On growth rates of hereditary permutation classes
(extended abstract)

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Abstract

A class of permutations X is called hereditary if $\sigma \prec \pi \in X$ implies $\sigma \in X$; the relation $\prec$ is a natural containment of permutations. Let $f(n)$ be the number of all permutations of $1, 2, \ldots, n$ lying in X. We investigate the counting functions $f(n)$. Our main result says that if $f(n) < 2^{n-1}$ for at least one $n \geq 1$, then there are constants $k \in \mathbf{N}$ and $c > 0$ such that $F_{n,k} \leq f(n) \leq n^c F_{n,k}$ for all $n \geq 1$, where $F_{n,k}$ are the generalized Fibonacci numbers. We also characterize constant and polynomial growths of hereditary permutation classes.

This text is an extended abstract of the article [17]. Only sketches of the proofs are therefore given.

We begin with a general setting for the problem of hereditary classes and their counting functions, and then we turn to one important instance, to the permutation containment. Let U be a countably infinite set of some combinatorial structures and $(U, \prec)$ be a partial ordering. Let $s : U \to \mathbf{N}_0 = \{0, 1, 2, \ldots\}$ be a *size function*. A subset $X \subset U$ is called a *hereditary class* if for $A \in U$ and $B \in X$, $A \prec B \in X$ always implies $A \in X$. Hereditary classes 1-1 correspond to the antichains of $(U, \prec)$; X corresponds to the set of minimal elements of $U \backslash X$. The *counting function* $f : \mathbf{N}_0 \to \mathbf{N}_0$ of X is defined by

$$f(n) = \#\{A \in X : \; s(A) = n\}.$$

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Which function can and which cannot be the counting function of a hereditary class? To obtain some interesting answers on this level of generality, it would be certainly necessary to postulate further properties of the ordering $(U, \prec)$ and the function s but this is not our aim here. We want to investigate the permutation containment. Before proceeding to it, we give an example for graphs.

The set

$$U = \{G = ([n], E) :\ n \in \mathbf{N}_0\},$$

where $[n] = \{1, 2, \ldots, n\}$, consists of all finite simple graphs whose vertex sets are initial segments of $\mathbf{N} = \{1, 2, \ldots\}$. $(U, \prec)$ is the subgraph relation, which means this: For $G_1 = ([n_1], E_1)$ and $G_2 = ([n_2], E_2)$, $G_1 \prec G_2$ if and only if there is an injection $h : [n_1] \to [n_2]$ such that for every $\{x, y\} \in E_1$ also $\{h(x), h(y)\} \in E_2$. The size function s is the number of vertices: $s(([n], E)) = n$. Let

$$X = \{G \in U :\ 2K_{1,1} \not\prec G\},$$

where $2K_{1,1} = ([4], \{\{1, 2\}, \{3, 4\}\})$, be the set of all graphs having no two independent edges. X is a hereditary class. The reader will easily verify that its counting function is

$$f(n) = 1 - \binom{n}{2} + \binom{n}{3} + \sum_{k=2}^{n} k\binom{n}{k} = n2^{n-1} + \binom{n}{3} - \binom{n+1}{2} + 1.$$

For results on counting functions of graph hereditary classes, see Scheinerman and Zito [27] and Balogh, Bollobás and Weinreich [7, 8, 9]. (Graph hereditary classes are usually closed to induced subgraphs. Graph classes closed to all subgraphs are called *monotone*.)

For the rest of the extended abstract we set

$$U = \bigcup_{n=0}^{\infty} S_n,$$

where S_n is the set of $n!$ permutations of $[n]$ ($S_0 = \{\emptyset\}$), and $s(\pi) = n$ for every $\pi \in S_n$. For two permutations $\pi = a_1a_2\ldots a_m$ and $\rho = b_1b_2\ldots b_n$ in U we define $\pi \prec \rho$ iff ρ has a (not necessarily contiguous) subsequence $b_{i_1}b_{i_2}\ldots b_{i_m}$ such that for every $1 \le j, k \le m$

$$b_{i_j} < b_{i_k} \iff a_j < a_k.$$

If $\pi \prec \rho$, we say that π is *contained* in ρ. We shall investigate counting functions of hereditary permutation classes. One has plenty of them.

Theorem 1 *1. There exist $2^{\aleph_0}$ hereditary permutation classes. 2. More strongly, there is a set R of $2^{\aleph_0}$ hereditary permutation classes such that for every two distinct classes X and Y in R, both sets $X\backslash Y$ and $Y\backslash X$ are infinite.*

Proof. (Sketch) Both results hold more generally for every countably infinite poset if it has an infinite antichain A. That this is the case for the permutation containment was shown by Pratt [25], Tarjan [32], Laver [22] (see also Kruskal [21, p. 304]), Spielman and Bóna [29], and implicitly perhaps already by Jenkyns and Nash-Williams [16].

1. Take the hereditary classes corresponding to the subsets of A. 2. Identify A with $\mathbf{N}$ and take the hereditary classes corresponding to the sets in a system $\{A_c \subset \mathbf{N} : \ c \in (0, 1/2)\}$, where A_c has lower and upper asymptotic density equal to c and $1 - c$, respectively. $\square$

If X and Y are hereditary classes with counting functions f_X and f_Y and such that the difference $X\backslash Y$ is finite, then f_Y *eventually dominates* f_X, which means that $f_Y(n) \geq f_X(n)$ for all $n > n_0$. By the last result, there are $2^{\aleph_0}$ hereditary permutation classes such that this trivial comparison cannot be applied to any two of them. Are there two hereditary permutation classes whose counting functions are incomparable by the eventual dominance? Theorem 4 implies that the at most polynomially growing counting functions form a linear (quasi)order with respect to the eventual dominance. Are there $2^{\aleph_0}$ hereditary permutation classes whose counting functions are mutually incomparable by the eventual dominance? For hereditary graph classes this question has positive answer, see [8, Theorem 11].

For a permutation π, consider the counting function $f(n, \pi)$ of the hereditary class

$$\{\sigma \in U : \ \sigma \not\succ \pi\}.$$

Arratia [4] showed that the limit $\lim_{n\to\infty} f(n, \pi)^{1/n}$ always exists. The *Stanley–Wilf conjecture* (that appeared in print first in Bóna [10, 11, 12]) says that the limit is never infinite, that is, for every π there exists a constant $c > 1$ such that $f(n, \pi) < c^n$ for all $n \in \mathbf{N}$. The conjecture can be reformulated in terms of hereditary classes: Every hereditary permutation class X such that $X \neq U$ has only exponentially growing counting function. (The counting function of $X = U$ is $f(n) = n!$.) For partial results towards the Stanley–Wilf conjecture, see Alon and Friedgut [3], Bóna [12, 13], and Klazar [19].

The next theorem was proved by Pavel Valtr [33]. We reproduce it with his kind permission.

Theorem 2 *Let c be any constant such that $0 < c < \mathrm{e}^{-3}$ $(= 0.04978\ldots)$. For every permutation $\pi \in S_k$, where $k > k_0 = k_0(c)$, one has*

$$\lim_{n\to\infty} f(n,\pi)^{1/n} > ck^2.$$

Proof. (Sketch) Let $\pi \in S_k$ be fixed. We may assume that π, taken as a sequence, cannot be split into two nonempty sequences $\pi = IJ$ so that every term in I is smaller than every term in J. (Else we take the reversed π.) A straightforward application of the probabilistic method shows that if $0 < d < \mathrm{e}^{-2}$ and $m = \lfloor dk^2 \rfloor$ then almost every $\sigma \in S_m$ does not contain π. In particular, $f(m,\pi) > m!/2$ for large k. Now take all $\rho \in S_n$ of the form $\rho = I_1 I_2 \ldots I_{n/m}$ where $I_1 < I_2 < \ldots < I_{n/m}$ and every I_i is appropriately shifted permutation from S_m not containing π. It follows that $\rho \not\succ \pi$. Thus $f(n,\pi) \geq (m!/2)^{n/m}$, which gives the stated bound. $\square$

For every π we denote $c(\pi) = \lim_{n\to\infty} f(n,\pi)^{1/n}$. It is easy to prove that for the identical permutation $id_k = 1,2,\ldots,k$ one has $c(id_k) \leq (k-1)^2$. (By Regev's asymptotics [26], $c(id_k) = (k-1)^2$.) This and the previous theorem give the estimate

$$0.04978\ldots \leq \liminf_{k\to\infty}\ k^{-2} \cdot \min_{\pi \in S_k} c(\pi) \leq 1.$$

Can these bounds be improved?

In the next four theorems we characterize possible growth rates of those counting functions $f(n)$ which satisfy $f(n) < 2^{n-1}$ for at least one $n \in \mathbf{N}$.

Theorem 3 *Let $f(n)$ be the counting function of a hereditary permutation class. If $f(n)$ is bounded then it is eventually constant. If $f(n)$ is unbounded then $f(n) \geq n$ for all $n \in \mathbf{N}$.*

Proof. (Sketch) For $r \in \mathbf{N}$ and $\pi \in S_n$, we say that π has the *r-intrusion property* if there are subsets $A, B \subset [n]$ and an element $x \in [n]$ such that $A < x < B$, $|A| \geq r$, $|B| \geq r$, and $\pi|(A \cup B)$ is monotone but $\pi|(A \cup B \cup \{x\})$ is not. We say that π has the *r-union property* if there are subset $A, B \subset [n]$ such that $A < B$, $|A| \geq r$, $|B| \geq r$, and both restrictions $\pi|A$ and $\pi|B$ are monotone but $\pi|(A \cup B)$ is not.

Let X be a hereditary permutation class and $f(n)$ be its counting function. We distinguish three cases. In the case 1 for every $r \in \mathbf{N}$ there is a $\pi \in X$ such that π or π^{-1} has the r-intrusion property. In the case 2 for every $r \in \mathbf{N}$ there is a $\pi \in X$ such that π has the r-union property. In the case 3 there is an r such that for every $\pi \in X$ neither π nor π^{-1} has the r-intrusion property and no $\pi \in X$ has the r-union property. It can be deduced that in the cases 1 and 2 one has $f(n) \geq n$ for all $n \in \mathbf{N}$, and in the case 3 the function $f(n)$ is eventually constant. $\square$

Every constant $c \in \mathbf{N}_0$ can be realized as a counting function. For $c = 0$ we take the empty class. For $c \geq 1$ we take the permutations

$$\{1, 2, \ldots, i, n, n-1, \ldots, i+2, i+1 :\ 0 \leq i \leq c-1, n \in \mathbf{N}\}.$$

This is a hereditary class whose counting function equals c for all $n \geq c$.

Theorem 4 *Let $f(n)$ be the counting function of a hereditary permutation class. If $f(n)$ satisfies $f(n) < n^c$ for all $n \in \mathbf{N}$ with a constant $c > 0$, then there is a number $M \in \mathbf{N}$ and $(M+1)^2$ integers $a_{i,j} \in \mathbf{Z}$, $0 \leq i, j \leq M$, such that for all $n > n_0$,*

$$f(n) = \sum_{i,j=0}^{M} a_{i,j} \binom{n-i}{j}.$$

Proof. (Sketch) This can be deduced by investigating in more details the case 1a of the proof of Theorem 5. $\square$

The sequence of *Fibonacci numbers* $(F_n)_{n \geq 1} = (1, 2, 3, 5, 8, 13, 21, 34, \ldots)$ satisfies the recurrence $F_0 = F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. Although Theorem 5 is subsumed in the more general Theorem 6, we state it to identify explicitly Fibonacci numbers as the first superpolynomial growth rate of counting functions. Also, its proof is considerably easier than the proof of Theorem 6. Note that the permutations

$$\{2, 1, 4, 3, 6, 5, \ldots, 2n, 2n-1 :\ n \in \mathbf{N}\}$$

and the permutations contained in them form a hereditary class whose counting function is $f(n) = F_n$.

Theorem 5 *Let $f(n)$ be the counting function of a hereditary permutation class. Then, for all $n \in \mathbf{N}$, either $f(n) < n^c$ with a constant $c > 0$ or $f(n) \geq F_n$.*

Proof. (Sketch) We split any permutation $\pi \in S_n$ uniquely as $\pi = I_1 I_2 \dots I_m$ where I_1 is the longest initial monotone segment, I_2 is the following longest monotone segment, and so on. We mark the elements in I_i by i and read the marks in π from bottom to top (that is, from left to right in π^{-1}). In this way we obtain a word $u(\pi)$ of length n over the alphabet $[m]$, where $m = m(\pi)$ is the number of the monotone segments I_i. For example, for $\pi = 3, 5, 4, 2, 1, 7, 8, 6, 9$ we have $m(\pi) = 4$ and $u(\pi) = 2, 2, 1, 2, 1, 4, 3, 3, 4$, because $I_1 = 3, 5$, $I_2 = 4, 2, 1$, $I_3 = 7, 8$, and $I_4 = 6, 9$. For every pair I_i, I_{i+1} we consider the interval $T_{\pi,i} = [\min\{a, b, c\}, \max\{a, b, c\}] \subset \mathbf{N}$ where a, b, c is a nonmonotone subsequence of $I_i I_{i+1}$ (such a subsequence exists). In our example, for $i = 3$ we may set $a, b, c = 7, 8, 6$ and $T_3 = [6, 8] = \{6, 7, 8\}$. For a word u, the symbol $\ell(u)$ denotes the length of the longest alternating (not necessarily contiguous) subsequence of two distinct symbols in u. In our example, $\ell(u(\pi)) = 4$.

Now let X be a hereditary permutation class and $f(n)$ be its counting function. We let π range over X and distinguish four cases. Case 1a: $m(\pi)$ is bounded and so is $\ell(u(\pi))$. Case 1b: $m(\pi)$ is bounded and $\ell(u(\pi))$ is unbounded. Case 2a: $m(\pi)$ is unbounded and so is the maximum number of mutually intersecting intervals in $\{T_{\pi,1}, T_{\pi,3}, T_{\pi,5}, \dots\}$. Case 2b: $m(\pi)$ is unbounded and so is the maximum number of mutually disjoint intervals in $\{T_{\pi,1}, T_{\pi,3}, T_{\pi,5}, \dots\}$.

In the case 1a it can be deduced that $f(n)$ grows at most polynomially. In the case 1b it follows that for every $k \in \mathbf{N}$ there is a $\pi \in X \cap S_k$ such that $\pi(i) < \pi(j)$ whenever i is odd and j is even. This implies that $f(n) \geq 2^{n-1} \geq F_n$ for all $n \in \mathbf{N}$. Similar argument applies to the case 2a and again $f(n) \geq 2^{n-1} \geq F_n$ for all $n \in \mathbf{N}$. In the case 2b it can be deduced that either for all $n \in \mathbf{N}$ we have $(2, 1, 4, 3, 6, 5, \dots, 2n, 2n-1) \in X$ or for all $n \in \mathbf{N}$ we have $(2n-1, 2n, 2n-3, 2n-2, \dots, 1, 2) \in X$. This implies that $f(n) \geq F_n$ for all $n \in \mathbf{N}$. $\square$

The *generalized Fibonacci numbers* $F_{n,k} \in \mathbf{N}_0$, where $k \in \mathbf{N}$ and $n \in \mathbf{Z}$, are defined as $F_{n,k} = 0$ for $n < 0$, $F_{0,k} = 1$, and by the recurrence $F_{n,k} = F_{n-1,k} + F_{n-2,k} + \cdots + F_{n-k,k}$ for $n > 0$. Thus $F_{n,1} = 1$ for all $n \in \mathbf{N}$ and $F_{n,2} = F_n$. The exponential growth rate of $F_{n,k}$ is α_k^n where α_k is the largest positive real root of the polynomial $x^k - x^{k-1} - x^{k-2} - \cdots - 1$. We list the first few α_k to 5 digits after the decimal point.

k	2	3	4	5	6	10
α_k	1.61803	1.83928	1.92756	1.96594	1.98358	1.99901

It follows that $\alpha_k \to 2^-$ as $k \to \infty$. Egge [14] and Egge and Mansour [15] enumerated hereditary classes given by antichains of forbidden permutations and found that in many cases the counting function expresses in terms of the Fibonacci numbers or the generalized Fibonacci numbers. Theorem 6 provides an explanation for this phenomenon—the only exponential growth rates of counting functions below 2^{n-1} are those of the generalized Fibonacci numbers. (We learned about the results in [14] and [15] long after [17] had been finished and submitted.)

Theorem 6 *Let $f(n)$ be the counting function of an infinite hereditary permutation class. Then*

- *either there are constants $k \in \mathbf{N}$ and $c > 0$ such that $F_{n,k} \le f(n) \le n^c F_{n,k}$ for all $n \in \mathbf{N}$ or*
- *$f(n) \ge 2^{n-1}$ for all $n \in \mathbf{N}$.*

Proof. (Sketch) This proof is involved, we refer the reader to [17] for details. An important role in it play the *irreducible* permutations. A permutation $\pi \in S_n$ is irreducible, more precisely *up-irreducible*, if there is no $m \in [n-1]$ such that $\pi(\{1, 2, \ldots, m\}) < \pi(\{m+1, m+2, \ldots, n\})$. Every $\pi \in S_n$ has a unique decomposition $[n] = I_1 \cup I_2 \cup \ldots \cup I_m$, where $I_1 < I_2 < \ldots < I_m$ are intervals, such that each restriction $\pi | I_i$ is irreducible. The proof is based on properties of these decompositions. $\square$

Conclusion. The enumeration of permutations not containing a given permutation or a given set of permutations was initiated by Schmidt and Simion [28]. In the last years, many articles on this problem or related topics appeared, to those already mentioned we may add, to name a few, Adin and Roichman [1], Albert et al. [2], Atkinson [5], Atkinson, Murphy and Ruškuc [6], Mansour [23], Mansour and Vainshtein [24], Stankova-Frenkel and West [30, 31], and West [34]. We list a few research directions suggested by our results. What are the growth rates of hereditary permutations classes above 2^{n-1}? What are the growth rates of hereditary classes of *ordered* graphs (now the injection h of our initial graph example is required to be increasing)? What are the growth rates of hereditary classes of set partitions (some results were obtained in Klazar [18, 20] and by B. Sagan)? What general classification theorems on counting functions of hereditary classes, which would put permutations, set partitions, ordered graphs etc. under one roof, can be obtained?

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