonal between these i and j boxes, labeled in D_1 by a, must be labeled in S itself by n_a distinct entries k with $i < k < j$. Similarly all n_b boxes on the $(q+1)$th diagonal to the right of i and above j, labeled in D_1 by b, must also be labeled in S by distinct entries k with $i < k < j$. Since $n_b = n_a + 1$ it follows that at least one b-label must be distinct from all a-labels. If this label is k, then a connected component of $\mathrm{str}_k(S)$ must start in the $(q+1)$th column with no component in the qth column. This leads in the kth row of $\psi(S)$ to a $-$ followed by a $+$, and hence to an element -1 in the qth column of UA, between the two 1's associated with the boxes i and j.

Similarly, between any two -1's in the qth column of UA there must exist an element 1. The proof is based on the diagram D_2 above. The boxes labeled i and j in the $(q+1)$th diagonal of S specify the start of connected components of $\mathrm{str}_i(S)$ and $\mathrm{str}_j(S)$ not present in the qth diagonal. Once again in D_2 the set of n_a boxes labeled by a and the set of n_b boxes labeled by b must each have distinct labels k with $i < k < j$ in S itself. Then $n_b = n_a - 1$ so that there exists k, with $i < k < j$ such that $\mathrm{str}_k(S)$ terminates in the qth column of S, leading to a rightmost $+$ in $\psi(S) = \phi(UA)$ and hence to a 1 in the qth column of UA lying between the two -1's associated with the i and j boxes of S.

This is not sufficient to prove that (A3) holds. It is necessary to prove further that the lowest non-zero entry in every column of UA is 1. The argument is very much as before. It should be noted that top and bottom are reversed in passing from S to $\psi(S)$. We consider the case of an entry -1 in the row corresponding to the label j of the qth column of UA and argue that there must exist an entry 1 in the row corresponding to the label k of the qth column of UA with $k < j$. This follows from the schematic diagram D_3 which is truncated at its top end by the boundary of S. Again $n_b = n_a - 1$ so that there exists k, with $k < j$ such that $\mathrm{str}_k(S)$ terminates in the qth column of S, leading to a rightmost $+$ in $\psi(S) = \phi(UA)$ and hence to a 1 in the qth column of UA lying below the -1 associated with the j box of S. This applies to any element -1 in UA so the lowest non-zero element of UA must be 1. In combination with the fact that, as we have proved, the signs of the non-zero elements are alternating in the columns of UA, this serves to complete the proof that (A3) holds.

The final argument in respect of (A5) is very similar. The relevant diagrams are as follows.

$$D_4 = \begin{array}{cccccc} a & b & & & & \\ & a & b & & & \\ & & a & b & & \\ & & & a & b & \\ & & & & a & * \end{array} \qquad D_5 = \begin{array}{cccccc} a & b & & & & \\ & a & b & & & \\ & & a & b & & \\ & & & a & b & \\ & & & & a & b \\ & & & & & * \end{array} \tag{5.4}$$

First we consider those diagonals q of S which end with the rightmost box of some row, that is those diagonals q such that $q = \mu_k$ for some k with $1 \le k \le n$. As can be seen from the diagram D_4, in which the entry $*$ indicates an empty box just beyond the end of the kth row containing the final a on the qth diagonal, the lengths $n_a = \mu'_q$ and $n_b = \mu'_{q+1}$ of the qth and $(q+1)$th diagonals of S, respectively, are such that $n_b = n_a - 1$. This is in accordance with (3.2). Since the actual entries in S corresponding to the n_a a's are all distinct, as are the entries corresponding to the n_b b's, it follows that there exists precisely one more connected component of the strips $\mathrm{str}_i(S)$ that terminate in the qth diagonal of S than the number of connected components of strips $\mathrm{str}_j(S)$ that start in the $(q+1)$th diagonal. Since it is the former that lead to all the 1's in the qth column of UA and the latter to all the -1's in the same column, the sum of the entries in this column must be 1.

Similarly, we consider those diagonals q of S which do not end with the rightmost box of any row, that is those diagonals q such that $q \neq \mu_k$ for any k. In this case the relevant schematic diagram is D_5 in which the entry $*$ signifies an empty box just beyond the final a in the qth diagonal. Since this a is not at the end of any row, it has a right hand neighbour b, which must lie at the end of the $(q+1)$th diagonal. As can be seen from D_5, in accordance with (3.2), the lengths $n_a = \mu'_q$ and $n_b = \mu'_{q+1}$ of the qth and $(q+1)$th diagonals of S, respectively, are such that $n_b = n_a$. This ensure that the number of connected components of strips $\mathrm{str}_i(S)$ that terminate in the qth diagonal of S is equal to the number of connected components of strips $\mathrm{str}_j(S)$ that start in the $(q+1)$th diagonal. Once again, since it is the former that lead to all the 1's in the qth column of UA and the latter to all the -1's in the same column, the sum of the entries in this column must be 0.

Taken together these last two results imply that (A5) holds in all cases, thereby completing the proof that for all $S \in \mathcal{ST}^\mu(n,\overline{n})$ we have $UA = \Psi(S) \in \mathcal{UA}^\mu(2n)$.

To save space we omit the proof of the inverse of the bijection. The arguments are similar to those already presented.

6 Weighted Enumeration

Propp [Pr00] has provided data for and made a number of conjectures about weighted enumeration of UASM. Eisenkölbl [E02] has proved several of these conjectures, and many are derivable from Kuperberg [K02]. Here we delineate a new family of weighted enumerations of UASM through μ–ASM and symplectic shifted tableaux and show their overlap with the results of Propp.

Traditionally tableaux are weighted by a product of x's chosen from $\{x_1, \ldots, x_n\}$ such that there is one x_i in the product for every i in the tableau. Symplectic tableau additionally have weights chosen from $\{x_1^{-1}, \ldots, x_n^{-1}\}$ where x_i^{-1} corresponds to $\bar{\imath}$ (as in Section 3). These weightings make it possible to define symmetric functions and symmetric function identities using tableaux.

However, there is an additional type of weighting that can be assigned to tableaux, one that is useful for determining t–deformations of symmetric function formulae, e.g. Tokuyama's formula. In our recent paper [HK02], we found the following t–dependence useful in describing a symplectic version of Tokuyama's formula. The t–dependence translates quite nicely to μ–UASM and we make connections between it and other known ASM weightings and derive a UASM counting formula.

The t–dependence of a shifted symplectic tableau is easily described. Each entry k belongs to one ribbon strip as defined in Definition 3.2. The t–weight of k is 1 if in its strip the box immediately preceding it is to its left, and t if in its strip the box immediately preceding it is below. If the element k is at the beginning of a connected component of a strip then it is given a weighting of 1, and if the strip does not start on the main diagonal then it is given a boldface weighting of **1** signifying a weighting of $(1+t)$. Similarly, the t–weight of $\overline{k}$ is t if in its strip the box immediately preceding it is to its left, and t^2 if in its strip the box immediately preceding it is below. If the element $\overline{k}$ is at the beginning of a connected component of a strip then it is given a weighting of t, and if the strip does not start on the main diagonal then it is given a boldface weighting of $\mathbf{t}$ signifying a weighting of $(t+t^2)$.

In terms of μ–UASM the t–weighting may be read off in the following manner. First, replace each -1 by a boldface **0** signifying a weighting of $(1+t)$. Second, in odd rows (counted from the top) associated with k for $1 \le k \le n$, replace each entry 1 and the immediately preceding continuous string of 0's (the positive zeros) by 1 unless the entry to the immediate left of the 1 in question is a 0 with nearest nonzero neighbours to the right and below both equal to 1, in which case the entry is t. Third, in the even rows (counted from the top) associated with $\overline{k}$ for $1 \le k \le n$, replace each entry 1 and the immediately preceding continuous string of 0's (positive zeros) by t^2 unless the entry is either in the first column or immediately to the right of a $\overline{1}$, or immediately to the right of a 0 with nearest nonzero neighbours to the right and below both equal to 1, in which case the entry is t rather than t^2. Note that the sites of special interest correspond to the location of the t's in this t–weighting (e.g. compare the boldface entries in equation (2.7) and the t's in Figure 3). Using square ice we can further refine the t–weighting on the zeros so as to distinguish not just positive and negative zeros but also NE positive zeros and SE positive zeros. This makes the result even more transparent as we just need to identify $\overline{1}$'s, 1's, NE 0's, and SE 0's rather than considering continuous strings and nearest neighbours. This refined weighting translates to:

$$(1+t)^{\#\overline{1}} t^{\#NE} (x_k)^{\#SE+\#NE+\#1}$$

for rows associated with x_k (odd rows counted from the top), and

$$(1+t)^{\#\overline{1}} t^{\#SE} (tx_k)^{\#SE+\#NE+\#1}$$

for rows associated with $\overline{x}_k$ (even rows counted from the top).

In our previous paper [HK02]we derived the following extension to Tokuyama's formula:

Theorem 6.1 *Let λ be a partition into no more that n parts and let δ be the partition $(n, n-1, \ldots, 1)$, then*

$$D_{sp(2n)}(x;t)\ sp_\lambda(x;t) \quad = \sum_{S \in \mathcal{ST}^{\lambda+\delta}(n,\overline{n})} t^{\mathrm{hgt}(S)+2\mathrm{bar}(S)}\ (1+t)^{\mathrm{str}(S)-n}\ x^{\mathrm{wgt}(S)} \tag{6.2}$$

where the summation is taken over all $sp(2n)$-standard shifted tableaux S of shape $\lambda+\delta$. The notation is such that $\mathrm{bar}(S)$ *is the total number of barred entries in S,* $\mathrm{str}(S)$ *is the total number*

$$S(t) = \begin{array}{ccccccccc} t & \mathbf{1} & t & t & \mathbf{t} & t^2 & t & t & \mathbf{1} \\ & t & t^2 & \mathbf{1} & \mathbf{1} & \mathbf{t} & t^2 & t & \\ & & 1 & \mathbf{t} & t & 1 & 1 & 1 & \\ & & & 1 & 1 & & & & \\ & & & & t & & & & \end{array} \quad \Longleftrightarrow \quad UA(t) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{0} & 1 \\ t & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & t & 1 & 1 & 1 & t & t & 0 \\ \mathbf{0} & t & 0 & \mathbf{0} & t & t^2 & t & 0 & 0 \\ 1 & 0 & \mathbf{0} & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{0} & t & t^2 & 0 & 0 & 0 \\ 0 & \mathbf{0} & 1 & t & 0 & 0 & 0 & 0 & 0 \\ t & t^2 & t & 0 & 0 & 0 & 0 & 0 & 0 \\ \mathbf{0} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ t & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Figure 3: Our example with a t–weighting.

of connected components of all the ribbon strips of S and $\mathrm{hgt}(S) = \sum_{k=1}^{n} \big(\mathrm{row}_k(S) - \mathrm{con}_k(S) - \mathrm{row}_{\overline{k}}(S)\big)$*, where* $\mathrm{row}_k(S)$ *and* $\mathrm{row}_{\overline{k}}(S)$ *are the numbers of rows of S containing an entry k and $\overline{k}$, respectively, and* $\mathrm{con}_k(S)$ *is the number of connected components of the ribbon strip of S consisting of all the entries k, while $x^{\mathrm{wgt}(S)}$ is defined as in Section 3 with the tableau T replaced by the shifted tableau S.*

Recall sp_λ is the t–weighted symplectic Schur function and note that

$$D_{sp(2n)}(x;t) = \prod_{1\le i\le n} x_i^{n-i+1} \prod_{1\le i\le n} (1+tx_i^{-2}) \prod_{1\le i<j\le n} (1+tx_i^{-1}x_j)(1+tx_i^{-1}x_j^{-1}). \tag{6.3}$$

In terms of UASM this theorem becomes:

Theorem 6.2 *Let λ be a partition into no more than n parts and let δ be the partition $(n, n-1, \ldots, 1)$, then*

$$D_{sp(2n)}(x;t)\ sp_\lambda(x;t) \quad = \sum_{A\in\mathcal{A}^{\lambda+\delta}(2n)} t^{\mathrm{ssi}(A)+2\mathrm{bar}(A)}\ (1+t)^{\mathrm{neg}(A)}\ x^{\mathrm{wgt}(A)} \tag{6.4}$$

where the summation is taken over all μ–U–turn alternating sign matrices of shape $2n\times(\lambda_1+n+1)$, whose non-vanishing column sums are 1 or 0 according as the column number is or is not a part of $\lambda+\delta$.

By setting $x_1 = x_2 = x_3 = \ldots = x_n = 1$ in this formula we derive the following results for μ–UASM and UASM:

$$(1+t)^{n^2} = \frac{\sum_{A\in\mathcal{UA}^{\lambda+\delta}} t^{ssi(A)+2bar(A)}(1+t)^{neg(A)}}{sp_\lambda(1;t)}; \qquad (1+t)^{n^2} = \sum_{A\in\mathcal{UA}^{\delta(\in\backslash)}} t^{ssi(A)+2bar(A)}(1+t)^{neg(A)}. \tag{6.5}$$

The UASM part of equation (6.5) with $t=1$ reduces to $\sum_{A\in\mathcal{UA}} 2^{neg(A)} = 2^{n^2}$.

The μ–UASM part of equation (6.5) could further be reduced with the evaluation of $sp_\lambda(1;t)$. It is known [W25, W26, ElSK79] that $sp_\lambda(1;1)$ simplifies to:

$$\prod_{1\le i<j\le m} \frac{\lambda_i - i - \lambda_j + j}{j-i} \prod_{1\le i\le j\le m} \frac{\lambda_i+\lambda_j+m-i-j+2}{m+2-i-j},$$

where m is the number of x variables set to 1. However, as far as we know no comparable formula for $sp_\lambda(1;t)$ has yet been found.

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