FACTORIZATIONS OF SIGNED PERMUTATIONS

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ABSTRACT. In this paper we consider a problem related to the factorizations of elements of the wreath product of the symmetric group $\mathfrak{S}_n$ by $\mathbb{Z}/k\mathbb{Z}$. More precisely, for a given integer k, we give a combinatorial construction relating factorizations of elements in the wreath product of $\mathfrak{S}_n$ by $\mathbb{Z}/k\mathbb{Z}$ and factorizations in $\mathfrak{S}_n$. Our proof relies on the encoding of such factorizations as maps with signed edges and can be generalized to the factorization of permutations of any cycle type.

RÉSUMÉ. Dans cet article, nous considérons un problème concernant les factorisations d'éléments du produit en couronne du groupe symétrique $\mathfrak{S}_n$ par $\mathbb{Z}/k\mathbb{Z}$. Plus précisément, pour un entier k donné, nous présentons une construction combinatoire reliant factorisations dans le produit en couronne de $\mathfrak{S}_n$ par $\mathbb{Z}/k\mathbb{Z}$ et factorisations dans $\mathfrak{S}_n$. Notre preuve repose sur le codage de factorisations par des cartes aux arêtes signées et peut être généralisée aux factorisations de permutations de type cyclique quelconque.

1. INTRODUCTION

Integer partitions. A *partition* $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ is a finite non-increasing sequence of positive integers λ_i such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0$. The terms of λ are called the *parts* of λ and the number ℓ of parts is the *length* of λ, denoted by $\ell(\lambda)$. We also write $\lambda = 1^{\alpha_1} 2^{\alpha_2} \ldots n^{\alpha_n}$ when α_i parts of λ are equal to i for any i in $\{1, \ldots, n\}$. The *weight* n of λ is the sum of its parts $\lambda_1 + \lambda_2 + \cdots + \lambda_\ell$, and we write $\lambda \vdash n$ or $|\lambda| = n$. For any two partitions λ and μ, $\lambda + \mu$ denotes the unique partition for which the multiset of parts is the union of the sets of the parts of λ and μ. For example, $(4,3,1,1) + (3,2,1) = (4,3,3,2,1,1,1)$. For an integer k and a partition λ, we call *k-decomposition* of λ every k-tuple of partitions $\vec{\lambda} = (\lambda^0, \ldots, \lambda^{k-1})$ such that $\lambda = \lambda^0 + \cdots + \lambda^{k-1}$.

Factorizations of cycles in the symmetric group. It is well known that the *conjugacy classes* of the symmetric group $\mathfrak{S}_n$ are indexed by the partitions of n [6]: we denote by $\mathcal{C}_\lambda$ the conjugacy class indexed by λ, which is called the *cycle type* of the permutations $\sigma \in \mathcal{C}_\lambda$ (a cycle of length j in σ induces a part of size j in λ). Let n be a given positive integer, λ, μ and ν three partitions of weight n, and π a permutation of $\mathcal{C}_\nu$: the number of pairs (σ, τ) of permutations in $\mathcal{C}_\lambda \times \mathcal{C}_\mu$ such that $\sigma\tau = \pi$ is denoted by $c^\nu_{\lambda,\mu}$. We call such a pair of permutations (σ, τ) a *factorization of* π. The coefficients $c^\nu_{\lambda,\mu}$, that express the number of ways a permutation can be factorized as a product of two permutations with given cycle types are known as *connection coefficients* or *structure constants* of the symmetric group.

Efforts for computing special values of $c^\nu_{\lambda,\mu}$ have been made by several authors, mostly in the case $\nu = (n)$ or restricted values of λ and μ (see the discussion in [4, 5]). In particular, Goupil and Schaeffer give the following explicit expression for $c^{(n)}_{\lambda,\mu}$, valid for any partitions λ and μ of weight n:

Theorem 1. [5, Theorem 2.1] *Let λ and μ be two partitions of weight n, with $\lambda = 1^{\alpha_1} \ldots n^{\alpha_n}$ and $\mu = 1^{\beta_1} \ldots n^{\beta_n}$. Then*

$$c^{(n)}_{\lambda,\mu} = \frac{n}{\left(\prod_{i=1}^n \alpha_i! \beta_i!\right) 2^{2g(\lambda,\mu)}} \sum_{g_1+g_2=g(\lambda,\mu)} S_{\ell(\lambda),g_1}(\lambda) S_{\ell(\mu),g_2}(\mu),$$

Work supported by grants from UQÀM, NATEQ (Québec) and NSERC (Canada). Part of this work was done while the second author visited the LaCIM (Spring 2002) and while the first author visited the LIAFA (Fall 2002).

where $g(\lambda,\mu)$ is the genus of the pair (λ,μ), defined by $\ell(\lambda)+\ell(\mu)=n+1-2g(\lambda,\mu)$, and

$$S_{k,g}(x_1,\dots,x_k)=(k+2g-1)!\sum_{(p_1,\dots,p_k)\vdash g}\prod_i \frac{1}{2p_i+1}\binom{x_i-1}{2p_i}$$

is a symmetric polynomial of degree $2g$ in the x_i.

The notion of genus $g(\lambda,\mu)$ of a pair (λ,μ) of partitions of same weight, which is central in this paper, is indeed directly related to the notion of topological genus of maps (see Section 2).

The group $\mathcal{W}_n^k$. In this paper, we are interested in the problem of the enumeration of connection coefficients in the groups $\mathcal{W}_n^k$, the wreath products of the symmetric groups by $\mathbb{Z}/k\mathbb{Z}$, also called *complete monomial groups over* $\mathbb{Z}/k\mathbb{Z}$ [6, Chapter 4].

Let n and k be two fixed integers, $\zeta_k=e^{2i\pi/k}$ (a root of unity), and $\mathcal{Z}_k$ be the set of the powers of ζ_k. An element σ of $\mathcal{W}_n^k$ is a permutation on the underlying set $U_n^k=\{\xi.i|\ \xi\in\mathcal{Z}_k,\ i\in[n]\}$, where $[n]=\{1,2,\dots,n\}$, such that for every $\xi\in\mathcal{Z}_k$ and $x\in U_n^k$, $\sigma(\xi.x)=\xi.\sigma(x)$. Elements of $\mathcal{W}_n^k$ are called *k-signed permutations*. For any element $x=\xi.i$ of U_n^k, we say that i is its *absolute value*, denoted by $|x|$, and ξ is its *sign*, denoted by $\zeta(x)$.

Remark 1. We use the terminology sign and absolute value in analogy with the case $k=2$ where $\zeta_k=-1$. From now on, the word sign will always refer to an element in $\mathcal{Z}_k$.

Following the representation of permutations of $\mathfrak{S}_n$ as a set of cycles, we call *cycle representation* of a k-signed permutation of $\mathcal{W}_n^k$ the set of cycles defined as follows. Let σ be a permutation of $\mathcal{W}_n^k$ and π be the permutation of $\mathfrak{S}_n$ defined by $\pi(i)=|\sigma(i)|$ for $i\in[n]$. For every cycle $\gamma=(\gamma_1\dots\gamma_\ell)$ of π, one defines the extension δ of γ to σ as the k-signed cycle $\delta=(\delta_1\dots\delta_\ell)$ defined by $\delta_j=\sigma(\pi^{-1}(\gamma_j))$ for every γ_j of γ. The cycle representation of σ is the set of k-signed cycles composed of the extensions to σ of the cycles of π. It is immediate to see that the cycle representation of a k-signed permutation is unique.

Example 1. Let $k=3$ and $n=5$. For $i\in[5]$, we use the notation $\bar{i}$ and $\underline{i}$ respectively for $\zeta_3.i$ and ${\zeta_3}^2.i$. The 3-signed permutation σ below in the two rows notation

$$\sigma=\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \bar{1} & \bar{2} & \bar{3} & \bar{4} & \bar{5} & \underline{1} & \underline{2} & \underline{3} & \underline{4} & \underline{5} \\ \bar{3} & 2 & \underline{5} & \bar{4} & 1 & \underline{3} & \bar{2} & 5 & \underline{4} & \bar{1} & 3 & \underline{2} & \bar{5} & 4 & \underline{1} \end{pmatrix}$$

has the cycle representation

$$\sigma=(1\ \bar{3}\ \underline{5})(2)(\bar{4}).$$

From now on, we consider k-signed permutations only through their cycle representation. For $i\in[n]$ and a k-signed permutation σ, we define the sign of i in σ, denoted by $\zeta(i,\sigma)$, as the sign of the element of absolute value i in the cycle representation of σ. The *sign of a cycle* $\gamma=(\xi_1.i_1\dots\xi_m.i_m)$ in a k-signed permutation is the product of the signs of its elements: $\zeta(\gamma)=\xi_1\dots\xi_m$. When $k=2$, cycles of sign 1 (resp. -1) are sometimes called positive (resp. negative) [1] or even (resp. odd) [7] cycles. The *cycle type* of a k-signed permutation σ is an ordered k-tuple $\vec{\lambda}=(\lambda^0,\dots,\lambda^{k-1})$ of partitions such that every part λ^i_j of λ^i is the size of a cycle of sign ${\zeta_k}^i$ in σ (λ^i is the cycle type of the restriction of σ to cycles of sign ${\zeta_k}^i$). Hence $\vec{\lambda}$ is a k-decomposition of a partition λ of weight n. The *sign of a k-decomposition* $\vec{\lambda}$ is given by $\zeta(\vec{\lambda})=\prod_{i=0}^{k-1}({\zeta_k}^i)^{\ell(\lambda^i)}$. In other words, if a k-signed permutation σ has cycle type $\vec{\lambda}$, then the sign of $\vec{\lambda}$ is the product of the signs of the cycles of σ.

Example 2. If $k=3$, $n=5$ and $\sigma=(1\ \bar{3}\ \underline{5})(2)(\bar{4})$, $(1\ \bar{3}\ \underline{5})$ and (2) are cycles of sign 1 and $(\bar{4})$ is a cycle of sign ζ_3. Hence the cycle type of σ is given by $\vec{\lambda}=(\lambda^0,\lambda^1,\lambda^2)$ where $\lambda^0=(3,1)$, $\lambda^1=(1)$ and $\lambda^2=\emptyset$, and the sign $\zeta(\vec{\lambda})$ is ζ_3.

It is known [6, Section 4.2] that the conjugacy classes of $\mathcal{W}_n^k$ are indexed by the k-decompositions of partitions of weight n. We denote by $\mathcal{C}_{\vec{\lambda}}$ the conjugacy class of $\mathcal{W}_n^k$ indexed by the k-decomposition $\vec{\lambda}$. Given an integer n, three k-decompositions of partitions of weight n, $\vec{\lambda}$, $\vec{\mu}$ and $\vec{\nu}$, and an element $\pi\in\mathcal{W}_n^k$ of cycle type $\vec{\nu}$, the number of pairs (σ,τ) of k-signed permutations in $\mathcal{C}_{\vec{\lambda}}\times\mathcal{C}_{\vec{\mu}}$ such that $\sigma\tau=\pi$ is denoted by $c^{\vec{\nu}}_{\vec{\lambda},\vec{\mu}}$.

From now on, we call *k-signed n-cycle* of sign ${\zeta_k}^j$ any k-signed permutation π with only one cycle, of length n and sign ${\zeta_k}^j$: its cycle type $\vec{\lambda}$ is a k-decomposition of (n) such that $\lambda^j = (n)$ and $\lambda^i = \emptyset$ for $i \neq j$. Such a cycle is said to be *canonic* if $|\sigma(i)| = i+1$ for any $i \in [n-1]$, and $|\sigma(n)| = 1$. The main part of this paper will be devoted to the description of a constructive proof of the following enumerative result on the factorization of a k-signed n-cycle.

Theorem 2. *Let k and n be two integers, $\lambda = 1^{\alpha_1}\dots n^{\alpha_n}$ and $\mu = 1^{\beta_1}\dots n^{\beta_n}$ be two partitions of weight n, $\vec{\lambda} = (\lambda^0,\dots,\lambda^{k-1})$ a k-decomposition of λ, $\vec{\mu} = (\mu^0,\dots,\mu^{k-1})$ a k-decomposition of μ (where $\lambda^i = 1^{\alpha^i_1}\dots n^{\alpha^i_n}$ and $\mu^i = 1^{\beta^i_1}\dots n^{\beta^i_n}$) and $\vec{\nu}$ a k-decomposition of (n). Then:*

$$c^{\vec{\nu}}_{\vec{\lambda},\vec{\mu}} = \begin{cases} 0 & \text{if } \zeta(\vec{\nu}) \neq \zeta(\vec{\lambda})\zeta(\vec{\mu}), \\ \left(\prod_{j=1}^{n} \binom{\alpha_j}{\alpha_j^0,\dots,\alpha_j^{k-1}} \binom{\beta_j}{\beta_j^0,\dots,\beta_j^{k-1}}\right) k^{2g(\lambda,\mu)}\, c^{(n)}_{\lambda,\mu} & \text{otherwise,} \end{cases}$$

where $g(\lambda,\mu)$ is defined by $\ell(\lambda) + \ell(\mu) = n + 1 - 2g(\lambda,\mu)$.

In Section 5, we propose, with a sketch of the proof, a generalization of this result to any cycle type $\vec{\nu}$.

Remark 2. Theorem 2 can be proved by an argument on the size of the conjugacy classes. But it can also be read in the following combinatorial way. Given

- two permutations σ and τ in $\mathfrak{S}_n$ of respective cycle types λ and μ, such that $\sigma\tau = (1\ 2\ \dots\ n)$,
- a canonic k-signed n-cycle π of sign ξ, and
- two k-decompositions $\vec{\lambda}$ and $\vec{\mu}$ respectively of λ and μ such that $\xi = \zeta(\vec{\lambda})\zeta(\vec{\mu})$,

there are exactly

$$\left(\prod_{i=1}^{n} \binom{\alpha_j}{\alpha_j^0,\dots,\alpha_j^{k-1}} \binom{\beta_j}{\beta_j^0,\dots,\beta_j^{k-1}}\right) k^{2g(\lambda,\mu)}$$

ways to sign the elements of σ and τ (i.e. to multiply every element in their cycle representation by a sign in $\mathcal{Z}_k$) in such a way that the resulting pair (σ',τ') of signed permutations is a factorization of π such that the cycle types of σ' and τ' are given respectively by $\vec{\lambda}$ and $\vec{\mu}$. If $\xi \neq \zeta(\vec{\lambda})\zeta(\vec{\mu})$, there is no way to give signs to the elements of σ and τ with respect to λ and μ and obtain a factorization of π.

Example 3. Let $k = 2$ ($\zeta_2.i$ will be denoted by -i), $\sigma = (1\ 5)(2\ 6)(4\ 7)(3\ 8)$, $\tau = (1\ 6\ 4)(2\ 8\ 5)(3\ 7)$, $\pi = (1\ \text{-}2\ 3\ 4\ 5\ \text{-}6\ 7\ 8)$ (we perform the product of permutations from right to left), $\vec{\lambda} = ((2,2,2,2),\emptyset)$ and $\vec{\mu} = ((3,3,2),\emptyset)$: we want all the cycles of σ' and τ' to have sign 1, that is an even number of elements of sign -1. Then $g(\lambda,\mu) = 1$, and there are four ways to assign an even number of -1 signs in every cycle of σ and τ, giving hence σ' and τ', in such a way that $\sigma'\tau' = \pi$ (for $k = 2$, we use the notation $\zeta_2.i = -i$):

- $\sigma' = (1\ 5)(\text{-}2\ \text{-}6)(4\ 7)(3\ 8)$ and $\tau' = (1\ 6\ 4)(2\ 8\ 5)(3\ 7)$,
- $\sigma' = (\text{-}1\ \text{-}5)(2\ 6)(4\ 7)(3\ 8)$ and $\tau' = (\text{-}1\ \text{-}6\ 4)(\text{-}2\ 8\ \text{-}5)(3\ 7)$,
- $\sigma' = (1\ 5)(2\ 6)(\text{-}4\ \text{-}7)(\text{-}3\ \text{-}8)$ and $\tau' = (1\ \text{-}6\ \text{-}4)(\text{-}2\ \text{-}8\ 5)(\text{-}3\ \text{-}7)$,
- $\sigma' = (\text{-}1\ \text{-}5)(\text{-}2\ \text{-}6)(\text{-}4\ \text{-}7)(\text{-}3\ \text{-}8)$ and $\tau' = (\text{-}1\ 6\ \text{-}4)(2\ \text{-}8\ \text{-}5)(\text{-}3\ \text{-}7)$.

But cycles of σ and τ cannot be transformed into positive k-signed cycles to obtain a factorization of $\pi' = (1\ \text{-}2\ \text{-}3\ 4\ 5\ \text{-}6\ 7\ 8)$.

This presentation of Theorem 2 leads to the question of a constructive proof, i.e. an algorithm which, given σ, τ, π and two k-decompositions of the cycle types of σ and τ, enumerates all the corresponding factorizations of π in $\mathcal{W}^k_n$. We propose such a proof in the next three sections. It relies on a generalization of a representation of products of permutations as maps, which induces an immediate combinatorial interpretation of the genus of such a product. This representation is described in Section 2. Section 3 describes our proof in the planar case ($g(\lambda,\mu) = 0$), and Section 4 extends this proof to the general case of unrestricted genus. Finally, in Section 5 we sketch an extension of this result in the case of factorizations of permutations of any cycle type.

2. Factorizations of k-signed n-cycles and k-signed 2-cacti

In this section we generalize the representation of the factorizations in $\mathfrak{S}_n$ of a n-cycle as a product of two permutations in terms of maps to the case of the group $\mathcal{W}_n^k$. This representation is interesting because it relates the parameter $g(\lambda, \mu)$ to the topological notion of *genus*.

A *map* $\mathcal{M} = (\mathcal{S}, \mathcal{G})$ on a compact oriented surface $\mathcal{S}$ without boundary is an embedding of a graph $\mathcal{G}$ into $\mathcal{S}$ such that connected components of the complement $\mathcal{S}\backslash\mathcal{G}$ of the embedding of $\mathcal{G}$ in $\mathcal{S}$, called the *faces* of the map, are homeomorphic to disks. The *genus* of a map $(\mathcal{S}, \mathcal{G})$ is the genus of the surface $\mathcal{S}$. The *Euler formula* states that for a map with e edges, v vertices and f faces,

$$v - e + f = 2 - 2g. \tag{1}$$

Two maps $(\mathcal{S}, \mathcal{G})$ and $(\mathcal{S}', \mathcal{G}')$ are isomorphic if there exists an orientation-preserving homeomorphism $f : \mathcal{S} \to \mathcal{S}'$ such that $f(\mathcal{G}) = \mathcal{G}'$. We shall consider maps up to isomorphism. A map is *k-signed* if each edge is weighted with a sign (chosen among $\mathcal{Z}_k$). The sign of an edge e of a map $\mathcal{M}$ is denoted by $\zeta(e, \mathcal{M})$. We say that two faces (resp. vertices) having a common edge e on their boundaries (resp. linked by an edge e) are *adjacent* through e, and that a face (resp. vertex) is *incident* to every edge and vertex on its boundary (resp. to every edge linking it to another vertex).

We now propose a natural generalization to signed maps of the notion of 2-*cacti* (see [4] for the notion of m-cacti and its relation with the factorizations of cycles in $\mathfrak{S}_n$). A *k-signed* 2-*cactus* of genus g can be seen as map of genus g with one single face (such maps are sometimes called g-trees) with bicolored (say black and white) vertices in which each edge is replaced by a 2-gon (a polygon with two edges, a white edge and a black edge) and each 2-gon is incident to a black vertex and a white vertex (see Figure 1 below). Formally, a k-signed 2-cactus with n 2-gons is a k-signed map whose vertices, edges and faces are bicolored (say in black and white) in such a way that

- there are n black faces (the 2-gons), and each is incident to exactly a white edge, a black edge, a white vertex and a black vertex;
- there is exactly one white face, and it is incident to all edges and adjacent to all black faces;
- every edge is incident to exactly one white vertex (called its white vertex) and one black vertex (called its black vertex);
- black (resp. white) edges follow immediately black (resp. white) vertices when turning counterclockwise around the white face.

The *degree* of a vertex or a face is the number of its incident 2-gons. A k-signed 2-cactus is *rooted* if one of its white edges (called the *root-edge*) is distinguished. From now on, *cactus* means *rooted cactus.* Classical 2-cacti, as defined in [4], are equivalent to 1-signed cacti.

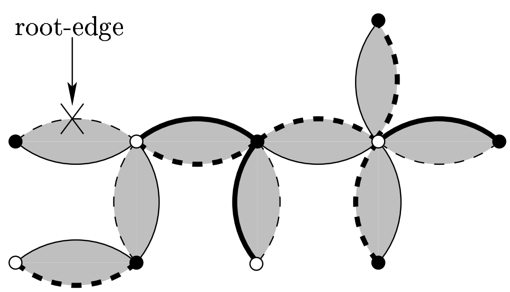


FIGURE 1. A 2-signed 2-cactus of genus 0: edges of sign -1 (resp. 1) are thick (resp. thin) edges, white (resp. black) edges are dashed (resp. solid).

For a given k-signed 2-cactus $\mathcal{C}$, we define the *sign of a vertex x*, denoted by $\zeta(x, \mathcal{C})$, as follows: let $e_1, \dots, e_m$ be the edges incident to x that have the same color as x; then $\zeta(x, \mathcal{C}) = \zeta(e_1, \mathcal{C}) \dots \zeta(e_m, \mathcal{C})$. The notion of sign of vertices induces a partition of the set of white vertices into an ordered k-tuple $\vec{W}(\mathcal{C}) = (W^0(\mathcal{C}), \dots, W^{k-1}(\mathcal{C}))$ of sets of vertices, where $W^i(\mathcal{C})$ is the set of white vertices whose sign is ${\zeta_k}^i$. Moreover, we associate to the set $W^i(\mathcal{C})$ the integer partition $\lambda^i = 1^{\alpha_1^i} \dots n^{\alpha_n^i}$ where α_j^i is the number of vertices in $W^i(\mathcal{C})$ of degree j. We call the k-tuple $\vec{\lambda} = (\lambda^0, \dots, \lambda^{k-1})$ the *white degree distribution* of $\mathcal{C}$. The set partition $\vec{B}(\mathcal{C}) = (B^0(\mathcal{C}), \dots, B^{k-1}(\mathcal{C}))$ of black vertices and the *black degree distribution* $\vec{\mu} = (\mu^0, \dots, \mu^{k-1})$ are defined accordingly. For example, the degree distributions of the cactus in Figure 1 are given by $\lambda^0 = (1)$, $\lambda^1 = (4, 3, 1)$, $\mu^0 = (3, 2, 1, 1, 1)$ and $\mu^1 = (1)$.

The relations between factorizations of permutations in the symmetric group and combinatorial maps have been well studied. For an account of the link between pairs of permutations and maps on oriented surfaces, the reader is referred to [3]. The next proposition is a natural extension, to the case of the group $\mathcal{W}_n^k$, of the relation between maps and pairs of permutations.

Proposition 1. *Let k and n be two integers, $\vec{\lambda}$ and $\vec{\mu}$ k-decompositions of partitions λ and μ of weight n, and π a k-signed n-cycle. There is a one-to-one correspondence between pairs of k-signed permutations (σ, τ) of $\mathcal{W}_n^k$, of cycle types $\vec{\lambda}$ and $\vec{\mu}$, such that $\sigma\tau = \pi$, and k-signed 2-cacti with n 2-gons, of genus $g(\lambda, \mu)$, with white and black degree distributions $\vec{\lambda}$ and $\vec{\mu}$.*

Sketch of proof. The proof we sketch here follows naturally from the discussion given in [3] (see also the proofs of [2, Proposition 2.2] and [4, Theorem 3.1]). The construction is a natural generalization of the constructions described in detail in the above cited papers.

Let us first introduce some notations and terminology. We call a *traversal* of a 2-cactus the process of following the edges of this cactus by turning counterclockwise around its white face, starting at the root-edge. For a k-signed 2-cactus $\mathcal{C}$ with n 2-gons and a permutation π of $\mathfrak{S}_n$, we call *π-labeling* of $\mathcal{C}$ the labeling of its 2-gons defined as follows: the 2-gon incident to the k^{th} visited white edge during a traversal of $\mathcal{C}$ is labeled by $|\pi^{k-1}(1)|$. We denote by w_i (resp. b_i) the white (resp. black) edge incident to the 2-gon labeled by i (hence the root-edge is always w_1).

The proposition relies on a construction relating vertices of a π-labeled k-signed 2-cactus $\mathcal{C}$ and cycles of elements in U_n^k: to any white (resp. black) vertex x of $\mathcal{C}$ of degree d corresponds the unique cycle $\gamma(x) = (\xi_1.i_1 \dots \xi_d.i_d)$ such that the cyclically ordered 2-gons incident to x (when turning counterclockwise around x) are labeled by $i_1, \dots, i_d$ and for each 2-gon i_ℓ ($\ell \in [d]$), $\zeta(w_{i_\ell}, \mathcal{C}) = \xi_\ell$ (resp. $\zeta(b_{i_\ell}, \mathcal{C}) = \xi_\ell$). □

Remark 3. Given a π-labeled k-signed 2-cactus $\mathcal{C}$ and $i \in [n]$, if b_j and w_ℓ are the two edges following w_i in a traversal of $\mathcal{C}$ ((b_j, w_ℓ) is a pair of consecutive edges), then $\pi(i) = \zeta(b_j, \mathcal{C})\zeta(w_\ell, \mathcal{C})\ell$.

Example 4. Let $k = 2$. The 2-signed 2-cactus of Figure 2 below corresponds to the factorization (σ, τ), where σ = (-1)(2 3 4 -5)(-6 7 -8) and τ = (-1 -5 8)(-2)(3)(4)(6)(-7), of the cycle π = (1 -2 -3 4 -5 -6 7 8).

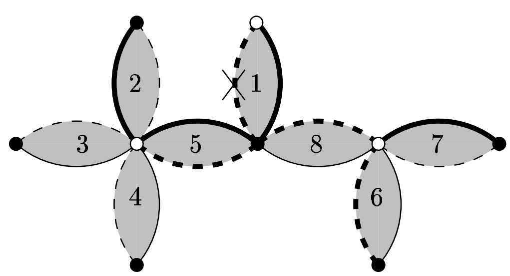


FIGURE 2. A labeled 2-signed 2-cactus of genus 0.

3. Proof of Theorem 2 in the planar case

Our proof relies on an algorithm that takes as input a 4-tuple of objects, called an *unsigned input*, described below.

Definition 1. An unsigned input of genus g is a 4-tuple $(\mathcal{C}, \vec{W}, \vec{B}, \pi)$ where π is a k-signed n-cycle (for some positive integer k), $\mathcal{C}$ is a π-labeled (1-signed) 2-cactus of genus g, $\vec{W}$ (resp. $\vec{B}$) is a partition of the white (resp. black) vertices of $\mathcal{C}$ into k sets $W^0, \ldots, W^{k-1}$ (resp. $B^0, \ldots, B^{k-1}$).

We call k-decomposition $\vec{\lambda}$ (resp. $\vec{\mu}$) *induced by* $\vec{W}$ (resp. $\vec{B}$) the unique k-decomposition such that, for every positive integer d, the number of vertices of W^i (resp. B^i) of degree d is equal to the number of parts of size d in λ^i (resp. μ^i), for any i in $\{0, \ldots, k-1\}$.

Definition 2. A k-signed 2-cactus $\mathcal{D}$ is said to be *consistent* with an unsigned input $(\mathcal{C}, \vec{W}, \vec{B}, \pi)$ if:

- its underlying unsigned 2-cactus is $\mathcal{C}$,
- for any i in $\{0, \ldots, k-1\}$, $W^i(\mathcal{D}) = W^i$ and $B^i(\mathcal{D}) = B^i$,
- $\mathcal{D}$ corresponds, according to Proposition 1, to a factorization of π.

We first focus on the case where the sign of the factorized n-cycle π is different from $\zeta(\vec{\lambda})\zeta(\vec{\mu})$ (Lemma 1), then we consider the planar case, that is when $g(\lambda, \mu) = 0$. We extend our algorithm for the planar case to the general case in Section 4.

Lemma 1. *Let $(\mathcal{C}, \vec{W}, \vec{B}, \pi)$ be an unsigned input, where π is an n-cycle of sign ξ, $\vec{\lambda}$ and $\vec{\mu}$ be the k-decompositions induced by $\vec{W}$ and $\vec{B}$, and $\mathcal{D}$ be a k-signed 2-cactus consistent with $(\mathcal{C}, \vec{W}, \vec{B}, \pi)$. Then $\xi = \zeta(\vec{\lambda})\zeta(\vec{\mu})$.*

Proof. First we notice that, by definition, the white and black degree distributions of $\mathcal{D}$ are given respectively by $\vec{\lambda}$ and $\vec{\mu}$. By definition of the sign of a vertex and of a degree distribution, we have

$$\zeta(\vec{\lambda})\zeta(\vec{\mu}) = \prod_{i=0}^{k-1} ({\zeta_k}^i)^{|W^i|+|B^i|} = \prod_{e \text{ edge of } \mathcal{D}} \zeta(e, \mathcal{D}).$$

Now the sign of a n-cycle is the product of the signs of the elements in its cycle representation, and for any $j \in [n]$, if b_ℓ is the edge that precedes w_j in a traversal of $\mathcal{D}$, then according to Proposition 1 $\zeta(j, \pi) = \zeta(b_\ell, \mathcal{D})\zeta(w_j, \mathcal{D})$. As the sign of π is ξ, it follows that

$$\xi = \prod_{e \text{ edge of } \mathcal{D}} \zeta(e, \mathcal{D}),$$

hence $\xi = \zeta(\vec{\lambda})\zeta(\vec{\mu})$. □

From now, we suppose that for every unsigned input $(\mathcal{C}, \vec{W}, \vec{B}, \pi)$, the decompositions $\vec{\lambda}$ and $\vec{\mu}$ respectively induced by $\vec{W}$ and $\vec{B}$ are such that the sign of π is equal to $\zeta(\vec{\lambda})\zeta(\vec{\mu})$. We now describe an algorithm that, given such an input $(\mathcal{C}, \vec{W}, \vec{B}, \pi)$, produces a k-signed 2-cactus consistent with this input.

Algorithm 1. (Input: an unsigned input $(\mathcal{C}, \vec{W}, \vec{B}, \pi)$ of genus 0, where π is a k-signed n-cycle. Output: a k-signed 2-cactus $\mathcal{D}$ with n 2-gons.)
Let $\mathcal{D} = \mathcal{C}$. As long as all the edges of $\mathcal{D}$ have not received a sign, traverse $\mathcal{D}$ and, for any pair of unsigned consecutive edges (b_i, w_j) (*i.e.* w_j follows immediately b_i),

(a). if b_i is the last unsigned black edge incident to its black vertex x, with $x \in B^\ell$, then:
- b_i receives the only possible sign such that $\zeta(x, \mathcal{D}) = {\zeta_k}^\ell$,
- $\zeta(w_j, \mathcal{D}) = \zeta(j, \pi)/\zeta(b_i, \mathcal{D})$.

(b). if w_j is the last unsigned white edge incident to its white vertex y, with $y \in W^\ell$, then:
- w_j receives the only possible sign such that $\zeta(y, \mathcal{D}) = {\zeta_k}^\ell$,
- $\zeta(b_i, \mathcal{D}) = \zeta(j, \pi)/\zeta(w_j, \mathcal{D})$.

Example 5. Let $k = 2$, and $\mathcal{C}$, π and a partition (W^0, W^1, B^0, B^1) of the vertices of $\mathcal{C}$ as in Figure 3 below. Then the output $\mathcal{D}$ of Algorithm 1 on this input is the k-signed 2-cactus displayed in Figure 2.

Lemma 2. *Let $(\mathcal{C}, \vec{W}, \vec{B}, \pi)$ be an unsigned input of genus 0. During an execution of Algorithm 1 with $(\mathcal{C}, \vec{W}, \vec{B}, \pi)$, either Rule (a) or (b) is applied to any pair of consecutive edges of $\mathcal{D}$.*

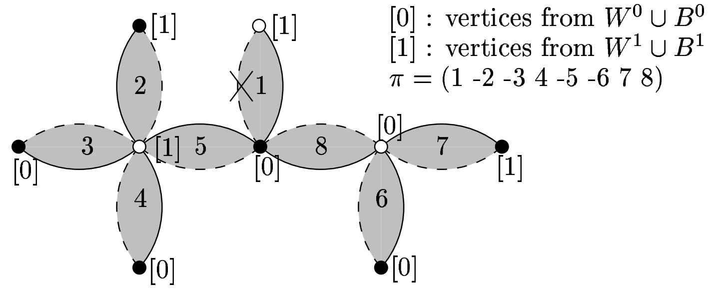


FIGURE 3. An input for Algorithm 1.

Proof. The *depth* of the vertices of $\mathcal{D}$ is defined recursively in the following way: vertices of degree 1 have depth 0, and vertices of depth $i > 0$ are those vertices whose all neighbors (vertices adjacent through an edge) but one have depth less than i, with at least one neighbor of depth $i - 1$.

Now let (b_i, w_j) be a pair of consecutive edges such that b_i is incident to vertices x and y, and assume, without loss of generality, that y is deeper than x. If x has depth 0, then both edges b_i and w_j receive a sign at their first visit. Otherwise, all x's neighbors but y have lower depth than x (denote by ℓ the depth of x), and by induction after $\ell - 1$ visits, all the black edges incident to x but b_i are signed. Hence b_i and w_j are signed after ℓ visits. □

Notation. For a pair (b_i, w_j) of consecutive edges in a 2-cactus $\mathcal{D}$, we denote by $\mathcal{D}_i$, $\mathcal{D}_j$ and $\mathcal{D}_{i,j}$ the subcacti defined as shown in Figure 4 below.

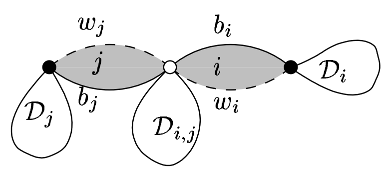


FIGURE 4. The three subcacti induced by a pair of consecutive edges (b_i, w_j).

Claim 1. *Consider an execution of Algorithm 1, and (b_i, w_j) a pair of consecutive edges in the 2-cactus $\mathcal{D}$ processed during this execution. Then, at any time during this process, the following property holds: if the hypothesis of Rule (a) (resp. (b)) is satisfied by b_i (resp. w_j) then all the edges in $\mathcal{D}_i$ (resp. $\mathcal{D}_{i,j}$) did receive a sign.*

Proof. We proceed by induction on the number of 2-gons in $\mathcal{D}_i$ and $\mathcal{D}_{i,j}$. The property clearly holds if $\mathcal{D}_i$ (resp. $\mathcal{D}_{i,j}$) is empty. Now assume that the property holds if $\mathcal{D}_i$ and $\mathcal{D}_{i,j}$ each have at most p 2-gons ($p \geq 0$), and suppose that $\mathcal{D}_i$ has $(p+1)$ 2-gons. Let x be the black vertex of b_i and (b_ℓ, w_m) be a pair of consecutive edges of $\mathcal{D}_i$ such that x is also the black vertex of b_ℓ (such edges exist since $\mathcal{D}_i$ is not empty). If Rule (a) can be applied to (b_i, w_j), then b_ℓ and w_m necessarily received their signs previously trough Rule (b). As, in this case, $\mathcal{D}_\ell$ and $\mathcal{D}_{\ell,m}$ have each less than p 2-gons, the property holds for $\mathcal{D}_i$ by induction. Similarly, the property for Rule (*b*) holds for $\mathcal{D}_{i,j}$, which ends the proof. □

Claim 2. *Consider an execution of Algorithm 1, and (b_i, w_j) a pair of consecutive edges in the 2-cactus $\mathcal{D}$ processed during this execution. If Rules (a) and (b) can be applied to (b_i, w_j) during the same traversal of $\mathcal{D}$, then b_i and w_j are the last unsigned edges in $\mathcal{D}$.*

Proof. Let $b_i = (x, y)$ and $w_j = (y, z)$ where x and z are black vertices and y is a white vertex. It follows immediately from Claim 1 that all the edges in $\mathcal{D}_i$ and $\mathcal{D}_{i,j}$ are signed. But as all edges of $\mathcal{D}_{i,j}$ are signed and w_j is not, one knows that b_j is already signed and that when it received a sign it was the last unsigned black edge incident to z (the corresponding white edge belongs to $\mathcal{D}_{i,j}$ and is signed due to the fact that w_j is not). It implies that the edges in $\mathcal{D}_j$ are all signed. Hence all the edges but b_i and w_j are already signed. □

Lemma 3. *Let $(\mathcal{C}, \vec{W}, \vec{B}, \pi)$ be an unsigned input of genus* 0*. The output of Algorithm 1 applied to $(\mathcal{C}, \vec{W}, \vec{B}, \pi)$ is consistent with the unsigned input $(\mathcal{C}, \vec{W}, \vec{B}, \pi)$.*

Proof. Let $\mathcal{D}$ be the k-signed 2-cactus resulting from Algorithm 1 and (b_i, w_j) be a pair of consecutive edges of $\mathcal{D}$. If it is not the last unsigned pair, Claim 2 states that only one of Rules (a) and (b) determines their signs, and it follows from Remark 3 that the signs given at this step do not violate any constraint induced by the unsigned input, and that $\zeta(j, \pi) = \zeta(b_i, \mathcal{D})\zeta(w_j, \mathcal{D})$. Hence it remains to verify that the same happens when processing the last pair of consecutive edges

Let ξ be the sign of the cycle π, $\vec{\lambda}$ and $\vec{\mu}$ the k-decompositions induced respectively by $\vec{W}$ and $\vec{B}$, and (b_i, w_j) the last pair of (consecutive) unsigned edges. It follows immediately from the definition of the sign of a k-signed n-cycle that the product of the signs of all the signed edges of $\mathcal{D}$ (that is all the edges but b_i and w_j) is equal to $\xi/\zeta(j, \pi)$. Moreover, we can deduce from the assumption that $\xi = \zeta(\vec{\lambda})\zeta(\vec{\mu})$ and from the fact that $\zeta(\vec{\lambda})\zeta(\vec{\mu}) = \prod_{e \in \mathcal{D}} \zeta(e, \mathcal{D})$ (proof of Lemma 1), that $\zeta(j, \pi)$ should be equal to $\zeta(b_i, \mathcal{D})\zeta(w_j, \mathcal{D})$, which shows that the signs given to (b_i, w_j) make the signed 2-cactus $\mathcal{D}$ consistent with the unsigned input. □

Lemma 4. *Let $(\mathcal{C}, \vec{W}, \vec{B}, \pi)$ be an unsigned input of genus* 0 *and $\mathcal{D}$ the corresponding output by Algorithm 1. Then $\mathcal{D}$ is the only k-signed 2-cactus consistent with $(\mathcal{C}, \vec{W}, \vec{B}, \pi)$.*

Proof. Let $\mathcal{E}$ be another consistent k-signed 2-cactus for the input. Perform a parallel traversal of $\mathcal{D}$ and $\mathcal{E}$, and let e be the first edge with different signs in $\mathcal{D}$ and $\mathcal{E}$. There can not be a white edge, because $\mathcal{E}$ would necessarily violate Remark 3.

So $e = b_i = (x, y)$ is a black edge; we show by induction on $|\mathcal{D}_i|$ that this situation leads to a contradiction. If x has degree 1 (*i.e.* $|\mathcal{D}_i| = 0$), the contradiction is immediate. Else, one of the other black edges b_j (x, y') incident to x has not the same sign in $\mathcal{D}$ and $\mathcal{E}$, which implies that the white edge $w_\ell = (y', z')$ that forms a consecutive pair with b_j has not the same sign in $\mathcal{D}$ and $\mathcal{E}$. If y' has degree 1 $(z' = x)$, then the sign of y' should be the same in $\mathcal{D}$ and $\mathcal{E}$, and we have a contradiction. Else, there should be another black edge $b_m = (x', y')$ in $\mathcal{D}_i$ with different signs in $\mathcal{D}$ and $\mathcal{E}$. As $|\mathcal{D}_m| < |\mathcal{D}_\ell|$, the induction hypothesis leads to a contradiction. □

Proposition 2. *Let $(\mathcal{C}, \vec{W}, \vec{B}, \pi)$ be an unsigned input of genus* 0*. There is only one signed* 2*-cactus $\mathcal{D}$ consistent with $(\mathcal{C}, \vec{W}, \vec{B}, \pi)$ and Algorithm 1 computes $\mathcal{D}$.*

Proof. Proposition 2 follows immediately from Lemmas 2, 3 and 4. □

Remark 4. For a k-signed 2-cactus $\mathcal{C}$ with n 2-gons, a k-signed n-cycle π, and two k-decompositions $\vec{\lambda}$ and $\vec{\mu}$ of partitions λ and μ of weight n (where $\lambda^i = 1^{\alpha_1^i} \dots n^{\alpha_n^i}$ and $\mu^i = 1^{\beta_1^i} \dots n^{\beta_n^i}$), there are exactly $\prod_{i=1}^n \binom{\alpha_i}{\alpha_i^0, \dots, \alpha_i^{k-1}} \binom{\beta_i}{\beta_i^0, \dots, \beta_i^{k-1}}$ distinct unsigned inputs $(\mathcal{C}, \vec{W}, \vec{B}, \pi)$ such that $\vec{\lambda}$ and $\vec{\mu}$ are respectively induced by $\vec{W}$ and $\vec{B}$.

Proof of Theorem 2 in the planar case. Given a signed 2-cactus $\mathcal{D}$ of genus 0, there is clearly only one unsigned input $(\mathcal{C}, \vec{W}, \vec{B}, \pi)$ of genus 0 such that $\mathcal{D}$ is consistent with $(\mathcal{C}, \vec{W}, \vec{B}, \pi)$. This, together with Proposition 2, implies that there is a bijection between unsigned inputs of genus 0 and signed 2-cactus of genus 0. This fact, Remark 4 and Lemma 1 prove Theorem 2 in the planar case. □

4. Proof of Theorem 2 in the general case

We now turn to the general case when an unsigned input $(\mathcal{C}, \vec{W}, \vec{B}, \pi)$ has no restriction on the genus g of $\mathcal{C}$. We want to extend Algorithm 1 so that it produces $k^{2g(\lambda,\mu)}$ k-signed 2-cacti consistent with $(\mathcal{C}, \vec{W}, \vec{B}, \pi)$. The general principle of this extension is to use Algorithm 1 on a planar substructure of a 2-cactus. Hence we introduce a natural notion of *planar subcactus* of a cactus.

Definition 3. Let $\mathcal{C}$ be a signed 2-cactus with m vertices. A *planar subcactus of* $\mathcal{C}$ is any (connected) submap of $\mathcal{C}$ that is a 2-cactus of genus 0 with m vertices.

A subcactus can clearly be obtained by removing 2-gons from $\mathcal{C}$ and it follows immediately from Euler formula (1) that, given any 2-cactus $\mathcal{C}$ of genus g and any planar subcactus $\mathcal{C}'$ of $\mathcal{C}$, there are exactly $2g$ 2-gons of $\mathcal{C}$ that do not belong to $\mathcal{C}'$. Any 2-cactus of genus g can be decomposed in a set of $2g$ 2-gons (called a *non-planar subset of* $\mathcal{C}$) and a planar subcactus.

We now describe an algorithm that, provided any unsigned input $(\mathcal{C}, \vec{W}, \vec{B}, \pi)$ of genus g, any non-planar subset $(i_1, \dots, i_{2g})$ of $\mathcal{C}$ (where the i_js are labels of 2-gons of $\mathcal{C}$) and any set of $2g$ signs $(s_1, \dots, s_{2g})$, produces a k-signed 2-cactus consistent with the input $(\mathcal{C}, \vec{W}, \vec{B}, \pi)$. The main idea behind this algorithm is that, once each edge b_{i_j} for $j \in [2g]$ has received the sign s_j, signs of all other edges are uniquely determined by Algorithm 1. For the clarity of the proof, we present it (Algorithm 2) in a slightly different way: we remove the 2-gons $i_1, \dots, i_{2g}$, which produces a planar 2-cactus. Next, in order to apply Algorithm 1 on an unsigned input with this planar 2-cactus, we modify π and the partitions $\vec{W}$ and $\vec{B}$, according to this planar subcactus and $(s_1, \dots, s_{2g})$, so that the assumption that the sign of π is equal to $\zeta(\vec{\lambda})\zeta(\vec{\mu})$ still holds (step (1)). Algorithm 1 can then be applied on the resulting unsigned input (step (2)). This gives the signs for most of the edges of the resulting signed 2-cactus and some last modifications needed to take into account the modifications done in step (1) (step (3)).

Algorithm 2. (Input: an unsigned input $(\mathcal{C}, \vec{W}, \vec{B}, \pi)$ of genus g, where π is a k-signed n-cycle, a non-planar subset $(i_1, \dots, i_{2g})$ for $\mathcal{C}$ and an ordered list $(s_1, \dots, s_{2g})$ of signs. Output: a k-signed 2-cactus $\mathcal{D}$ of genus g with n 2-gons.)

(1) Let $\pi_0 = \pi$, $\mathcal{C}_0 = \mathcal{C}$, $\vec{W_0} = \vec{W}$ and $\vec{B_0} = \vec{B}$. For j from 1 to $2g$:
  (a) let x and y be the two vertices incident to the 2-gon i_j, with $x \in B^{\ell}_{j-1}$ and $y \in W^{m}_{j-1}$, and w_p be the white edge following immediately b_{i_j} during a traversal of $\mathcal{C}_{j-1}$;
  (b) remove the 2-gon i_j from $\mathcal{C}_j$ and let $\mathcal{C}_{j+1}$ be the resulting cactus;
  (c) $B^{\ell}_j = B^{\ell}_{j-1} \setminus \{x\}$, $B^{\ell/s_j}_j = B^{\ell/s_j}_{j-1} \cup \{x\}$,
  (d) $W^m_j = W^m_{j-1} \setminus \{y\}$, $W^{ms_j/\zeta(p,\pi)}_j = W^{ms_j/\zeta(p,\pi)}_{j-1} \cup \{y\}$;
  (e) remove the element of absolute value i_j from π_{j-1}, give to p the sign of $\zeta(i_j, \pi)$ and let π_j be the resulting k-signed cycle.

  Root $\mathcal{C}_{2g}$ at the white edge incident to the 2-gon with the smallest label.

(2) Perform Algorithm 1 on the unsigned input $(\mathcal{C}_{2g}, \vec{W_{2g}}, \vec{B_{2g}}, \pi_{2g})$. Let $\mathcal{D}_{2g}$ be the resulting cactus.

(3) For j from $2g$ to 1:
  (a) insert in $\mathcal{D}_j$ a 2-gon labeled with i_j in the same position than in $\mathcal{C}$ and let $\mathcal{D}_{j-1}$ be the resulting cactus (as an unsigned cactus, $\mathcal{D}_{j-1}$ is equal to $\mathcal{C}_{j-1}$);
  (b) let (b_{i_j}, w_p) and (b_ℓ, w_{i_j}) be the two consecutive pairs in $\mathcal{D}_{j-1}$ involving b_{i_j} and w_{i_j};
  (c) $\zeta(b_{i_j}, \mathcal{D}_{j-1}) = s_j$, $\zeta(w_{i_j}, \mathcal{D}_{j-1}) = \zeta(w_p, \mathcal{D}_{j-1})$, $\zeta(w_p, \mathcal{D}_{j-1}) = \zeta(p, \pi)/s_j$.

  Root $\mathcal{D}_0$ at w_1 and let $\mathcal{D} = \mathcal{D}_0$.

A detailed example of this algorithm is given in Figure 5 below.

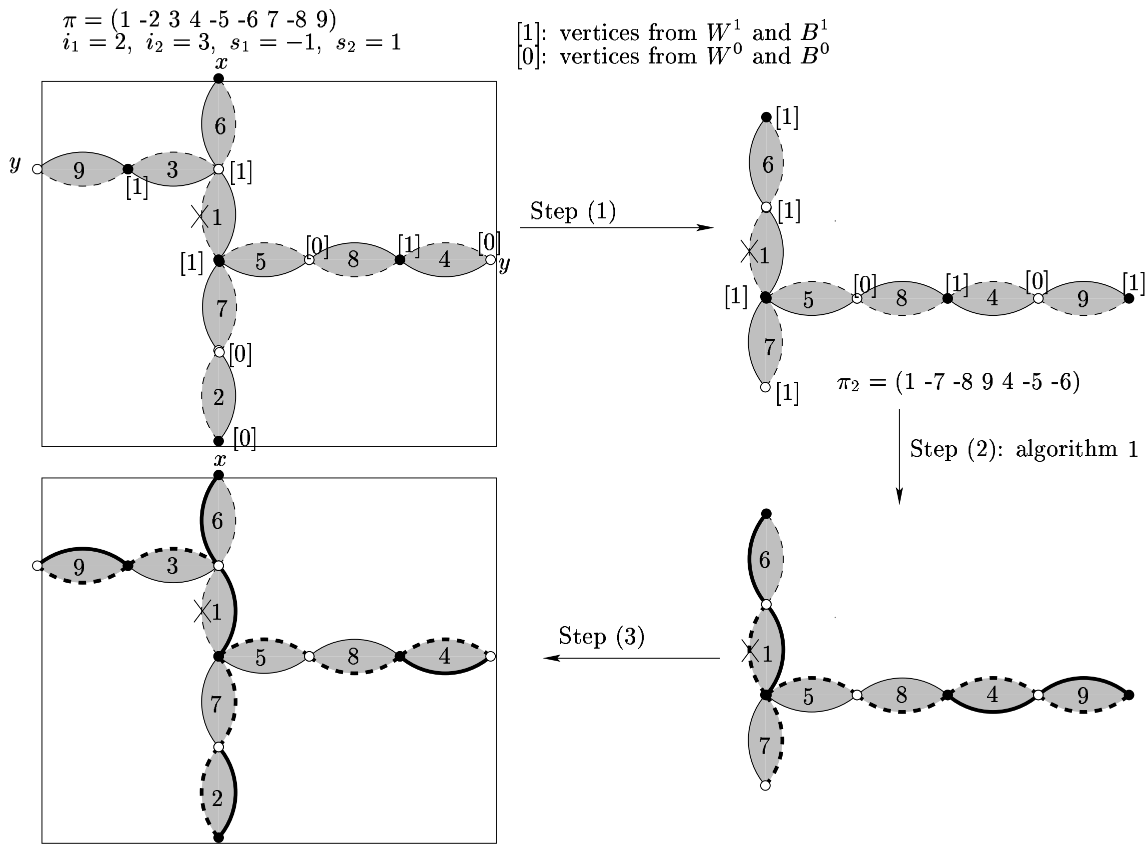


FIGURE 5. Running Algorithm 2 with $k = 2$ and $g = 1$

Lemma 5. *Let $(\mathcal{C}, \vec{W}, \vec{B}, \pi)$ be an unsigned input, $(i_1, \ldots, i_{2g})$ be any non-planar subset for $\mathcal{C}$ and $(s_1, \ldots, s_{2g})$ be any tuple of signs. The k-signed 2-cactus $\mathcal{D}$ resulting from Algorithm 2 applied on $((\mathcal{C}, \vec{W}, \vec{B}, \pi), (i_1, \ldots, i_{2g}), (s_1, \ldots, s_{2g}))$ is consistent with $(\mathcal{C}, \vec{W}, \vec{B}, \pi)$.*

Proof. It is easy to verify that after step (1) of Algorithm 2 (the modification that produced $(\mathcal{C}_{2g}, \vec{W}_{2g}, \vec{B}_{2g}, \pi_{2g})$), the sign of π_{2g} is equal to the product of the sign of the k-decompositions induced by $\vec{W}_{2g}$ and $\vec{B}_{2g}$. Then we can apply Algorithm 1 on $(\mathcal{C}_{2g}, \vec{W}_{2g}, \vec{B}_{2g}, \pi_{2g})$, and by Proposition 2, the resulting cactus $\mathcal{D}_{2g}$ is the only signed 2-cactus consistent with $(\mathcal{C}_{2g}, \vec{W}_{2g}, \vec{B}_{2g}, \pi_{2g})$. Now step (3) can be seen as the reverse of step (1) and clearly leads to a signed 2-cactus consistent with $(\mathcal{C}, \vec{W}, \vec{B}, \pi)$. □

Lemma 6. *Let $(\mathcal{C}, \vec{W}, \vec{B}, \pi)$ be an unsigned input, $(i_1, \ldots, i_{2g})$ be any non-planar subset for $\mathcal{C}$ and $(s_1, \ldots, s_{2g})$ be any tuple of signs. The k-signed 2-cactus $\mathcal{D}$ resulting from Algorithm 2 applied on $((\mathcal{C}, \vec{W}, \vec{B}, \pi), (i_1, \ldots, i_{2g}), (s_1, \ldots, s_{2g}))$ is the only one that is consistent with $(\mathcal{C}, \vec{W}, \vec{B}, \pi)$ and such that, for every $j \in [2g]$, the sign of b_{i_j} is s_j.*

Proof. Let $\mathcal{E}$ be a signed 2-cactus consistent with $(\mathcal{C}, \vec{W}, \vec{B}, \pi)$, with $\zeta(b_{i_j}, \mathcal{E}) = s_j$ for $j \in [2g]$. When one removes the 2-gons labeled with $i_1, \ldots, i_{2g}$ and one modifies the signs of the edges according to step (3) of Algorithm 2, one obtains a planar signed 2-cactus $\mathcal{E}'$ consistent with $(\mathcal{C}_{2g}, \vec{W}_{2g}, \vec{B}_{2g}, \pi_{2g})$. Indeed, let (b_ℓ, w_m) be a pair of consecutive edges in $\mathcal{E}'$. If neither b_m nor w_ℓ has been modified when removing the 2-gons $i_1, \ldots, i_{2g}$, then, as $\mathcal{E}$ is consistent with $(\mathcal{C}, \vec{W}, \vec{B}, \pi)$, $\zeta(m, \pi_{2g}) = \zeta(m, \pi) =$

$\zeta(b_\ell,\mathcal{E}')\zeta(w_m,\mathcal{E}')m$. Otherwise, the two edges are incident in $\mathcal{E}$ to a 2-gon i_j. Now step (3) sets the sign of w_{i_j} of $\mathcal{E}$ equal to the sign of w_m in $\mathcal{E}'$, and $\mathcal{E}$ is consistent with $(\mathcal{C},\vec{W},\vec{B},\pi)$. Hence $\zeta(m,\pi_{2g}) = \zeta(b_\ell,\mathcal{E}')\zeta(w_m,\mathcal{E}')j$. Now, if $\mathcal{E}'$ is consistent with $(\mathcal{C}',\vec{W}',\vec{B}',\pi')$, then $\mathcal{E}' = \mathcal{D}_{2g}$, which implies that $\mathcal{E} = \mathcal{D}$, due to the deterministic nature of step (3). □

Proof of Theorem 2. The case $\zeta(\vec{\nu}) \neq \zeta(\vec{\lambda})\zeta(\vec{\mu})$ follows immediately from Lemma 1.

Now, let (σ,τ) be a pair of permutations of $\mathfrak{S}_n$ such that $\sigma\tau$ is a canonic n-cycle, with respective cycle types λ and μ; let π be a canonic k-signed n-cycle of cycle type $\vec{\nu}$, and $\vec{\lambda}$ and $\vec{\mu}$ two k-decompositions of λ and μ such that $\zeta(\vec{\nu}) = \zeta(\vec{\lambda})\zeta(\vec{\mu})$. Let us denote by $\mathcal{C}$ the 2-cactus corresponding to (σ,τ) (according to Proposition 1 in the case $k=1$), g its genus and $(i_1,\ldots,i_{2g})$ any non-planar subset for $\mathcal{C}$.

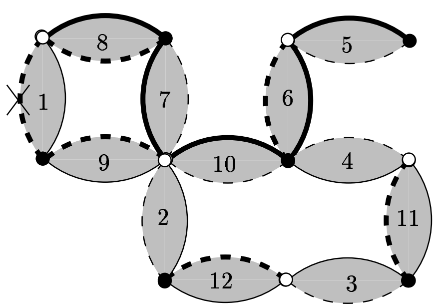
Lemmas 5 and 6 imply that Algorithm 2 defines an injective application from the set of all possible $((\mathcal{C},\vec{W},\vec{B},\pi),(i_1,\ldots,i_{2g}),(s_1,\ldots,s_{2g}))$ such that $\vec{W}$ and $\vec{B}$ induce $\vec{\lambda}$ and $\vec{\mu}$ and $(s_1,\ldots,s_{2g})$ is an ordered list of signs to the set of factorizations (σ',τ') of π of cycle types $\vec{\lambda}$ and $\vec{\mu}$. As every such factorization induces clearly an input for Algorithm 2, it is also surjective, and then bijective. This fact, together with Remark 4 and the fact that there are exactly k^{2g} different ordered sets $(s_1,\ldots,s_{2g})$ of signs ends the proof. □

5. Extension to factorizations of unrestricted permutations

We proved in the previous sections an interesting relationship between factorizations of cycles in the symmetric group and factorizations of cycles in the groups $\mathcal{W}_n^k$. In this last section, we sketch an extension of this result to more general factorizations.

Indeed, we worked here on one of the simplest cases of factorizations of permutations, that is factorizing cycles as product of two permutations. A natural extension of our result is to consider factorizations of any kind of signed permutations. This problem was solved, in the symmetric group, but only in the planar case, by Bousquet-Mélou and Schaeffer [2]. They described a family of planar maps, called planar *constellations*, that generalizes unsigned 2-cacti of genus 0 and gives a combinatorial model for the study of such factorizations.

2-constellations of genus g are *bipartite maps* (vertices are colored black and white and every edge is incident to a black vertex and a white vertex) of genus g where edges have been replaced by 2-gons (see Figure 6 below). It follows immediately from the work of Bousquet-Mélou and Schaeffer (see also the papers by Cori and Machì [3]) and Section 2 of the present paper that k-signed 2-constellations are in one-to-one correspondence with factorizations of k-signed permutations as a product of two k-signed permutations, in such a way that the cycles of the product permutation are in correspondence with the white faces of the k-signed 2-constellation.



σ = (-1 -8)(2 10 7 -9)(3 -12)(4 -11)(5 -6)
τ = (1 9)(2 12)(3 11)(4 -6 -10)(-5)(-7 -8)
$\pi = \sigma\tau$ = (1 2 3 4 -5 6 -7)(-8 9)(10 -11 -12)

FIGURE 6. A 2-constellation with three white faces and the corresponding factorization $\sigma\tau = \pi$.

Moreover, a k-signed 2-constellation with f faces can be seen as a k-signed 2-cactus augmented by $f-1$ 2-gons. The technique of Algorithm 2 (removing 2-gons from a k-signed 2-constellations in order to obtain a k-signed 2-cactus that can be processed with Algorithm 2) leads to the following generalization of Theorem 2 and [2].

Theorem 3. *Let k and n be two integers, $\lambda = 1^{\alpha_1} \dots n^{\alpha_n}$, $\mu = 1^{\beta_1} \dots n^{\beta_n}$ and ν be three partitions of weight n, $\vec{\lambda} = (\lambda^0, \dots, \lambda^{k-1})$ a k-decomposition of λ, $\vec{\mu} = (\mu^0, \dots, \mu^{k-1})$ a k-decomposition of μ (where $\lambda^i = 1^{\alpha_1^i} \dots n^{\alpha_n^i}$ and $\mu^i = 1^{\beta_1^i} \dots n^{\beta_n^i}$) and $\vec{\nu}$ a k-decomposition of ν. Then*

$$c^{\vec{\nu}}_{\vec{\lambda},\vec{\mu}} = \begin{cases} 0 & \text{if } \zeta(\vec{\nu}) \neq \zeta(\vec{\lambda}).\zeta(\vec{\mu}) \\ \left(\prod_{j=1}^{n} \binom{\alpha_j}{\alpha_j^0, \dots, \alpha_j^{k-1}} \binom{\beta_j}{\beta_j^0, \dots, \beta_j^{k-1}} \right) k^{2g(\lambda,\mu,\nu)+\ell(\nu)-1} c^{\nu}_{\lambda,\mu} & \text{otherwise,} \end{cases}$$

where $g(\lambda, \mu, \nu)$ is defined by $\ell(\lambda) + \ell(\mu) + \ell(\nu) = n + 2 - 2g(\lambda, \mu)$.

6. Conclusion

The results of this article are, as far as we know, the first combinatorial results on the enumeration of factorizations in $\mathcal{W}_n^k$. The (constructive) proof of our result relies strongly (and intuitively) on the representation of such factorizations as maps (k-signed 2-cacti).

We restricted here our study to the case of factorizations as a product of two permutations. A natural extension would consist in the study of factorizations as a product of m permutations. The case of cycles in $\mathfrak{S}_n$ has been studied by Poulalhon and Schaeffer [8], while the planar case for general permutations has been done by Bousquet-Mélou and Schaeffer [2]. The combinatorial model would be k-signed m-constellations (2-gons would be replaced by m-gons). It seems that with little more technicalities, our methods can be extended in this case, and we are currently working on this question.

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