PRODUCTION MATRICES

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ABSTRACT. We translate the concept of succession rule and the ECO method into matrix notation, introducing the concept of a *production matrix*. Among other things, this allows us to combine our method with other enumeration techniques using matrices, such as the method of Riordan matrices. Moreover, we show that certain operations on production matrices correspond to well known operations on the numerical sequences determined by them.

Nous transférons l'idée de règle de succession et la mèthode ECO dans la notation des matrices, en introduisant l'idée de *matrice de production*. Entre autres choses, ça nous a permis de concilier notre méthode avec d'autres techniques d'énumération qui utilisent des matrices, comme la méthode des matrices de Riordan. En autre, nous montrons que certaines opérations sur les matrices de production correspondent à des opérations bien connues sur les séquences numériques qu'elles déterminant.

1. INTRODUCTION

A succession rule is a formal system that defines a non-decreasing sequence of positive integers. Succession rules have been studied in various works, for example [BBDFGG] and [FPPR], and they are directly related to an enumerative method called ECO method [BDLPP]. The aim of the present paper is to provide a computational version of the ECO method and succession rules by encoding the latter by matrices, called *production matrices*. Indeed any succession rule has a simple representation in terms of production matrices, which allows us to use matrix notation to handle the rules from the enumerative point of view. The idea of translating a combinatorial theory into a theory of infinite matrices is actually a current trend in discrete mathematics. Riordan arrays [DS, R, SGWW, Sp], recursive matrices [BBN], Aigner's admissible matrices [A1, A2], to cite only a few, represent an explicit justification of the previous statement. A comparison between the Riordan array theory and the production matrices method is established; however it surely deserves to be further investigated. In particular, the new concept of exponential Riordan matrix [DS] is studied for what concerns its relations with the present topic. Another problem considered in the next pages is that of defining operations on succession rules, reflecting well-known operations on the numerical sequences they determine. It turns out that production matrices provide a very neat description of such operations; several examples and applications are scattered throughout the text to illustrate this fact.

2. BASIC DEFINITIONS

A succession rule is a formal system consisting of an *axiom* (a), $a \in \mathbf{N}^+$, and a set of *productions*:

$$\{(k_t) \rightsquigarrow (e_1(k_t))(e_2(k_t))\dots(e_{k_t}(k_t)) : t \in \mathbf{N}\},$$

where $e_i : \mathbf{N}^+ \longrightarrow \mathbf{N}^+$, which explains how to derive the *successors* $(e_1(k)), (e_2(k)), \dots, (e_k(k))$ of any given label (k), $k \in \mathbf{N}^+$. In general, for a succession rule Ω, we use the more compact notation:

$$\Omega : \begin{cases} (a) \\ (k) \rightsquigarrow (e_1(k))(e_2(k)) \dots (e_k(k)), \qquad \text{for } k \in \mathbf{N}^+. \end{cases} \tag{1}$$

The positive integers (a), (k), $(e_i(k))$, are called *labels* of Ω. The rule Ω can be represented by means of a *generating tree*, that is a rooted tree whose vertices are the labels of Ω; (a) is the label of the root and each node labelled (k) has k sons labelled by $e_1(k), \dots, e_k(k)$ respectively, according to the production of (k) in (1). A succession rule Ω defines a sequence of positive integers $(a_n)_{n\geq 0}$, a_n being the number of the nodes at level n in the generating tree defined by Ω. By convention the root is at level 0, so $a_0 = 1$. The function $f_\Omega(x) = \sum_{n\geq 0} a_n x^n$ is the *generating function* determined by Ω. We refer to [BDLPP] for further details on these topics.

In this paper we propose a new approach for the study of succession rules, based on linear algebra tools.

Instead of representing succession rules by generating trees, we represent them by infinite matrices $P = (p_{k,i})_{k,i\geq 0}$. Assume that the set of the labels of a succession rule is $\{(l_k)\}_k$, and in particular that l_0 is the label of the axiom. Then we define $p_{k,i}$ to be the number of labels l_i produced by label l_k. We call P the *production matrix* of the given succession rule. Observe that the first row of a production matrix gives precisely the production of the axiom.

The labels do not occur explicitly in this matrix representation of the succession rule. However, they are the row sums of the matrix. In particular, the label l_0 of the axiom is the first row sum of P.

Example. To the succession rule

$$\begin{cases} (2) \\ (2) \rightsquigarrow (3)^2 \\ (k) \rightsquigarrow (3)(4)\dots(k)(k+1)^2, \end{cases} \tag{2}$$

there corresponds the production matrix

$$P = \begin{pmatrix} 0 & 2 & 0 & 0 & \dots \\ 0 & 1 & 2 & 0 & \dots \\ 0 & 1 & 1 & 2 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{3}$$

Writing the succession rule as

$$\begin{array}{l} (2) \rightsquigarrow (2)^0(3)^2 \\ (3) \rightsquigarrow (2)^0(3)^1(4)^2 \\ (4) \rightsquigarrow (2)^0(3)^1(4)^1(5)^2 \\ \dots\dots\dots\dots\dots \ \dots\dots, \end{array}$$

the matrix P is nothing but the matrix of the exponents (where an exponent is zero if and only if the label it refers to does not appear in the production).

In the generating tree at level zero we have only one node with label $l_0(=2)$. This is represented by the row vector $r_0 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \end{pmatrix}$.

At the next levels of the generating tree the distribution of the labels $l_1, l_2, \dots$ is given by the row vectors

$$
\begin{aligned}
r_1 = r_0P &= \begin{pmatrix}0 & 2 & 0 & 0 & 0 & 0 & 0 & \ldots\end{pmatrix}\\
r_2 = r_1P &= \begin{pmatrix}0 & 2 & 4 & 0 & 0 & 0 & 0 & \ldots\end{pmatrix}\\
r_3 = r_2P &= \begin{pmatrix}0 & 6 & 8 & 8 & 0 & 0 & 0 & \ldots\end{pmatrix}\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}
$$

Stacking these row matrices, we obtain the matrix

$$
A_P = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & \ldots\\
0 & 2 & 0 & 0 & 0 & 0 & \ldots\\
0 & 2 & 4 & 0 & 0 & 0 & \ldots\\
0 & 6 & 8 & 8 & 0 & 0 & \ldots\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}.
$$

The row sums of the above matrix are $1, 2, 6, 22, 90, 394, 1806, \ldots$, i.e. the large Schröder numbers (A006318 in [Sl]). This is the sequence corresponding to the succession rule of our example. We also recall that matrices like A_P (where the entry (n, k) gives the number of nodes labelled l_k at level n of the generating tree) were also studied in [MV], where they have been called AGT matrices. In general, we will refer to A_P as the *ECO matrix* induced by P.

Remarks. Let P be the production matrix of a given succession rule Ω. Throughout the whole paper we will denote by $u^\top$ the infinite row vector $(1, 0, 0, \ldots, 0, \ldots)$ and by e the column vector $(1, 1, 1, \ldots)^\top$.

The following facts are easy to verify; we state them without any further explanation.

(i) The labels of the nodes of the generating tree corresponding to Ω are the row sums of P. If two row sums happen to be equal, then, as labels, they will be considered to be distinct. This can be achieved by using, for example, distinguishing subscripts; in the vocabulary of succession rules, these are called *colored succession rules* (see [FPPR]).

(ii) The distribution of the nodes having various labels at the various levels is given by the ECO matrix

$$
A_P = \begin{pmatrix} u^\top \\ u^\top P \\ u^\top P^2 \\ \ldots \end{pmatrix}
$$

(indeed, we have $r_0 = u^\top, \quad r_1 = r_0P = u^\top P, \quad r_2 = r_1P = u^\top P^2, \ldots$). The same thing can be expressed in a concise way by the matrix equality

$$
DA_P = A_P P, \tag{4}
$$

where $D = (\delta_{i,j+1})_{i,j\geq 0}$ (δ is the usual Kronecker delta). In some works [Sh] the matrix P is also called the *Stieltjes transform matrix of* A_P.

(iii) The sequence a_n induced by the succession rule is given by $a_n = u^\top P^n e$.

(iv) The bivariate generating function of the matrix A_P is

$$
G(t, z) = u^\top (I - zP)^{-1} \begin{pmatrix} 1 \\ t \\ t^2 \\ \vdots \end{pmatrix}.
$$

(v) The generating function of the sequence corresponding to the succession rule is $f_P(z) = u^\top (I - zP)^{-1} e$.

(vi) The exponential generating function of the sequence corresponding to the succession rule is $F_P(z) = u^\top \exp(zP) e$.

Example. We intend to find the sequence determined by the production matrix

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 1 & 0 & 0 & \dots \\ 0 & 0 & 2 & 1 & 0 & \dots \\ 0 & 0 & 0 & 3 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{5}$$

Simple computations show that the first row of $\exp(zP)$ is

$$\begin{pmatrix} 1 & e^z - 1 & \frac{1}{2!}(e^z-1)^2 & \frac{1}{3!}(e^z-1)^3 & \dots \end{pmatrix}.$$

For the exponential generating function induced by P we obtain $G_P(z) = e^{e^z - 1}$. The corresponding sequence is 1,1,2,5,15,52,203,876,... (A000110 in [Sl]; Bell numbers).

3. Operations on production matrices

In this section we will define some operations to be performed on production matrices in order to describe usual operations on numerical sequences. Several ideas developed in this section have been suggested by [FPPR, PPR]; we provide their translation into the vocabulary of production matrices with a probably more rigorous presentation.

In the sequel we will write $P \longrightarrow a_0, a_1, a_2, \ldots$ to mean that $(a_n)_{n\geq 0}$ is the numerical sequence determined by the production matrix P. Similar expressions (like $P \longrightarrow (a_n)_{n\geq 0}$, $P \longrightarrow f_P(z) = \sum_n a_n z^n$) are intended similarly.

Proposition 3.1. *If $P \longrightarrow f_P(z)$ and k is a positive integer, then $kP \longrightarrow f_P(kz)$.*

Proof. Using generating functions we have $f_{kP}(z) = u^\top (I - zkP)^{-1} e = f_P(kz)$. □

Proposition 3.2. *If $P \longrightarrow (a_n)_{n\geq 0}$, then*

$$M \stackrel{def}{=} P + I \longrightarrow \left(\sum_{k=0}^{n} \binom{n}{k} a_k \right)_{n\geq 0},$$

the binomial transform of (a_n).

Proof. Expanding the binomial we obtain

$$u^\top (P+I)^n e = \sum_{k=0}^{n} \binom{n}{k} u^\top P^k e = \sum_{k=0}^{n} \binom{n}{k} a_k. \quad \square$$

Proposition 3.3. *If $P \longrightarrow a_0, a_1, a_2, \ldots$, then $P^q \longrightarrow a_0, a_q, a_{2q}, a_{3q}, \ldots$. In particular, $P^2 \longrightarrow a_0, a_2, a_4, \ldots$.*

Proof. Let A and B be the matrices induced by P and P^q, respectively. Then the rows of B are $u^\top, u^\top P^q, u^\top P^{2q}, \ldots$, which are rows $1, q+1, 2q+1, \ldots$ of the matrix A. Consequently, the row sums of B are $a_0, a_q, a_{2q}, a_{3q}, \ldots$. □

Often in the sequel we will deal with two numerical sequences, and we would like to describe what happens to production matrices when we consider usual algebraic operations on the sequences. To do so, we need to tackle a technical problem. If the production matrices of the sequences under consideration are both infinite, it could be meaningless to consider block matrices in which some of the blocks are the production matrices above. For example, if P and Q are infinite production matrices, then the expression $\begin{pmatrix} 0 & P \\ 0 & Q \end{pmatrix}$ does not define a matrix (not even an infinite one), because of the presence of the infinite matrix P as a block in the upper part of the array. We will make up for this predicament by reshuffling the lines of the two production matrices. Observe that a sequence defined by a given production matrix P is determined up to a permutation of its rows, provided that:

(1) the first row remains fixed,
(2) every permutation of the rows is followed by the same permutation of the columns.

Proposition 3.4. *If* $P \longrightarrow 1, a_1, a_2, \dots,$ *and* $Q \longrightarrow 1, b_1, b_2, \dots,$ *then*

$$M \stackrel{def}{=} \begin{pmatrix} 0 & u^\top P & u^\top Q \\ 0 & P & 0 \\ 0 & 0 & Q \end{pmatrix} \longrightarrow 1, a_1 + b_1, a_2 + b_2, \dots . \tag{6}$$

Proof. Taking into account that $u^\top M^k = \begin{pmatrix} 0 & u^\top P^k & u^\top Q^k \end{pmatrix}$, for the matrix A_M induced by M we obtain

$$A_M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & u^\top P & u^\top Q \\ 0 & u^\top P^2 & u^\top Q^2 \\ \vdots & \vdots & \vdots \end{pmatrix}.$$

From here it follows at once that the row sums of A_M are $1, a_1 + b_1, a_2 + b_2, \dots$. □

Proposition 3.5. *If* $P \longrightarrow a_0, a_1, a_2, \dots,$ *and* $Q \longrightarrow b_0, b_1, b_2, \dots$ *, then*

$$M \stackrel{def}{=} \begin{pmatrix} P & eu^\top Q \\ 0 & Q \end{pmatrix} \longrightarrow c_0, c_1, c_2, \dots ,$$

where (c_n) *is the convolution of the sequences* (a_n) *and* (b_n)*.*

Proof. We have

$$(I - zM)^{-1} = \begin{pmatrix} (I - zP)^{-1} & (I - zP)^{-1} eu^\top [(I - zQ)^{-1} - I] \\ \star & \star \end{pmatrix},$$

where the entries not shown are irrelevant. Now,

$$\begin{aligned} f_M(z) &= u^\top (I - zM)^{-1} e \\ &= \begin{pmatrix} u^\top & 0 \end{pmatrix} \begin{pmatrix} (I - zP)^{-1} & (I - zP)^{-1} eu^\top [(I - zQ)^{-1} - I] \\ \star & \star \end{pmatrix} \begin{pmatrix} e \\ e \end{pmatrix} \\ &= u^\top (I - zP)^{-1} eu^\top (I - zQ)^{-1} e = f_P(z) f_Q(z). \quad \square \end{aligned}$$

Example. Taking for both P and Q the matrix $\begin{pmatrix}0&1\\1&1\end{pmatrix}$, we obtain the production matrix

$$\begin{pmatrix}0&1&0&1\\1&1&0&1\\0&0&0&1\\0&0&1&1\end{pmatrix}$$

that induces the convolution of the Fibonacci sequence with itself, i.e. 1, 2, 5, 10, 20, 38, 71, 130,... (A001629 in [Sl]).

Corollary 3.1. *If* $P \longrightarrow a_0, a_1, a_2, \dots$, *then*

$$M \stackrel{def}{=} \begin{pmatrix}1 & u^\top P\\0 & P\end{pmatrix} \longrightarrow a_0, a_0+a_1, a_0+a_1+a_2, \dots ,$$

the sequence of the partial sums of (a_n).

Proof. Just observe that the sequence of the partial sums of a sequence a_n is the convolution of that sequence with the sequence $(1, 1, 1, \dots)$, the latter having (1) as its production matrix. $\square$

Example. We take the production matrix P given in (**??**), inducing the Fibonacci sequence. Then

$$M = \begin{pmatrix}1&0&1\\0&0&1\\0&1&1\end{pmatrix}$$

induces the sequence of partial sums 1, 2, 4, 7, 12, 20, 33, 54, 88, 143,... (A000071 in [Sl]). Note that these are the Fibonacci numbers minus 1.

Proposition 3.6. *If* $P \longrightarrow a_0, a_1, a_2, \dots$, *and* $Q \longrightarrow b_0, b_1, b_2, \dots$, *then* $P \otimes Q \longrightarrow a_0b_0, a_1b_1, a_2b_2, \dots$, *where* $\otimes$ *denotes Kronecker product.*

Proof. We recall some simple properties of the Kronecker product, namely that $(U \otimes V)^n = U^n \otimes V^n$ and that the first row sum of a Kronecker product $U \otimes V$ is the product of the first row sum of U and the first row sum of V. Now, if $(c_n)_{n\geq 0}$ is the sequence induced by $P \otimes Q$, then

$$c_n = u^\top (P\otimes Q)^n e = u^\top (P^n \otimes Q^n) e = (u^\top P^n e)(u^\top Q^n e) = a_n b_n. \quad \square$$

Example. Taking $P = \begin{pmatrix}0&1\\1&1\end{pmatrix}$ and $Q = \begin{pmatrix}1&1\\2&1\end{pmatrix}$, the production matrices of the Fibonacci and Pell sequences, respectively, we obtain that

$$P \otimes Q = \begin{pmatrix}0&0&1&1\\0&0&2&1\\1&1&1&1\\2&1&2&1\end{pmatrix}$$

is the production matrix of the Hadamard (componentwise) product 1, 2, 10, 36, 145, 560, 2197, 8568, ... of the Fibonacci and Pell sequences (A001582 in [Sl]).

Proposition 3.7. *If* $P \longrightarrow f_P(z)$, *then*

$$M \stackrel{def}{=} \begin{pmatrix}0 & u^\top\\0 & P + eu^\top\end{pmatrix} \longrightarrow \frac{1}{1 - zf_P} .$$

Proof. Standard computation yields

$$(I - zM)^{-1} = \begin{pmatrix} 1 & zu^\top (I - zP - zeu^\top)^{-1} \\ \star & \star \end{pmatrix},$$

where the entries not shown are irrelevant. Denoting $X = (I - zP)^{-1}, \quad Y = (I - zP - zeu^\top)^{-1}$, we have $f_P(z) = u^\top Xe, \quad f_M(z) = 1 + zu^\top Ye$ and $X(I - zP) = I, \quad I + zeu^\top Y = (I - zP)Y$. Now

$$zf_P(z)f_M(z) = zu^\top Xe(1 + zu^\top Ye) = zu^\top X(I + zeu^\top Y)e = zu^\top X(I - zP)Ye = zu^\top Ye,$$

i.e. $zf_P(z)f_M(z) = f_M(z) - 1$, which is equivalent to the assertion of the theorem. $\square$

The next result uses the techniques developed throughout the paper to provide a new class of operations on production matrices (and so also on succession rules).

Proposition 3.8. *Let b, c, and r be nonnegative integers. If $P \longrightarrow f_P(z)$, then*

$$M \stackrel{def}{=} \begin{pmatrix} b & ru^\top \\ ce & P \end{pmatrix} \longrightarrow \frac{1 + rzf_P(z)}{1 - bz - rcz^2 f_P(z)}.$$

Proof. Let

$$(I - zM)^{-1} = \begin{pmatrix} \alpha & y^\top \\ \star & \star \end{pmatrix},$$

where the entries not shown are irrelevant. A relatively simple computation gives

$$\alpha = \frac{1}{1 - bz - rcz^2 f_P(z)}, \quad y^\top = \frac{rz}{1 - bz - rcz^2 f_P(z)} u^\top (I - zP)^{-1}.$$

Now,

$$f_M(z) = \alpha + y^\top e = \frac{1 + rzf_P(z)}{1 - bz - rcz^2 f_P(z)}. \quad \square$$

The above theorem has numerous applications.

Example. Consider the production matrix

$$M = \begin{pmatrix} 2 & 1 & 0 & \cdots \\ 2 & 1 & 1 & \cdots \\ 2 & 1 & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

It can be written as $M = \begin{pmatrix} 2 & u^\top \\ 2e & P \end{pmatrix}$, where

$$P = \begin{pmatrix} 1 & 1 & 0 & \cdots \\ 1 & 1 & 1 & \cdots \\ 1 & 1 & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

However, it is known that P induces the generating function $\frac{C(z)-1}{z}$ of the Catalan numbers, where $C(z) = \frac{1-\sqrt{1-4z}}{2z}$. Now, taking $b = 2, c = 2, r = 1$, and $f_P(z) = \frac{C(z)-1}{z}$ in Theorem 3.8, after some elementary computations we obtain $f_M(z) = \frac{1-\sqrt{1-4z}}{2z\sqrt{1-4z}}$, the generating function of the sequence $\binom{2n-1}{n}$ of half the central binomial coefficients.

The following, immediate corollary of Proposition 3.8 provides sequences for production matrices having a certain form.

Corollary 3.2. *Let P be an infinite production matrix of the form*

$$P = \begin{pmatrix} b & ru^{\top} \\ ce & P \end{pmatrix},$$

where b, c, r are nonnegative integers. Then the induced generating function $f_P(z)$ satisfies the quadratic equation

$$rcz^2 f_P^2 - (1 - bz - rz) f_P + 1 = 0.$$

Examples.
b=0, c=1, r=1 yields 1,1,2,4,9,21,51,127,... (the Motzkin numbers), A001006;
b=1, c=1, r=1 yields 1,2,5,14,42,132,429,1430,... (the Catalan numbers), A000108;
b=1, c=1, r=2 yields 1,3,11,45,197,903,4279,20793,... (the little Schroder numbers), A001003;
b=2, c=1, r=1 yields 1,3,10,36,137,543,2219,9285,... (number of restricted hexagonal polyominoes with n cells), A002212;

Remark. We would like to point out that Corollary 3.2 can be used also for finding production matrices for certain sequences. Namely, if the generating function f of a sequence satisfies a quadratic equation that can be identified with $rcz^2f^2 - (1 - bz - rz)f + 1 = 0$ for some nonnegative integers b, c, and r, then the production matrix P follows at once. For example, the generating function of the little Schröder numbers satisfies the equation $2z^2f^2 - (1 - 3z)f + 1 = 0$. Consequently, we look for nonnegative integers b, c, r such that $rc = 2$, $b + r = 3$. We obtain two solutions $b = 1$, $c = 1$, $r = 2$ and $b = 2$, $c = 2$, $r = 1$, leading to two production matrices:

$$\begin{pmatrix} 1 & 2 & 0 & 0 & \cdots \\ 1 & 1 & 2 & 0 & \cdots \\ 1 & 1 & 1 & 2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 2 & 1 & 0 & 0 & \cdots \\ 1 & 2 & 1 & 0 & \cdots \\ 1 & 1 & 2 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

4. Riordan production matrices

An infinite lower triangular matrix A is called a *Riordan matrix* if its column k ($k = 0, 1, 2, ...$) has generating function $d(z)(zh(z))^k$, where $d(z)$ and $h(z)$ are formal power series with $d(0) \neq 0$. If, in addition, $h(0) \neq 0$, then A is said to be a *proper Riordan matrix.* We may write $A = (d(z), h(z))$.

Riordan matrices were first introduced in [SGWW]. Proper Riordan matrices are characterized by the following fundamental property [R, Sp]: if $A = (d(z), h(z)) = (d_{n,k})$, then there exist unique sequences $\alpha = (\alpha_0, \alpha_1, \alpha_2, ...)$ ($\alpha_0 \neq 0$) and $\zeta = (\zeta_0, \zeta_1, \zeta_2, ...)$ such that

(i) every element in column 0 can be expressed as a linear combination of all the elements in the preceding row, the coefficients being the elements of the sequence ζ, i.e.

$$d_{n+1,0} = \zeta_0 d_{n,0} + \zeta_1 d_{n,1} + \zeta_2 d_{n,2} + \dots \quad ; \tag{7}$$

(ii) every element $d_{n+1,k+1}$, not lying in column 0 or row 0, can be expressed as a linear combination of the elements of the preceding row, starting from the preceding column on, the coefficients being the elements of the sequence α, i.e.

$$d_{n+1,k+1} = \alpha_0 d_{n,k} + \alpha_1 d_{n,k+1} + \alpha_2 d_{n,k+2} + \dots \quad . \tag{8}$$

Conversely, the existence of such sequences α and ζ ensure that the matrix A is a proper Riordan matrix.

By abuse of notation, by α, ζ we shall denote also the generating functions of these sequences. The sequences α and ζ will be called the α-sequence and the ζ-sequence of the Riordan matrix.

At this stage, it is natural to investigate the relationship between our theory of production matrices and the theory of Riordan matrices. It turns out that, if the ECO matrix A_P, induced by a production matrix P, is Riordan, then the matrix P has a very simple structure.

Proposition 4.1. *Let P be an infinite production matrix and let A_P be the matrix induced by P. Then A_P is a proper Riordan matrix if and only if P is of the form*

$$P = \begin{pmatrix} \zeta_0 & \alpha_0 & 0 & 0 & 0 & \dots \\ \zeta_1 & \alpha_1 & \alpha_0 & 0 & 0 & \dots \\ \zeta_2 & \alpha_2 & \alpha_1 & \alpha_0 & 0 & \dots \\ \zeta_3 & \alpha_3 & \alpha_2 & \alpha_1 & \alpha_0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{9}$$

Moreover, columns 0 and 1 of the matrix P are the ζ- and α-sequences, respectively, of the proper Riordan matrix A_P.

The above theorem, formulated in terms of succession rules, is the main result of [MV]. Because of the above property, a production matrix P having the form (9) will be called a *Riordan production matrix.*

In the case of a given Riordan production matrix P, having $(\zeta_n)_{n\geq 0}$ and $(\alpha_n)_{n\geq 0}$ as its first two columns, one can easily determine the bivariate generating function $G(t,z)$ of the matrix A_P induced by P and then, obviously, also the generating function $f_P(z)$ of the sequence induced by P. For this result we use some known properties of Riordan matrices (see [Sp]).

Proposition 4.2. *Let P be a Riordan production matrix and let $\zeta(z)$ and $\alpha(z)$ be the generating functions of its first two columns, respectively. Then the bivariate generating function $G(t,z)$ of the matrix A_P induced by P and the generating function $f_P(z)$ of the sequence induced by P are given by*

$$G_P(t,z) = \frac{d(z)}{1-tzh(z)}, \quad f_P(z) = \frac{d(z)}{1-zh(z)}, \tag{10}$$

where $h(z)$ and $d(z)$ are determined from the equations

$$h(z) = \alpha(zh(z)), \qquad d(z) = \frac{1}{1-z\zeta(zh(z))}. \tag{11}$$

Example. Consider the Riordan production matrix

$$P = \begin{pmatrix} 3 & 1 & 0 & 0 & 0 & \dots \\ 7 & 3 & 1 & 0 & 0 & \dots \\ 15 & 7 & 3 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Note that the row sums of P, i.e. the labels of the generating tree, are the Eulerian numbers 4,11,26,57,120,... (A000295 in [Sl]). We have $\alpha(z) = \frac{1}{(1-z)(1-2z)}$ and $\alpha - z\zeta = 1$. Then, recalling some known results from the theory of Riordan matrices [Sp], A_P is the Riordan matrix $(d(z), h(z))$ such that $h = \frac{1}{(1-zh)(1-2zh)}$, $d = h$. Let us denote by f_P the generating function of the sequence determined by P. From the last theorem we get $f_P = \frac{h}{1-zh}$. Eliminating d and h from the last three equalities, we obtain $(1 + zf_P)^3 = f_P(1 - zf_P)$. The substitution $K = z + z^2 f_P$ leads to the equation $K^3 = (K - z)(2z - K)$, which is the equation giving the generating function for the number of noncrossing connected graphs. The sequence corresponding to f_P starts 1,4,23,156,1162,... (A007297 in [Sl]).

5. Exponential Riordan production matrices

In this section we outline the main result concerning exponential Riordan matrices as they are exposited in [DS], and then we study this concept from the point of view of production matrices. For the case of tridiagonal matrices, see also [A1, A2, PW].

Let $d(z), h(z)$ be two formal power series such that $d(0) \neq 0 \neq h(0)$. An *exponential Riordan* (briefly, *eR*) *matrix* is an infinite lower triangular array $A = (a_{n,k})_{n,k\geq 0}$ whose k-th column has exponential generating function $C_k(z) = \frac{1}{k!}d(z)(zh(z))^k$. We will use the notation $A = [d(z), h(z)]$ to denote the eR matrix determined by $d(z), h(z)$. In [DS] it is shown that eR matrices form a group (with respect to the usual multiplication operation), whose identity element is $[1, z]$. Looking at the above definition, one could expect many similarities with the theory of (classical) Riordan matrices. From the point of view of the present work, one of the main analogies with the ordinary case is the possibility of expressing every entry of an eR matrix as a linear combination of the elements of the preceding row. More precisely, we have the following result [DS].

Proposition 5.1. *The infinite array $A = (a_{n,k})_{n,k\geq 0}$ is an eR matrix if and only if there exist two formal power series $c(y) = \sum_{j\geq 0} c_j y^j$, $r(y) = \sum_{j\geq 0} r_j y^j$ such that, for any n, k, we have*

$$a_{n+1,k} = \sum_{i\geq k-1} a_{n,i} p_{i,k}, \tag{12}$$

where $p_{i,k} = \frac{i!}{k!}(c_{i-k} + kr_{i-k+1})$.

The sequences $(c_n)_{n\geq 0}, (r_n)_{n\geq 0}$ are called respectively the *c-sequence* and the *r-sequence* of A. There is a clear analogy with the α-sequence and the ζ-sequence of the classical case. However, whereas for an ordinary Riordan matrix the coefficients of the linear combinations in (8) do not depend on the column index, in the exponential case they do, as it is clear from formula (12).

Proof (sketch). Let $A = [d(z), h(z)]$ be an eR matrix and let $c(y), r(y)$ be two formal power series satisfying:

$$r(h(z)) = (zh(z))', \qquad c(zh(z)) = \frac{d'(z)}{d(z)}. \tag{13}$$

Then it can be shown [DS] that the generic element $a_{n+1,k}$ of A can be expressed as a linear combination of the elements of the above row, and the coefficients of such a combination can be determined from (13). Observe that we can arrange these coefficients in an infinite matrix $P = (p_{i,j})_{i,j\geq 0}$, so that formula (12) can be rewritten as the matrix equality $AP = DA$, where, as in formula (4), $D = (\delta_{i,j+1})_{i,j\geq 0}$ (δ is the usual Kronecker delta). The matrix P is "almost" lower triangular, meaning that $p_{i+1,i} = r_0$, whereas, for $k \geq 2$, $p_{i+k,i} = 0$.

Conversely, suppose we are given an infinite "almost" lower triangular matrix P, in the sense explained before. If $n \geq -1$, we will denote by $\mathrm{diag}(n)$ the sequence $(p_{n+k,k})_{k\geq 0}$. Assume that, for

$n \geq 0$, $\mathrm{diag}(n)$ is an arithmetic progression with first term c_n and ratio r_{n+1}, provided that we divide its k-th term by $k \cdot (k+1) \cdot \ldots \cdot (k+n-1)$ (so that the sequence $\left(\frac{(k-1)! p_{n+k,k}}{(n+k-1)!}\right)_{k \geq 0}$ is the desired arithmetic progression). Moreover, suppose that $\mathrm{diag}(-1)$ is the constant sequence r_0. Under these hypotheses, it is clear that $p_{i,j} = \frac{i!}{j!}(c_{i-j} + j r_{i-j+1})$.

If we denote $c(y) = \sum_{n \geq 0} c_n y^n$, $r(y) = \sum_{n \geq 0} r_n y^n$, then we can consider the system of differential equations (13). Solving it with the initial condition $d(0) = d_{0,0}$, we obtain the two formal power series $d(z), h(z)$. Then the eR matrix $[d(z), h(z)]$ is precisely the matrix A from which we started. □

If the matrix P has nonnegative integer entries, then it can be viewed as a production matrix. In such a case it will be called an *exponential Riordan production matrix*. Clearly, in order that $[d(z), h(z)]$ be an ECO matrix, we have to set $d(0) = 1$ as the remaining initial condition of the system (13).

From the expression of the entries $p_{n,k}$ of P given in the statement of Proposition 5.1 it is quite easy to determine the bivariate generating function of P, which is

$$\varphi_P(t,z) = \sum_{n,k} p_{n,k} t^k \frac{z^n}{n!}. \tag{14}$$

Observe that φ_P has been defined to be ordinary with respect to the variable t (tracking the rows of P) and exponential with respect to the variable z (tracking the columns of P). By simply replacing the values of the $p_{n,k}$'s in (14) we immediately obtain:

$$\begin{aligned}
\varphi_P(t,z) &= \sum_{n,k} \frac{n!}{k!}(c_{n-k} + k r_{n-k+1}) t^k \frac{z^n}{n!} \\
&= \sum_{n,k} \frac{t^k z^k}{k!} c_{n-k} z^{n-k} + \sum_{n,k} \frac{t^k z^k}{k!} k r_{n-k+1} z^{n-k} \\
&= \sum_{k} \frac{t^k z^k}{k!} \cdot \sum_n c_n z^n + t \sum_k \frac{t^k z^k}{k!} \cdot \sum_n r_n z^n = e^{tz}(c(z) + t r(z)).
\end{aligned}$$

Clearly, setting $t = 1$ gives the exponential generating function of the row sums of P. If P is an eR production matrix, these are the *labels* of the succession rule induced by P, and their exponential generating function is $\varphi_P(1,z) = e^z(c(z) + r(z))$.

Example. Let P be the matrix of the falling factorials:

$$\begin{pmatrix}
1 & 1 & 0 & 0 & 0 & \ldots \\
2 & 2 & 1 & 0 & 0 & \ldots \\
6 & 6 & 3 & 1 & 0 & \ldots \\
24 & 24 & 12 & 4 & 1 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}.$$

Clearly $p_{i,j} = \frac{(i+1)!}{j!} = (i+1)_{i-j+1}$, where by $(n)_k$ we denote the usual falling factorials. It is immediate to check that P is an eR production matrix, and precisely the one determined by $c(y) = \frac{1}{(1-y)^2}, r(y) = \frac{1}{1-y}$.

Solving the system (13) in this special case gives $h(z) = \frac{1-\sqrt{1-2z}}{z}$, $d(z) = \frac{1}{\sqrt{1-2z}}$, so that $A_P = \left[\frac{1}{\sqrt{1-2z}}, \frac{1-\sqrt{1-2z}}{z}\right]$. Using the definition of eR matrix, A_P turns out to be

$$A_P = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & \dots \\ 3 & 3 & 1 & 0 & \dots \\ 15 & 15 & 6 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The rows of A_P are the coefficients of the Bessel polynomials (see A001497 in [Sl]; the sequence of the row sums of A_P is A001515). The exponential generating function of the labels (=row sums of P) is $e^z(c(z)+r(z)) = \frac{(2-z)e^z}{(1-z)^2}$, which gives rise to sequence A000522 in [Sl].

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