THE DEGREE DISTRIBUTION IN BIPARTITE PLANAR MAPS: APPLICATIONS TO THE ISING MODEL

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Abstract. We characterize the generating function of bipartite planar maps counted according to the degree distribution of their black and white vertices. This result is applied to the solution of the hard particle and Ising models on random planar lattices with arbitrary degree distribution. We thus recover and extend some results previously obtained by means of matrix integrals.

Proofs are purely combinatorial and rely on the idea that planar maps are conjugacy classes of trees. In particular, these trees explain why the solutions of the Ising and hard particle models on maps of bounded degree are always algebraic.

Résumé. Nous calculons la série génératrice des cartes planaires biparties comptées selon la distribution des degrés de leurs sommets noirs et blancs. Ceci permet de résoudre le modèle des particules dures et le modèle d'Ising sur cartes planaires. Nous retrouvons ainsi et nous étendons des résultats auparavant obtenus à l'aide d'intégrales de matrices.

Nos preuves sont combinatoires et reposent sur l'idée selon laquelle les cartes planaires sont des classes de conjugaison d'arbres. En particulier ces arbres permettent de comprendre l'algébricité des séries génératrices du modèle des particules dures et du modèle d'Ising sur cartes de degré borné.

1. Introduction

The enumeration of planar maps (*i.e.*, connected graphs drawn on the sphere with non-intersecting edges) has received a lot of attention since the 60's. Four decades of exploration have resulted into three main enumeration techniques, and two types of results: first, a few remarkable families of maps are counted by very nice and simple numbers; second, and more generally, many families of maps share the characteristic of having an *algebraic generating function*. Meanwhile, many exciting connections have been established between maps and various branches of mathematics, like Grothendieck's theory of "dessins d'enfants" or knot theory, and, most importantly for this paper, between maps and certain branches of physics.

This paper presents a combinatorial approach for solving two types of physics models on maps: the *Ising* and *hard particle* models. In these models, the vertices of the maps carry *spins* or *particles*; their solution is equivalent to a weighted enumeration of maps. The weight is in general a specialization of the Tutte polynomial of the map.

The idea of putting such a model on maps originates in physics, and more precisely in two-dimensional *quantum gravity*: in this theory, maps arise as discrete models of geometries, or *random planar lattices*, and a lattice not carrying any spin or particle is only moderately interesting. As can be observed in numerous occasions, physicists are not *only* good at doing physics: in this case, Brézin *et al.*, in the steps of t'Hooft [26], developed a new approach for counting maps, completely different from what had been done previously in combinatorics. This approach, suggested by quantum field theory, is based on the evaluation of matrix integrals with well chosen potentials [3, 10]. An introduction to these techniques, intended for mathematicians, is presented in [31], and a far reaching account can be found in [14].

This approach is extremely powerful: it allows to produce quickly expressions for generating functions without requiring much invention at the combinatorial level. In particular, the Ising model on tri- and on tetravalent maps (maps with vertices of degree 3 or 4) was solved via this approach in the late 80's, yielding intriguing algebraic generating functions [4, 19, 21]; the same kind of technique was applied much more recently to the hard particle model on a larger variety of maps[1] [7]. However, the evaluation of the matrix integrals as presented in most papers is not satisfying from the mathematical point of view: justifications of the calculations are often omitted, whereas they involve non-trivial complex analysis and resummation

This is an extended abstract. The full text is available as `arXiv:math.CO/0211070`.

[1]The hard particle model can be seen as a specialization of the Ising model in a magnetic field, see below for details.

issues. Moreover, this approach gives little insight on the combinatorial structure of maps and does not explain the algebraic nature of the generating functions thus obtained.

The material presented in this paper provides an alternative approach to the Ising and hard particle models, and has none of the above mentioned drawbacks. It is essentially of a bijective nature, and only involves elementary combinatorial arguments. We show that certain families of plane trees are at the heart of both models, and this justifies the algebraicity of their solutions[2]. We do not study the singularities of the series we obtain, even though they are very significant from the physics point of view: this has already been done in [4, 7, 19], and is anyway routine for algebraic series.

The central objects of our paper are actually the bipartite maps, which we enumerate according to the degree distribution of their black and white vertices. We thus extend the results of [5] on the so-called constellations, and also the results of [8] on the degree distribution of (unicolored) planar maps (the latter being in one-to-one correspondence with bipartite maps in which all black vertices have degree two). Then, we show that the solutions of the Ising and hard particle models are closely related to the enumeration of bipartite maps.

Our approach provides a new illustration of the principle according to which planar maps are, in essence, *conjugacy classes of trees*. This idea was introduced by the second author of this paper in order to explain bijectively certain nice formulae for the number of maps [24, 25]. Then, it led to a common generalization of formulae of Tutte and Hurwitz [5], before being adapted to maps satisfying 2- and 3-connectivity constraints [22, 23]. A few months ago, Bouttier *et al.* cleverly built on the same idea to give a bijective derivation of Bender and Canfield's solution of the very general problem of counting maps by the degree distribution of their vertices [2, 8]. Moreover, a few days after the publication of [6], these authors proposed an independent but very similar combinatorial solution for the hard particle model on tetravalent maps [9].

Let us mention that the seminal work of Tutte on the enumeration of maps was not based on bijections, but on recursive decompositions of maps, which led to functional equations for their generating functions. This type of approach is usually fairly systematic; combined with the so-called quadratic method and its generalizations, it has provided many algebraicity results, including recent ones [2, 11, 16, 27, 28]. It yields short self-contained proofs, relying only on elementary decompositions and algebraic calculus in the realm of formal power series. However, as far as we know, there has been very few successful attempts to apply this method to the enumeration of maps carrying an additional structure, like an Ising model. Roughly speaking, the standard decompositions of maps still provide certain functional equations, but these are much harder to solve that in the previous cases. This is perfectly illustrated by the *tour de force* achieved by Tutte, who spent 10 years solving the equation he had established for the chromatic polynomial of triangulations [29].

For the sake of completeness, let us mention a few additional results and techniques. The first bijective proofs in map enumeration are due to Cori [12] and later to Arquès [1]. Some models involving self-avoiding walks on maps can essentially be solved without matrix integrals [15], by using enumeration results due to Tutte. Then, a cluster expansion method has been developed very recently for a general model of spins with neighbor interactions on maps [20]. However, this provides only partial qualitative results and does not allow to recover the critical exponents of the hard particle and Ising models (whereas they can be derived from the generating functions obtained either with our approach or with matrix integrals). Finally, the enumerative theory of ramified coverings of the sphere has led a number of authors to consider a very refined enumeration of maps on higher genus surfaces, in which the degrees of vertices *and* faces are taken into account. The relevant methods rely on the encoding of maps by permutations [13] and lead to expressions involving characters of the symmetric group [17, 18]. However, to the best of our knowledge, none of the simple generating functions of planar maps have ever been rederived from these expressions.

2. A glimpse at the results

To begin with, let us recall a few definitions and conventions. A planar map is a 2-cell decomposition of the oriented sphere into vertices (0-cells), edges (1-cells) and faces (2-cells). In more vernacular terms, it is a connected graph drawn on a sphere with non-intersecting edges. Loops and multiple edges are allowed. Two maps are isomorphic if there exists an orientation preserving homeomorphism of the sphere that sends one onto the other. A map is *rooted* if one of its edges is distinguished and oriented. In this case, the map is

[2]Indeed, what is more algebraic than a tree generating function?

drawn on the plane in such a way the infinite face lies to the right of the root edge. All the maps considered in this paper are planar, rooted, and considered up to isomorphisms. An example is provided in Figure 1.1; this map is tetravalent, meaning that all vertices have degree four. In the physics literature, authors often consider unrooted maps; but they count them with a symmetry factor which makes the problem equivalent to counting rooted maps, up to a differentiation of the generating function.

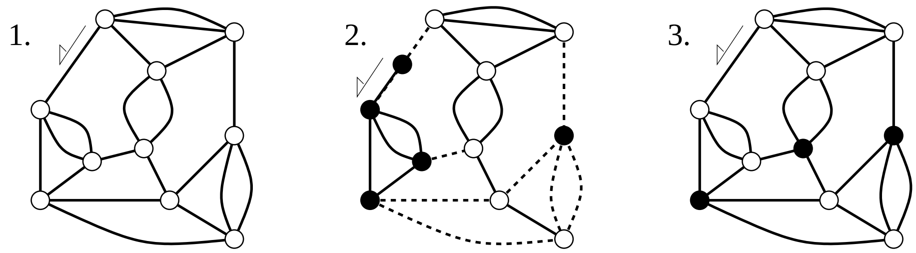

FIGURE 1. 1. A tetravalent map – 2. An Ising configuration on a quasi-tetravalent map with 8 frustrated edges (dashed edges) – 3. A hard particle configuration with 3 particles on a tetravalent map.

In combinatorial terms, the solution of the Ising model on planar maps (in a magnetic field) is equivalent to counting maps having vertices of two types (white and black): one has to count them, not only by their number of edges and vertices of each color, but also by the number of *frustrated edges*, that is, edges that are adjacent to a white vertex and to a black one.

The tools developed in this paper allow us to solve the Ising model on any class of maps subject to degree conditions. The associated generating functions are algebraic as soon as the degrees of the vertices are bounded. Here is, for instance, the result we obtain for "quasi-tetravalent" maps: maps having only vertices of degree 4, except for the root vertex which has degree 2 (Figure 1.2).

Proposition 1 (Ising on quasi-tetravalent maps). *Let $I(X,Y,u)$ be the generating function for bicolored quasi-tetravalent maps, rooted at a black vertex, where the variable X (resp. Y) counts the number of white (resp. black) vertices of degree 4, and u counts the number of frustrated edges. Let $P \equiv P(x,y,v)$ be the power series defined by the following algebraic equation:*

$$P = 1 + 3xyP^3 + v^2\,\frac{P(1+3xP)(1+3yP)}{(1-9xyP^2)^2}.$$

Then the Ising generating function $I(X,Y,u)$ can be expressed in terms of $P(x,y,v)$, with $x = X(u-\bar{u})^2, y = Y(u-\bar{u})^2$, and $v = \bar{u} = 1/u$. One possible expression is:

$$\frac{I(X,Y,u)}{1-\bar{u}^2} = xP^3 + \frac{P(1+3xP-2xP^2-6xyP^3)}{1-9xyP^2} - \frac{yv^2P^3(1+3xP)^3}{(1-9xyP^2)^3}.$$

As P itself, the series $I(X,Y,u)$ is algebraic of degree 7.

Remarks
1. Replacing X by tX and Y by tY gives a generating function in which tetravalent vertices are counted by t. The expansion of the Ising generating function then begins as follows:

$$I(tX,tY,u) = 1 + t(2Xu^2+2Y) + t^2\left(9X^2u^2 + 9Y^2 + XY(12u^2+6u^4)\right) + O(t^3).$$

These terms correspond to maps having at most two tetravalent vertices.
2. It is easy to check that the parametrization by P is equivalent to the one given by Boulatov and Kazakov for the free energy of the Ising model on tetravalent maps [4, Eq. (17)].

Proposition 1 will only be proved in Section 7. At the heart of the proof is the fact that the series P is the generating function of certain trees, as suggested by the equation defining P. In [6] we also prove that the Ising generating function for truly tetravalent maps belongs to the same algebraic extension of $\mathbb{Q}[X,Y,u]$ as P. But its expression in terms of P is messier.

Observe that the series $I(X,Y,u)$ contains all the information we need to count also the number of *uniform* (non-frustrated) edges. In particular, the series $uI(X,Yu^2,1/\sqrt{u})$ counts bicolored quasi-tetravalent maps by their white and black vertices (variables X and Y), and by the number of uniform black edges (variable u). By setting u to zero in this series, we forbid such edges. In particular, both neighbours of the black root vertex are white. Let us erase the root vertex: the root edge is now uniformly white. Let $H(X,Y)$ denote the limiting series of $uI(X,Yu^2,1/\sqrt{u})$ as u goes to zero: it counts bicolored planar maps, rooted at a uniformly white edge, in which two adjacent vertices cannot be both black. Say that a white (resp. black) vertex is vacant (resp. occupied by a particle): we have just solved the so-called *hard particle model* on tetravalent maps. A hard particle configuration is shown on Figure 1.3.

Corollary 2 (Hard particles on tetravalent maps). *The hard particle generating function for tetravalent maps rooted at an edge whose ends are vacant is algebraic of degree 7 and can be expressed as:*

$$H(x,y) = xP^3 + \frac{xP^2(3-2P)}{1-9xyP^2} - \frac{27x^3yP^6}{(1-9xyP^2)^3}$$

where $P \equiv P(x,y)$ is the power series defined by

$$P = 1 + 3xyP^3 + \frac{3xP^2}{(1-9xyP^2)^2}.$$

More generally, our techniques will allow us to solve the hard particle model on any class of maps defined by degree conditions. This includes the case of tri- and tetravalent maps solved in [7] via matrix integrals, but not the case of tri- and tetravalent *bipartite* maps (all cycles have an even length) also solved in [7]. These bipartite models seem to mimic more faithfully the phase transitions observed on the square and honeycomb lattices (which are of course bipartite).

Our method for solving the Ising and hard particle models uses a detour via the enumeration of *fully frustrated* maps, better known as *bipartite* maps. Their enumeration will be based on a correspondence between these maps and certain trees, called *blossom trees*. These trees have an algebraic generating function as soon as the degrees of their vertices are bounded. In particular, the series P of Proposition 1 and Corollary 2 will be shown to count certain blossom trees.

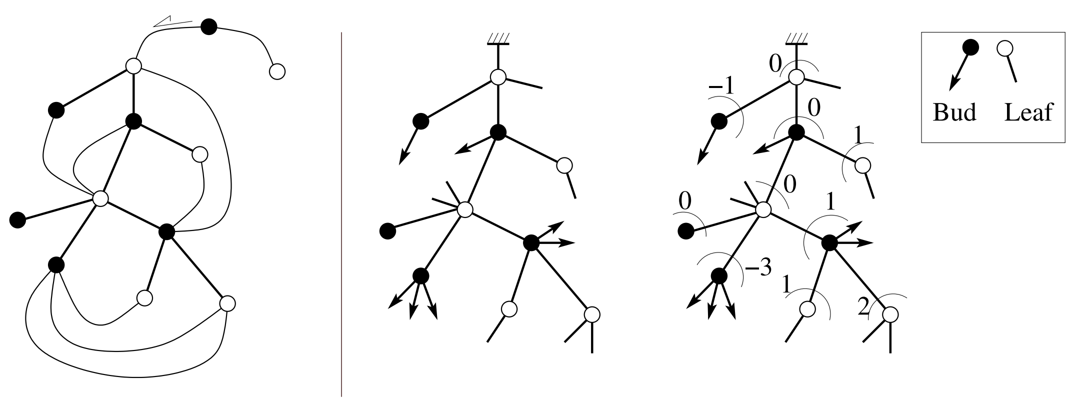

FIGURE 2. A bipartite map; a blossom tree rooted at a leaf, and the charge at each subtree.

Let us give a brief description of these blossom trees that are at the heart of bipartite maps. An example is provided by Figure 2. As one can expect, these trees are themselves bicolored: all the neighbours of a black vertex are white, and vice-versa. In addition to the standard vertices and edges, blossom trees carry *half-edges*, which are called *leaves* when they hang from a white vertex, and *buds* otherwise. Leaves are represented in our figures by short segments, while buds are represented by black arrows. The trees are rooted at a leaf or a bud, and the vertex attached to this half-edge is called the root vertex. Finally, we define the *charge* of a tree to be the difference between the number of leaves and the number of buds it contains; the root half-edge does not count. The charge at a vertex is the charge of the subtree rooted at

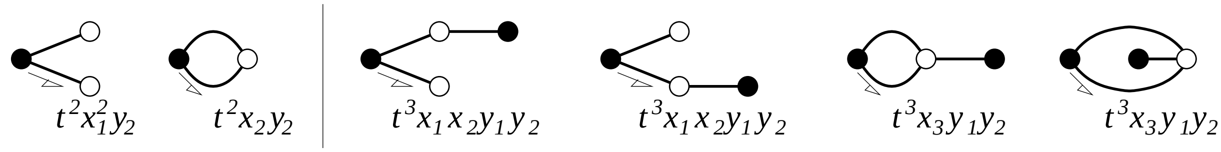

FIGURE 3. The bipartite maps rooted on a black vertex of degree two and having at most 3 edges, together with their degree distribution.

this vertex. We impose the following charge conditions on the vertices of blossom trees (except at the root vertex): all white vertices have a nonnegative charge, while all black vertices have a charge at most one. The notion of charge was introduced by Bouttier *et al.* in [8] for the enumeration of usual planar maps, and turns out to be also useful in the bipartite case.

Given the neat recursive structure of trees, it is not difficult to write functional equations that govern a very detailed generating function for blossom trees. Let $\boldsymbol{x} = (x_1, x_2, \dots)$ and $\boldsymbol{y} = (y_1, y_2, \dots)$. For $i \in \mathbb{Z}$, let $W_i(\boldsymbol{x}, \boldsymbol{y})$ be the generating function for blossom trees rooted at a leaf such that the charge at the (white) root vertex is i. In this series, the variable x_k (resp. y_k) counts white (resp. black) vertices of degree k. Similarly, let $B_i(\boldsymbol{x}, \boldsymbol{y})$ be the generating function for blossom trees rooted at a bud such that the charge at the (black) root vertex is i. Let us form the following generating functions:

$$W(\boldsymbol{x}, \boldsymbol{y}; z) \equiv W(z) = \sum_{i\geq 0} W_i(\boldsymbol{x}, \boldsymbol{y})\, z^i \quad \text{and} \quad B(\boldsymbol{x}, \boldsymbol{y}; z) \equiv B(z) = \sum_{i\leq 1} B_i(\boldsymbol{x}, \boldsymbol{y})\, z^i. \tag{1}$$

Then

$$W(z) = [z^{\geq 0}] \sum_{k\geq 0} x_{k+1} \Big(z + B(z)\Big)^k, \tag{2}$$

and

$$B(z) = [z^{\leq 1}] \sum_{k\geq 0} y_{k+1} \left(\frac{1}{z} + W(z)\right)^k. \tag{3}$$

We have used the following notation: for any power series S in $\boldsymbol{x}$ and $\boldsymbol{y}$ having coefficients in $\mathbb{Z}[z, 1/z]$, the series $[z^{\geq 0}]S(\boldsymbol{x}, \boldsymbol{y}; z)$ is obtained by selecting from S the terms with a nonnegative power of z. The notation $[z^{\leq 1}]$ naturally corresponds to the extraction of terms in which the exponent of z is at most one. More generally, for any $i \in \mathbb{Z}$,

$$B_i(\boldsymbol{x}, \boldsymbol{y}) = [z^i] \sum_{k\geq 0} y_{k+1} \left(\frac{1}{z} + W(z)\right)^k. \tag{4}$$

Our central theorem gives an expression for the *degree generating function* of bipartite planar maps. This series, denoted by $M(\boldsymbol{x}, \boldsymbol{y})$, enumerates maps by their number of white and black vertices having a given degree k (variables x_k and y_k). The theorem below thus solves a combinatorial problem that has an interest of its own. Moreover, we shall see that the solutions of the Ising and hard particle models on planar maps have close connections to it.

Theorem 3. *The degree generating function of bipartite planar maps rooted at a black vertex of degree* 2 *is related to the generating functions of blossom trees as follows:*

$$M(\boldsymbol{x}, \boldsymbol{y}) = y_2((W_0 - B_2)^2 + W_1 - B_3 - B_2^2)$$

where $W_i \equiv W_i(\boldsymbol{x}, \boldsymbol{y})$ *and* $B_i \equiv B_i(\boldsymbol{x}, \boldsymbol{y})$ *are defined by Equations* (1–4).

Remarks

1. One can naturally write $M(\boldsymbol{x}, \boldsymbol{y}) = y_2(W_0^2 + W_1 - B_3 - 2W_0B_2)$, and this expression is better suited to the applications. However, the expression of Theorem 3 will be shown to reflect more faithfully the combinatorics of bipartite maps. Theorem 3 will be proved in Section 5.

2. Replacing x_k by $t^k x_k$ gives a generating function in which edges are counted by t. The first few terms of M then read:

$$t^2y_2(x_1^2+x_2)+2t^3y_2y_1(x_1x_2+x_3)+t^4y_1^2y_2(2x_1x_3+3x_4+x_2^2)+t^4y_2^2(4x_1x_3+2x_1^2x_2+2x_4+x_2^2)+O(t^5).$$

The bipartite maps corresponding to the first two terms are shown in Figure 3.

3. Let us rephrase the above result in terms of permutations. A bipartite planar map with n labelled edges can be encoded by a pair of permutations (σ,τ) of $\mathcal{S}_n$ satisfying the following conditions: the group generated by σ and ρ acts transitively on $\{1,\dots,n\}$, and the three permutations σ, ρ and $\sigma\rho$ have a total of $n+2$ cycles [13]. The enumeration of such *minimal transitive factorizations* is the subject of a vast litterature; see [5, 17] and references therein. For λ and μ two partitions of n, let $m(\lambda,\mu)$ be the number of pairs of permutations (σ,ρ) of respective cyclic type λ and μ, satisfying the two conditions indicated above. Theorem 3 gives an expression for the generating function $M(\boldsymbol{x},\boldsymbol{y})=\sum_{n\geq 1}\sum_{\lambda,\mu} m(\lambda,\mu)\cdot 2m_2\cdot \boldsymbol{x}^\lambda\boldsymbol{y}^\mu/n!$, where the inner summation is on all partitions $\lambda=1^{\ell_1}\dots n^{\ell_n}$ and $\mu=1^{m_1}\dots n^{m_n}$ of n, and the weights $\boldsymbol{x}^\lambda\boldsymbol{y}^\mu$ are defined by $\boldsymbol{x}^\lambda=x_1^{\ell_1}\cdots x_n^{\ell_n}$, and $\boldsymbol{y}^\mu=y_1^{m_1}\cdots y_n^{m_n}$.

The paper is organized as follows: in Section 3, we describe a general connection between maps and trees: one transforms a map into a tree by cutting certain of its edges into two half-edges. Conversely, merging half-edges of a tree gives a planar map. In Section 4, we show that these transformations, restricted to certain classes of maps and trees (called balanced trees) are one-to-one. Balanced trees are then enumerated in Section 5: this proves our central Theorem 3 above. Finally, Sections 6 and 7 are respectively devoted to applications of this theorem to the solution of the hard particle and Ising models.

3. Maps and trees

Take a bipartite planar map M rooted at a black vertex v of degree 2. These maps are exactly those we wish to count (Theorem 3). Let us consider that each of the edges that start from v is made of two half-edges. Delete v and the two half-edges attached to it. Two cases occur (Figure 4):

- either we get an ordered pair of maps, each of them carrying a half-edge (or *leg*), on which it is rooted,
- or we get a single map with two half-edges, and root the map at the half-edge belonging to the root edge of M.

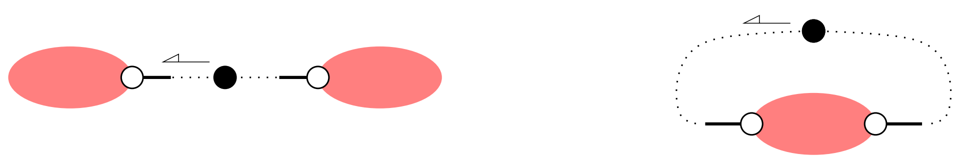

FIGURE 4. The deletion of a black root vertex of degree two.

The degree generating function of bipartite maps rooted at a black vertex of degree 2 can thus be written as

$$M(\boldsymbol{x},\boldsymbol{y})=y_2(L_1(\boldsymbol{x},\boldsymbol{y})^2+L_2(\boldsymbol{x},\boldsymbol{y})), \tag{5}$$

where the series L_1 and L_2 respectively count by their degree distribution maps with one or two legs, rooted at a leg. The reason why we keep the legs is that we do not want to modify the degrees of the vertices. Compare Equation (5) to Theorem 3: this section and the next two are devoted to proving that $L_1=W_0-B_2$ and $L_2=W_1-B_3-B_2^2$.

The above decomposition explains why, as in [8], we shall be interested in generalized bipartite maps, that do not only consist of the traditional edges and vertices, but also contain a number of half-edges in their infinite face (when drawn on the plane), and are rooted at one of these half-edges. Let us insist on the distinction between a half-edge, which is incident to only one vertex, and a dangling edge, which is incident to two vertices, one of which has degree one. *From now on, in this section and the next two, all maps are bipartite and rooted at a half-edge.* This requires, unfortunately, to introduce a bit of terminology. In passing, we shall reformulate slightly the definition of blossom trees given in Section 2.

The vertex adjacent to the root half-edge is called the root vertex of the map. Half-edges that hang from black vertices are called *buds* and are represented in our figures by black outgoing arrows. Half-edges that hang from white vertices are called *leaves* and are represented by short segments. The *degree distribution* of a map is the pair of partitions (λ, μ) such that λ gives the degree distribution of white vertices and μ gives the degree distribution of black vertices. Half-edges are included in the degree of the vertex they are attached to. A map with b buds and ℓ leaves obviously satisfies $b + |\lambda| = \ell + |\mu|$ where $|\lambda|$ denotes the sum of the parts of λ. The 1-leg map of Figure 5 has degree distribution $\lambda = 2^2345$, $\mu = 24^25$.

Let us now define two important subclasses of maps. A *k-leg map* is a map with exactly k leaves and no bud; hence a k-leg map is rooted at one of its leaves. A *tree* is a map with only one face (and an arbitrary number of buds and leaves). The *total charge* of a tree is the difference between its number of leaves and its number of buds; the root half-edge is counted. The *charge* of a tree is the same difference, but the root half-edge is not counted. Hence the charge and total charge always differ by ± 1.

Take a tree T and an edge e of this tree. Cut e into two half-edges: this leaves two subtrees, rooted at these half-edges. The subtree that does not contain the root of T is called the *lower subtree* of T at e. Let $T_e^\bullet$ denote the subtree containing the black endpoint of e and T_e° the other one. The charges $c_\circ$ of T_e° and $c_\bullet$ of $T_e^\bullet$ satisfy $c_\circ + c_\bullet = c$, where c is the total charge of T. The *black charge rule* at e is satisfied if the subtree $T_e^\bullet$ has charge at most one. The *white charge rule* at e is satisfied if the subtree T_e° has a nonnegative charge.

A tree is called a *blossom tree* if all its *lower* subtrees satisfy the charge rule corresponding to their color. An example is displayed on Figure 2. The series W_i and B_i defined by (1–4) respectively count blossom trees of charge i rooted at a leaf and a bud. The following lemma is immediate.

Lemma 4. *Let T be a tree of total charge one, and let e be an edge of T. Then the black charge rule is satisfied at e if and only if the white charge rule is satisfied at e.*

If, in addition, T is a blossom tree, then both charge rules are satisfied at every edge; re-rooting T at another half-edge yields again a blossom tree.

3.1. **Balanced trees and their closure.** In this section we define the *closure* as a mapping ϕ from certain trees with total charge $k \geq 1$ to k-leg maps. This closure is the same as in earlier texts using the idea of conjugacy classes of trees [5, 8, 22, 23, 24, 25].

Let T be a tree with total charge $k \geq 1$. Its half-edges form a cyclic sequence around the tree in counterclockwise direction. Buds and leaves can be matched in this cyclic sequence as if they were respectively opening and closing brackets. More precisely, first match the pairs made of a bud and a leaf that are immediately consecutive in the cyclic sequence. Then, forget matched buds and leaves, and repeat until there is no more possible matching. In view of the number of buds and leaves in the original cyclic sequence, k leaves remain unmatched. These leaves are called the *single leaves* of T.

A tree rooted at a leaf is said to be *balanced* if its root is single. The *closure* of a balanced tree T is obtained as follows: match buds and leaves into pairs as previously explained and fuse each pair into an edge (in counterclockwise direction around the tree). The root remains unchanged. We thus have:

Proposition 5. *Let T be a balanced tree having total charge k and degree distribution (λ, μ). Then $\phi(T)$ is a k-leg map with degree distribution (λ, μ).*

There is an alternative description of the closure as an iterative process, illustrated by Figure 5: start from a balanced tree T and walk around the border of the infinite face in counterclockwise order; each time a bud is immediately followed by a leaf in the cyclic sequence of half-edges, fuse them into an edge in counterclockwise direction (this creates a new finite face that encloses no unmatched half-edges); stop the course when all buds have been matched. Observe that this process may in general require to turn several times around the tree.

3.2. **The opening of a k-leg map.** In this section we define the opening ψ as a mapping from k-leg maps to trees with total charge k.

The opening of a map is the result of an iterative procedure: start from a k-leg map M and walk around the border of the infinite face in counterclockwise order, starting from the root; each time a non-separating[3]

[3]An edge is *separating* if its deletion disconnects the map, *non-separating* otherwise; a map is a tree if and only if all its edges are separating.

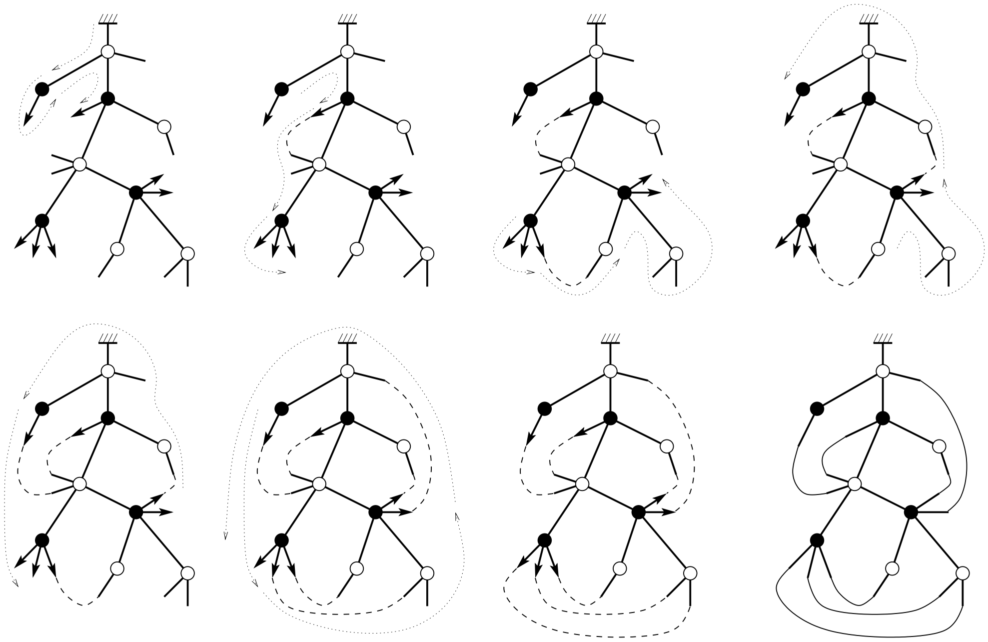

FIGURE 5. The closure of a balanced tree.

edge has just been visited from its black endpoint to its white endpoint, cut this edge into two half-edges: a bud at the black endpoint and a leaf at the white endpoint; proceed until all edges are separating edges. An example is shown on Figure 6. The opening process cannot get stuck: as long as there are non-separating edges, a positive even number of them are incident to the infinite face; half of them are oriented from black to white in the counterclockwise direction. The final result is then a tree rooted at a leaf.

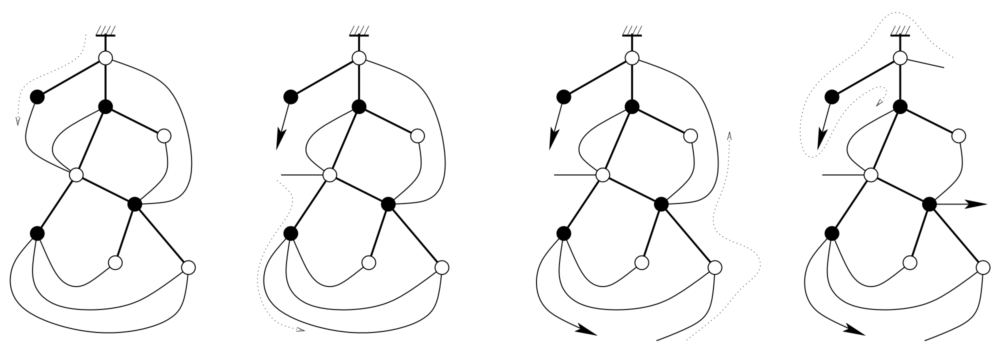

FIGURE 6. The first few steps of the opening of a 1-leg map. The final tree will be the one of Figure 5.

In view of the number of buds and leaves created, the image $T = \psi(M)$ of a k-leg map is a tree with total charge k. Moreover, the pairs of buds and leaves created by the opening are in correspondence in the matching procedure of the tree, so that the tree is balanced. Hence the proposition:

Proposition 6. *The closure is inverse to the opening: for any k-leg map M, $\phi(\psi(M)) = M$.*

We shall prove below that, under some conditions, the tree $\psi(M)$ created by the opening of a map is not only balanced, but is also a blossom tree; that is, all its lower subtrees satisfy the charge rules.

4. Bijections between maps and balanced blossom trees

4.1. The fundamental case of one-leg maps.

Theorem 7. *Closure and opening are inverse bijections between balanced blossom trees of total charge* 1 *and* 1*-leg maps. Moreover they preserve the degree distribution* (λ, μ).

In order to prove Theorem 7, we first exhibit a bijective decomposition of 1-leg maps into one or two smaller 1-leg maps. Then, we present a related decomposition of balanced blossom trees of total charge 1. We observe that the two decompositions are *isomorphic*, so that they induce a recursive bijection between 1-leg maps and balanced blossom trees of total charge 1. This bijection preserves the degree distribution. Once the existence of a bijection has thus been established, we want to identify it as the opening of maps. We observe that the opening transforms the decomposition rules of maps into the decomposition rules of trees. In view of Propositions 5 and 6, this proves that closure and opening realize this recursive bijection.

Details can be found in [6, Section 4.1].

4.2. **2-leg maps and other extensions.** Theorem 7 extends verbatim to balanced blossom trees of total charge 2 and 2-leg maps. Details can be found in [6, Section 4.2].

Theorem 8. *There is a bijection between balanced blossom trees of total charge* 2 *and* 2*-leg maps. Moreover this bijection preserves the degree distribution* (λ, μ).

The extension to maps with more than two legs is harder except in the following special case.

Theorem 9. *Let* $m \geq 3$ *be an integer. Closure and opening are inverse bijections between balanced blossom trees of total charge* m *in which the degrees of vertices are all multiples of* m*, and* m*-leg maps satisfying the same condition. Moreover the degree distribution* (λ, μ) *is preserved.*

The following interesting variation was observed, in the non-bipartite case, by P. Zinn-Justin [30] and by Bouttier *et al.* [8]. It describes a class of maps that are in bijection with trees *not subject to any balance conditions* – and hence, much easier to count.

Theorem 10. *There is a one-to-one correspondence between (not necessarily balanced) blossom trees of charge* 1 *rooted at a leaf and* 1*-leg maps having a marked black vertex of degree* 1.

Similarly, there is a one-to-one correspondence between blossom trees of charge 1 *rooted at a bud and* 1*-leg maps having a marked white vertex of degree* 1.

Proof. Consider a blossom tree of total charge 2 rooted at a leaf. Replace the root leaf by a marked black vertex of degree 1 and re-root the resulting tree of total charge one on its unique single leaf. The closure then yields bijectively a 1-leg map with a marked black vertex of degree 1. A similar argument proves the second statement. □

5. Counting balanced trees

In view of Eq. (5) and Theorems 7 and 8, our objective is now to count balanced blossom trees of total charge 1 or 2. In this section, we establish bijections that allow us to express their generating functions in terms of the generating functions W_i and B_i of (not necessarily balanced) blossom trees. These bijections are adaptations to the bipartite case of the bijections presented in [8].

For $i \in \mathbb{Z}$, let $\mathcal{W}_i$ be the set of blossom trees of charge i rooted at a leaf (so that the total charge is $i+1$). Similarly, let $\mathcal{B}_i$ be the set of blossom trees of charge i rooted at a bud (their total charge is $i-1$). These trees are respectively counted by the series W_i and B_i of Eqs. (1–4). Finally, for $i \geq 1$, let $\mathcal{W}_i^*$ be the subset of $\mathcal{W}_i$ formed of balanced trees. We are especially interested in the enumeration of the trees of $\mathcal{W}_0^*$ and $\mathcal{W}_1^*$. The theorem below implies our Theorem 3 on the enumeration of bipartite maps.

Theorem 11. *There exists a simple degree-preserving bijection between the sets* $\mathcal{W}_0$ *and* $\mathcal{W}_0^* \cup \mathcal{B}_2$. *There also exists a degree-preserving bijection between the sets* $\mathcal{W}_1$ *and* $\mathcal{W}_1^* \cup \mathcal{B}_3 \cup (\mathcal{B}_2)^2$.

Proof of the first bijection. The first bijection is extremely simple to describe: a tree T belonging to $\mathcal{W}_0$ is either balanced (that is, belongs to $\mathcal{W}_0^*$) or its root leaf is matched to a bud. In this case, T can be re-rooted at this bud; the resulting tree T' has still total charge 1 and Lemma 4 proves that it is a blossom tree, hence an element of $\mathcal{B}_2$. Conversely, if we take a tree T' in $\mathcal{B}_2$ and re-root it at the leaf matched to the root bud, we obtain a tree of $\mathcal{W}_0$ that is not balanced. □

The proof of the second bijection can be found in [6, Section 5], together with the proof of the following theorem.

Theorem 12. *Let $m \geq 1$, and let $\widehat{\mathcal{W}}_i$, $\widehat{\mathcal{W}}_i^*$ and $\widehat{\mathcal{B}}_i$ be the restriction of the sets $\mathcal{W}_i$, $\mathcal{W}_i^*$ and $\mathcal{B}_i$ to trees in which the degrees of all vertices are multiples of m. There exists a simple degree-preserving bijection between $\widehat{\mathcal{W}}_{m-1}$ and $\widehat{\mathcal{W}}_{m-1}^* \cup \widehat{\mathcal{B}}_{m+1}$.*

This theorem generalizes the results obtained in [5] for some bipartite maps called constellations.

6. Hard particle models on planar maps

In this section, we consider rooted planar maps in which some vertices are occupied by a particle, in such a way two adjacent vertices are not both occupied. The vacant (resp. occupied) vertices are represented by white (resp. black) cells. Moreover, we require the two ends of the root edge to be vacant. We shall say, for short, that the map is rooted at a vacant edge. Let $H(\boldsymbol{X}, \boldsymbol{Y})$ denote the generating function for these maps, in which X_k (resp. Y_k) counts white (resp. black) vertices of degree k. As above, $\boldsymbol{X}$ stands for $(X_1, X_2, \ldots)$ and $\boldsymbol{Y}$ for $(Y_1, Y_2, \ldots)$.

Theorem 13. *The hard particle generating function for planar maps rooted at a vacant edge can be expressed in terms of the generating function for blossom trees as follows:*

$$H(\boldsymbol{X}, \boldsymbol{Y}) = (W_0 - B_2)^2 + W_1 - B_3 - B_2^2,$$

where the series $W_i \equiv W_i(\boldsymbol{x}, \boldsymbol{y})$ and $B_i \equiv B_i(\boldsymbol{x}, \boldsymbol{y})$ are evaluated at $x_k = X_k$ for $k \geq 1$, $y_2 = 1 + Y_2$ and $y_k = Y_k$ for $k \neq 2$.

Proof. Take a map with hard particles. On every edge having both ends vacant, add a black vertex of degree 2, of a special shape: say, a square black vertex. One thus obtains a bipartite map satisfying the following conditions:

– the black vertices of degree 2 can be discs or squares,
– all other vertices are discs,
– the root vertex is a black square of degree 2.

We conclude using Theorem 3. ◻

In the rest of this section we apply Theorem 13 to tetravalent maps. That is, all the variables X_k and Y_k are zero, except for $k = 4$. As a preliminary result, we need to enumerate blossom trees having white vertices of degree 4 and black vertices of degree 2 and 4. We are actually going to solve a slightly more general enumeration problem, by counting blossom trees having black *and* white vertices of degree 2 and 4: first because this problem is nicely symmetric, then (and most importantly) because we shall need this result to solve the Ising model on tetravalent maps.

The corresponding series W_i and B_i depend on the four variables x_2, y_2, x_4 and y_4, which we denote below by v, w, x and y for the sake of simplicity. Observe that the charge at the root of such blossom trees is always odd: hence the series W_{2i} and B_{2i} are all zero. Equations (2) and (3) specialize to

$$\left\{\begin{array}{lcl} W_1 &=& v(1+B_1) + 3xB_{-1}(1+B_1)^2, \\ W_3 &=& x(1+B_1)^3, \end{array}\right. \qquad \left\{\begin{array}{lcl} B_{-1} &=& w + 3yW_1, \\ B_1 &=& wW_1 + 3y(W_3 + W_1^2), \end{array}\right.$$

and $W_5 = 0$, while Equation (4) gives:

$$B_3 = wW_3 + yW_1(6W_3 + W_1^2). \tag{6}$$

After a few reductions, we express all the series W_i and B_i in terms of the series $P = 1 + B_1$, which satisfies:

$$P = 1 + 3xyP^3 + \frac{P(v + 3xwP)(w + 3yvP)}{(1 - 9xyP^2)^2}. \tag{7}$$

In particular,

$$W_1 = \frac{P(v + 3xwP)}{1 - 9xyP^2} \quad \text{and} \quad W_3 = xP^3. \tag{8}$$

By Theorem 13, the hard particle generating function on tetravalent maps can be expressed in terms of the above series W_1 and B_3 with $v = 0$ and $w = 1$. We thus obtain a new proof of Corollary 2, independent of the (more general) solution of the Ising model.

The parametrization by P is equivalent the one given in [7] for the free energy of this hard particle model: more precisely, upon setting $x = g, y = zg$ and $P = V/g$, the equation defining the parameter P becomes Eq. (2.14) of the above reference.

7. The Ising model on planar maps

In this section, we consider maps with white and black vertices, and enumerate them according to their degree distribution (variables X_k and Y_k) and according to the number of frustrated edges (variable u). Let us recall that an edge is frustrated if it has a black end and a white one.

We first deal with the general case (Theorem 14 below), and then make it explicit in the quasi-tetravalent case (this is Proposition 1). The k-regular case and in particular the completely tetravalent case are made explicit in [6, Section 7.3].

Theorem 14. *The Ising generating function $I(\boldsymbol{X},\boldsymbol{Y},u)$ for planar maps whose root vertex is black and has degree 2 can be expressed in terms of the generating functions for blossom trees:*

$$I(\boldsymbol{X},\boldsymbol{Y},u) = Y_2(u-\bar{u})\left((W_0-B_2)^2 + W_1 - B_3 - B_2^2\right),$$

where the series $W_i \equiv W_i(\boldsymbol{x},\boldsymbol{y})$ and $B_i \equiv B_i(\boldsymbol{x},\boldsymbol{y})$ are evaluated at

$$\left\{\begin{array}{lcl} x_2 &=& \bar{u} + X_2(u-\bar{u}), \\ x_k &=& X_k(u-\bar{u})^{k/2}, \end{array}\right. \qquad \left\{\begin{array}{lcl} y_2 &=& \bar{u} + Y_2(u-\bar{u}), \\ y_k &=& Y_k(u-\bar{u})^{k/2}, \end{array}\right. \quad \textit{for } k \neq 2$$

with $\bar{u} = 1/u$.

Proof. Take a bicolored map rooted at a black vertex of degree 2. On each edge, add a (possibly empty) sequence of square vertices of degree 2, in such a way the resulting map is bipartite. Note that every frustrated edge receives an even number of square vertices, while every uniform edge receives an odd number of these squares. The resulting map remains rooted at its black vertex of degree 2, which is not a square.

Let $\tilde{M}(\boldsymbol{x},\boldsymbol{y},v)$ denote the degree generating function for the maps one obtains in that way: in this series, the square vertices are counted by v, while the other vertices are, as usually, counted by the variables x_k and y_k. The above construction gives

$$\tilde{M}(\boldsymbol{x},\boldsymbol{y},v) = I(\boldsymbol{X},\boldsymbol{Y},u) \tag{9}$$

where

$$X_k = \frac{x_k}{(\bar{v}-v)^{k/2}}, \quad Y_k = \frac{y_k}{(\bar{v}-v)^{k/2}} \quad \text{and} \quad u = 1/v = \bar{v}.$$

By Theorem 3, we also have

$$\tilde{M}(\boldsymbol{x},\boldsymbol{y},v) = y_2((W_0-B_2)^2 + W_1 - B_3 - B_2^2) \tag{10}$$

where the series W_i and B_i are evaluated at $x_1, x_2{+}v, x_3, \ldots, y_1, y_2{+}v, y_3, \ldots$ The result follows by comparison of (9) and (10). □

The general result above applies in particular to quasi-regular maps, that is, maps in which all vertices have degree $m \geq 3$, except for the root which has degree 2. Let us make Theorem 14 explicit for the case $m = 4$ (this is Proposition 1).

We first form the Ising generating function for planar maps, rooted at a black vertex of degree 2, having only vertices of degree 2 and 4. This series is given by Theorem 14, with $X_k = Y_k = 0$ for $k \neq 2, 4$. The blossom trees occurring in this theorem are the ones that have been counted in Section 6. Their enumeration has resulted in Eqs. (6–8). Theorem 14 yields:

$$I(\boldsymbol{X},\boldsymbol{Y},u) = Y_2(u-\bar{u})(W_1 - B_3),$$

where W_1 and W_3 are evaluated at $v = \bar{u}{+}X_2(u{-}\bar{u})$, $w = \bar{u}{+}Y_2(u{-}\bar{u})$, $x = X_4(u{-}\bar{u})^2$, $y = Y_4(u{-}\bar{u})^2$. The Ising generating function for quasi-tetravalent maps is obtained by setting $X_2 = 0$ in the series $I(\boldsymbol{X},\boldsymbol{Y},u)$, and then by extracting the coefficient of Y_2: in other words, by setting $v = w = \bar{u}$ in the series W_1 and W_3. Proposition 1 follows.

References

[1] D. Arquès. Les hypercartes planaires sont des arbres très bien étiquetés. *Discrete Math.*, 58(1):11–24, 1986.

[2] E. A. Bender and E. R. Canfield. The number of degree-restricted rooted maps on the sphere. *SIAM J. Discrete Math.*, 7(1):9–15, 1994.

[3] D. Bessis, C. Itzykson, and J. B. Zuber. Quantum field theory techniques in graphical enumeration. *Adv. in Appl. Math.*, 1(2):109–157, 1980.

[4] D. V. Boulatov and V. A. Kazakov. The Ising model on a random planar lattice: the structure of the phase transition and the exact critical exponents. *Phys. Lett. B*, 186(3-4):379–384, 1987.

[5] M. Bousquet-Mélou and G. Schaeffer. Enumeration of planar constellations. *Adv. in Appl. Math.*, 24(4):337–368, 2000.

[6] M. Bousquet-Mélou and G. Schaeffer. The degree distribution in bipartite planar maps: applications to the Ising model, 2002. 32pp. `arXiv:math.CO/0211070`.

[7] J. Bouttier, P. Di Francesco, and E. Guitter. Critical and tricritical hard objects on bicolourable random lattices: exact solutions. *J. Phys. A*, 35(17):3821–3854, 2002.

[8] J. Bouttier, P. Di Francesco, and E. Guitter. Census of Planar Maps: From the One-Matrix Model Solution to a Combinatorial Proof. to appear in *Nucl. Phys. B*. `arXiv:cond-mat/0207682`.

[9] J. Bouttier, P. Di Francesco, and E. Guitter. Combinatorics of Hard particles on Planar Graphs. preprint SPhT/02-160. `arXiv:cond-mat/0211168`.

[10] E. Brézin, C. Itzykson, G. Parisi, and J. B. Zuber. Planar diagrams. *Comm. Math. Phys.*, 59(1):35–51, 1978.

[11] W. G. Brown and W. T. Tutte. On the enumeration of rooted non-separable planar maps. *Canad. J. Math.*, 16:572–577, 1964.

[12] R. Cori. *Un code pour les graphes planaires et ses applications*. Société Mathématique de France, Paris, 1975. With an English abstract, Astérisque, No. 27.

[13] R. Cori and A. Machì. Maps, hypermaps and their automorphisms: a survey. I, II, III. *Exposition. Math.*, 10(5):403–427, 429–447, 449–467, 1992.

[14] P. Di Francesco, P. Ginsparg, and J. Zinn-Justin. 2D gravity and random matrices. *Phys. Rep.*, 254(1-2), 1995. 133 pp.

[15] B. Duplantier and I. Kostov. Geometrical critical phenomena on a random surface of arbitrary genus. *Nuclear Phys. B*, 340(2–3):491–541, 1990.

[16] Z. J. Gao, I. M. Wanless, and N. C. Wormald. Counting 5-connected planar triangulations. *J. Graph Theory*, 38(1):18–35, 2001.

[17] I. P. Goulden and D. M. Jackson. Connexion coefficients for the symmetric group, free products in operator algebras and random matrices. In *Free probability theory (Waterloo, ON, 1995)*, volume 12 of *Fields Inst. Commun.*, pages 105–125. Amer. Math. Soc., Providence, RI, 1997.

[18] D. M. Jackson and T. I. Visentin. Character theory and rooted maps in an orientable surface of given genus: face-colored maps. *Trans. Amer. Math. Soc.*, 322(1):365–376, 1990.

[19] V. A. Kazakov. Ising model on a dynamical planar random lattice: exact solution. *Phys. Lett. A*, 119(3):140–144, 1986.

[20] V. A. Malyshev. Dynamical triangulation models with matter: high temperature region. *Comm. Math. Phys.*, 226(1):163–181, 2002.

[21] M. L. Mehta. A method of integration over matrix variables. *Comm. Math. Phys.*, 79(3):327–340, 1981.

[22] D. Poulalhon. *Factorisations dans le groupe symétrique et triangulations*. PhD thesis, École Polytechnique, Palaiseau, France, 2002.

[23] D. Poulalhon and G. Schaeffer. Loopless triangulations of a polygon with interior points. *Theoret. Comput. Sci.*, 2002. To appear.

[24] G. Schaeffer. Bijective census and random generation of Eulerian planar maps with prescribed vertex degrees. *Electron. J. Combin.*, 4(1):Research Paper 20, 14 pp. (electronic), 1997.

[25] G. Schaeffer. *Conjugaison d'arbres et cartes combinatoires aléatoires*. PhD thesis, Université Bordeaux 1, France, 1998.

[26] G. 't Hooft. A planar diagram theory for strong interactions. *Nucl. Phys. B*, 72:461–473, 1974.

[27] W. T. Tutte. A census of planar maps. *Canad. J. Math.*, 15:249–271, 1963.

[28] W. T. Tutte. On the enumeration of planar maps. *Bull. Amer. Math. Soc.*, 74:64–74, 1968.

[29] W. T. Tutte. Chromatic sums revisited. *Aequationes Math.*, 50(1–2):95–134, 1995.

[30] P. Zinn-Justin. Personnal communication. 2002.

[31] A. Zvonkin. Matrix integrals and map enumeration: an accessible introduction. *Math. Comput. Modelling*, 26(8-10):281–304, 1997. Combinatorics and physics (Marseilles, 1995).

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