LATTICE STRUCTURES FROM PLANAR GRAPH

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ABSTRACT. The set of all orientations of a planar graph with prescribed outdegrees carries the structure of a distributive lattice. This general theorem is proven in the first part of the paper. In the second part the theorem is applied to show that interesting combinatorial sets related to a planar graph have lattice structure: Eulerian orientations, spanning trees and Schnyder woods. For the Schnyder wood application some additional theory has to be developed. In particular it is shown that a Schnyder wood for a planar graph induces a Schnyder wood for the dual.

RÉSUMÉ. L'ensemble de tous les orientations d'un graphe planaire sur lesquelles on a fixé le dégrée sortant des sommets porte une structure de treillis distribué. Ce théorème général est démontré dans la première partie de ce papier. Dans la seconde partie le théorème est appliqué pour démontrer que des ensembles combinatoires intéressantes qui sont liés a un graphe planaire ont une structure de treillis : orientations Eulériennes, arbres couvrantes et forêts de Schnyder. Pour l'application au forêts de Schnyder une théorie supplémentaire doit être développée. En particulier on montre qu'une forêt de Schnyder d'un graphe planaire induit une forêt de Schnyder pour le graphe dual.

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1. Introduction

This work originated in the study of rigid embeddings of planar graphs and the connections with Schnyder woods. These connections were discovered by Miller [8] and further investigated in [3]. The set of Schnyder woods of a planar triangulation has the structure of a distributive lattice. This was independently shown by Brehm [1] and Mendez [9]. My original objective was to generalize this and prove that the set of Schnyder woods of a 3-connected planar graph also has a distributive lattice structure. The theory developed to this aim turned out to work in a more general situation. In the first half of this paper we present a theory of α-orientations of a planar graph and show that they form a distributive lattice. As noted in [4] this result was already obtained in the thesis of Mendez [9]. Another source for related results is a paper of Propp [12] where he describes lattice structures in the dual setting. The cover relations in Propp's lattices are certain *pushing-down* operations. These operations were introduced by Mosesian and further studied by Pretzel [10] as reorientations of diagrams of ordered sets.

This is an extended abstract. The full paper (26 pages) is electronically available as: `http://page.inf.fu-berlin.de/~felsner/Paper/alpha-or.ps.gz`

The second part of the paper deals with special instances of the general result. In particular we find lattice structures on the following combinatorial sets related to a planar graph: Eulerian orientations, spanning trees and Schnyder woods. While the application to Eulerian orientations is rather obvious already the application of spanning trees requires some ideas. To connect spanning trees to orientations we introduce the completion of a plane graph which can be thought of as superposition of the primal and the dual which is planarized by introducing a new edge-vertex at every crossing pair of a primal edge with its dual edge. The lattice structure on spanning trees of a planar graph has been discovered in the context of knot theory by Gilmer and Litherland [5] and by Propp [12] as an example of his lattice structures. A closely related family of examples concerns lattices on matchings and more generally f-factors of plane bipartite graphs. This is related to a combinatorial correspondence between trees and matchings which is used in [6].

To show that the Schnyder woods of a 3-connected plane graph have a distributive lattice structure some additional theory has to be developed. We prove that a Schnyder wood for a planar graph induces a Schnyder wood for the dual. A primal dual pair of Schnyder woods can be embedded on a completion of the plane graph, i.e., on a superposition of the primal and the dual as described above. In the next step it is shown that the orientation of the completion alone allows to recover the Schnyder wood. As in the case of spanning trees the lattice structure comes from orientations of the completion.

2. Lattices of Fixed Degree Orientations

A *plane graph* is a planar graph $G = (V, E)$ together with a fixed planar embedding. In particular there is a designated outer (unbounded) face F^* of G. Given a mapping $\alpha : V \to \mathbb{N}$ an orientation X of the edges of G is called an *α-orientation* if α records the out-degrees of all vertices, i.e., $\mathsf{outdeg}_X(v) = \alpha(v)$ for all $v \in V$. We call α feasible if α-orientation of G exists. The main result of this section is the following theorem.

Theorem 1. *Let G be a plane graph and $\alpha : V \to \mathbb{N}$ be feasible. The set of α-orientations of G carries an order-relation which is a distributive lattice.*

2.1. **Reorientations and essential cycles.** Let X be an α-orientation of G. Given a directed cycle C in X we let X^C be the orientation obtained from X by reversing all edges of C. Since the out-degree of a vertex is unaffected by the reversal of C the orientation X^C is another α-orientation of G. The plane embedding of G allows us to classify a directed simple cycle as clockwise (*cw-cycle*) if the interior, $\mathsf{Int}(C)$, is to the right of C or as counterclockwise (*ccw-cycle*) if $\mathsf{Int}(C)$ is to the left of C. If C is a ccw-cycle of X then we say that X^C is *left of X* and X is *right of X^C*. It will turn out that the transitive closure of this 'left of' relation is the order relation which makes the set of α-orientations of G a distributive lattice.

Let X and Y be α-orientations of G and let D be the set of edges with oppositional orientations in X and Y. Every vertex is incident to an even number of edges in D, hence, the subgraph with edge set D is Eulerian. If we impose the orientation of X on the edges of D the subgraph is a directed Eulerian graph. Consequently, the edge set D can be decomposed into simple cycles $C_1, .., C_k$ which are directed cycles of X. We restate a consequence of this observation as a lemma.

Lemma 1. *If $X \neq Y$ are α-orientations of G then for every edge e which is oppositionally directed in X and Y there is a simple cycle C with $e \in C$ and C is oppositionally directed in X and Y.*

An edge of G is *α-rigid* if it has the same direction in every α-orientation. Let $R \subseteq E$ be the set of α-rigid edges. Since directed cycles in X can be reversed, rigid edges never belong to directed cycles.

With $A \subset V$ we consider two sets of edges, the set $E[A]$ of edges with two ends in A, i.e., edges induced by A, and the set $E_{\mathsf{Cut}}[A]$ of edges in the cut $(A, \overline{A})$, i.e., the set of edges connecting a vertex on A to a vertex in the complement $\overline{A} = V \setminus A$.

Given A and a α-orientation X, then exactly $\sum_{v \in A} \alpha(v)$ edges have their tail in A. The number of edges incident to vertices in A is $|E[A]| + |E_{\mathsf{Cut}}[A]|$. The *demand* of A in X is the number of edges pointing from $\overline{A}$ into A.

Lemma 2. *The demand of A is $\mathsf{dem}_\alpha(A) = |E[A]| + |E_{\mathsf{Cut}}[A]| - \sum_{v \in A} \alpha(v)$, thus $\mathsf{dem}_\alpha(A)$ only depends on α and not on X.*

By looking at demands we can identify certain sets of rigid edges. If for example $\mathsf{dem}_\alpha(A) = 0$, then all the edges in $E_{\mathsf{Cut}}[A]$ point away from A in every α-orientation and, hence, $E_{\mathsf{Cut}}[A] \subseteq R$ in this case. Symmetrically, if $\mathsf{dem}_\alpha(A) = |E_{\mathsf{Cut}}[A]|$, then all the edges in $E_{\mathsf{Cut}}[A]$ point towards A and again $E_{\mathsf{Cut}}[A] \subseteq R$.

The set of vertices in the interior of a simple cycle C in G is denoted I_C. Of special interest to us will be cycles C with the property that $E_{\mathsf{Cut}}[I_C] \subset R$. In that case we say that *the interior cut of C is rigid.* This means that the orientation of all the edges connecting C to an interior vertex is fixed throughout all α-orientations. Note that the interior cut of a face cycle of G is always rigid because $E_{\mathsf{Cut}}[I_C] = \emptyset$ in this case.

Definition 1. *A cycle C of G is an* essential cycle *if*

- *C is simple and induced,*
- *the interior cut of C is rigid, i.e., $E_{\mathsf{Cut}}[I_C] \subseteq R$,*
- *there exists an α-orientation X such that C is a directed cycle in X.*

With lemmas 3–6 we show that with reorientations of essential cycles we can commute between any two α-orientations. In fact reorientations of essential cycles represent the cover relations in the 'left of' order on α-orientations.

A cycle C has a *chordal path* in X if there is a directed path consisting of edges interior to C whose first and last vertex are vertices of C. We allow that the two end vertices of a chordal path coincide.

Lemma 3. *If C has a chordal path in an α-orientation X, then C has a chordal path in every α-orientation.*

Lemma 4. *A cycle C has no chordal path iff the interior cut of C is rigid.*

Lemma 5. *If C and C' are essential cycles then either the interior regions of the cycles are disjoint or one of the interior regions is contained in the other and the two cycles are vertex disjoint.*

Lemma 6. *If C is a directed cycle in X, then X^C can be obtained by a sequence of reversals of essential cycles.*

Lemma 7. *If C is a simple directed ccw-cycle in X, then X^C can be obtained by a sequence of reversals of essential cycles from ccw to cw. Moreover, the set of essential cycles of such a sequence is the unique minimal set such that the interior regions of the essential cycles cover the interior region of C.*

2.2. **Interlaced flips in sequences of flips.** A *flip* is the reorientation of an essential cycle from ccw to cw. A *flop* is the converse of a flip, i.e., the reorientation of an essential cycle from cw to ccw.

A *flip sequence on X* is a sequence $(C_1, .., C_k)$ of essential cycles such that C_1 is flipable in X, i.e., C_1 is a ccw-cycle of X, and C_i is flipable in $X^{C_1 \ldots C_{i-1}}$ for $i = 2, .., k$.

Recall that an edge e is contained in at most two essential cycles. If we think of e as directed, then there can be an essential cycle $C^{l(e)}$ left of e and another essential cycle $C^{r(e)}$ right of e.

Lemma 8. *If $(C_1, .., C_k)$ is a flip sequence on X then for every edge e the essential cycles $C^{l(e)}$ and $C^{r(e)}$ alternate in the sequence, i.e., if $i_1 < i_2$ with $C_{i_1} = C_{i_2} = C^{l(e)}$ then there is a j with $i_1 < j < i_2$ and $C_j = C^{r(e)}$. The same holds with left and right exchanged.*

Lemma 9. *For every edge e there is a $t_e \in \mathbb{N}$ such that for all α-orientations X a flip sequence on X implies at most t_e reorientations of e.*

Lemma 10. *The length of any flip sequence is bounded by some $t \in \mathbb{N}$ and there is a unique α-orientation $X_{\min}$ with the property that all cycles in $X_{\min}$ are cw-cycles.*

From this lemma it follows that the 'left of' relation is acyclic. We now adopt a more order theoretic notation and write $Y \prec X$ if Y can be obtained by a sequence of flips starting at X. We summarize our knowledge about this relation.

Corollary 1. *The relation $\prec$ is an order relation with a unique minimal element $X_{\min}$.*

2.3. **Flip-sequences and potentials.** With the next series of lemmas we investigate properties of sequences of flips that lead from X to $X_{\min}$. It will be shown that any two such sequences contain the same essential cycles.

Lemma 11. *Suppose $Y \prec X$ and let C be an essential cycle. Every sequence $S = (C_1, \ldots, C_k)$ of flips that transforms X into Y contains the same number of flips at C.*

For a given α let $\mathcal{E} = \mathcal{E}_\alpha$ be the set of all essential cycles. Given an α-orientation X there is a flip sequence S from X to $X_{\min}$. For $C \in \mathcal{E}$ let $z_X(C)$ be the number of times C is flipped in a flip sequence S. The previous lemma shows that this independent of S and hence a well defined mapping $z_X : \mathcal{E} \to \mathbb{N}$. Moreover, if $X \neq Y$ then $z_X \neq z_Y$.

Definition 2. *An α*-potential *for G is a mapping $\wp : \mathcal{E}_\alpha \to \mathbb{N}$ such that*

- *$|\wp(C) - \wp(C')| \leq 1$, if C and C' share an edge e.*
- *$\wp(C) \leq 1$, if there is an edge $e \in C$ such that C is the only essential cycle to which e belongs.*
- *If $C^{l(e)}$ and $C^{r(e)}$ are the essential cycles left and right of e in $X_{\min}$ then $\wp(C^{l(e)}) \leq \wp(C^{r(e)})$.*

Lemma 12. *The mapping $z_X : \mathcal{E}_\alpha \to \mathbb{N}$ associated to an α-orientation X is an α-potential.*

Lemma 13. *For every α-potential $\wp : \mathcal{E}_\alpha \to \mathbb{N}$ there is an α-orientation X with $z_X = \wp$.*

Lemma 12 and Lemma 13 establish a bijection between α-orientations and α-potentials. With the following lemma we thus complete the proof of Theorem 1.

Lemma 14. *The set of all α-potentials $\wp : \mathcal{E} \to \mathbb{N}$ with the dominance order $\wp \prec \wp'$ if $\wp(C) \leq \wp'(C)$ for all $C \in \mathcal{E}$ is a distributive lattice. Join $\wp_1 \vee \wp_2$ and meet $\wp_1 \wedge \wp_2$ of two potentials $\wp_1$ and $\wp_2$ are given by $(\wp_1 \vee \wp_2)(C) = \max\{\wp_1(C), \wp_2(C)\}$ and $(\wp_1 \wedge \wp_2)(C) = \min\{\wp_1(C), \wp_2(C)\}$ for all $C \in \mathcal{E}$.*

Corollary 2. *Let G be a plane graph and $\alpha : V \to \mathbb{N}$ be feasible. The following sets carry isomorphic distributive lattices*

- *The set of α-orientations of G.*
- *The set of α-potentials $\wp : \mathcal{E}_\alpha \to \mathbb{N}$.*
- *The set of Eulerian subdigraphs of a fixed α-orientation X.*

3. Applications

Distributive lattices are beautiful and well understood structures and it is always nice to identify a distributive lattice on a finite set $\mathcal{C}$ of combinatorial objects. Such a lattice structure may then be exploited in theoretical and computational problems concerning $\mathcal{C}$.

Usually the cover relation in the lattice $\mathcal{L}_C$ corresponds to some minor modification (*move*) in the combinatorial object. In our example the moves are reorientations of essential cycles (flips and flops). In most cases it is easy to find all legal moves that can be applied to a given object from $\mathcal{C}$. In our example finding the applicable moves corresponds to finding the directed essential cycles of an α-orientation. This task is easy in the sense that it can be accomplished in time polynomial in the size of the plane graph G. By the fundamental theorem of finite distributive lattices: there is a finite partially ordered set $P_{\mathcal{C}}$ such that the elements of $\mathcal{L}_C$, i.e., the objects in $\mathcal{C}$, correspond to the order ideals (down-sets) of $P_{\mathcal{C}}$. The moves operating on the objects in $\mathcal{C}$ can be viewed as elements of $P_{\mathcal{C}}$. If $\mathcal{C}$ is the set of α-orientations the elements of $P_{\mathcal{C}}$ thus correspond to essential cycles, however, a single essential cycle may correspond to several elements of $P_{\mathcal{C}}$, Figure 1 illustrates this effect. The elements of $P_{\mathcal{C}}$ can be shown to be in bijection to the flips on a maximal chain from $X_{\max}$ to $X_{\min}$ in $\mathcal{L}_C$. Consequently, in the case of α-orientations of G the order $P_{\mathcal{C}}$ has size polynomial in the size of G and can be computed in time polynomial in the size of G.

We explicitly mention three applications of a distributive lattice structure on a combinatorial set $\mathcal{C}$ before looking at some specific instances of Theorem 1.

- Any two objects in $\mathcal{C}$ can be transformed into each other by a sequence of moves. Proof: Every element of $\mathcal{L}_C$ can be transformed into the unique minimum of $\mathcal{L}_C$ by a sequence (chain) of moves. Reverting the moves in one of the two chains gives a transformation sequence for a pair of objects.
- All elements of $\mathcal{C}$ can be generated/enumerated with polynomial time complexity per object. The idea is as follows: Assign different priorities to the elements of $P_{\mathcal{C}}$. Use these priorities in a tree search (e.g., depth-first-search) on $\mathcal{L}_C$ starting in the minimal element. An object is output/count only when visited for the first time, i.e., with the lexicographic minimal sequence of moves that generate it.
- To generate an element of $\mathcal{C}$ from the uniform distribution a Markov chain combined with the coupling from the past method can be used. This very elegant approach gives a process that stops itself in the perfect uniform distribution. Although this stop can be observed to happen quite fast in many processes of the described kind, only few of these processes have been analyzed satisfactorily. For more on this subject we recommend the work of Propp and Wilson [11] and [13].

3.1. **Eulerian orientations.** Let G be a plane graph, such that every vertex v has even degree $d(v)$. An Eulerian orientation of G is an orientation with $\mathsf{indeg}(v) = \mathsf{outdeg}(v)$ for every vertex v. Hence, Eulerian orientations are just the α-orientations with $\alpha(v) = \frac{d(v)}{2}$ for all $v \in V$. By Theorem 1 the Eulerian orientations of a planar graph form a distributive lattice.

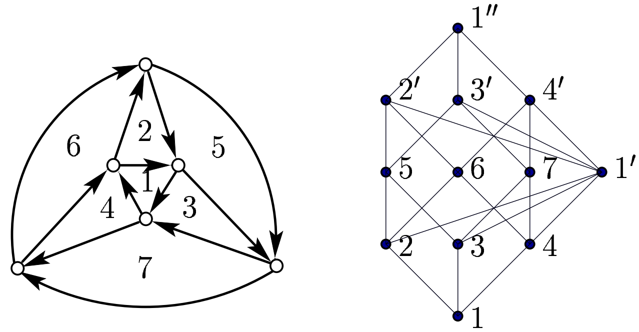

FIGURE 1. Left: A graph G with its minimal Eulerian orientation and a labeling of the faces. Right: The ordered set P such that the set of ideals of P is the lattice of Eulerian orientations of G.

3.2. **The primal dual completion of a plane graph.** For later applications we need the primal dual completion of a plane graph G. With G there is the dual graph G^*, the primal dual completion $\widetilde{G}$ of G is constructed as follows: Superimpose plane drawings of G and G^* such that only the corresponding primal dual pairs of edges cross. The completion $\widetilde{G}$ is obtained by adding a new vertex at each of these crossings. The construction is illustrated in Figure 2. If G has

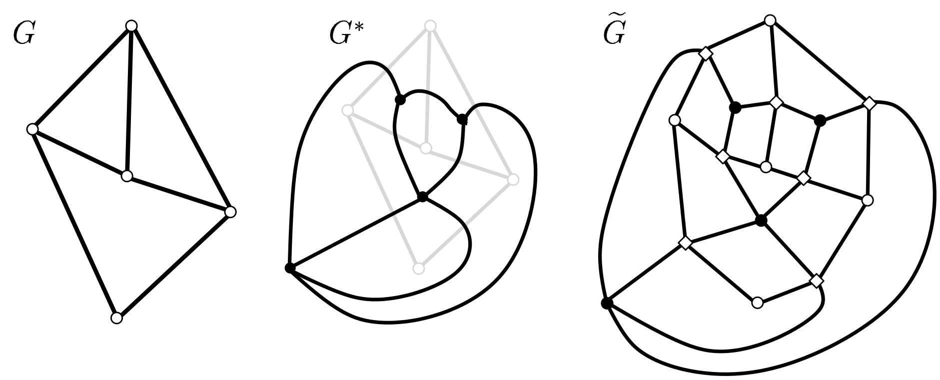

FIGURE 2. A plane graph G with its dual G^* and completion $\widetilde{G}$.

n vertices, m edges and f faces, then the corresponding numbers $\widetilde{n}$, $\widetilde{m}$ and $\widetilde{f}$ for $\widetilde{G}$ can be expressed as follows:

- $\widetilde{n} = n + m + f$. We denote the vertices of $\widetilde{G}$ originating in vertices of G, G^* and crossings of edges as *primal-vertices*, *dual-vertices* and *edge-vertices*.

- $\widetilde{m} = 4m$.
- $\widetilde{f} = 2m$: This follows since every face of $\widetilde{G}$ is a quadrangle with a primal- and a dual-vertex at opposite corners and edge-vertices at the remaining corners. Thus, there is a bijection between angles of G and faces of $\widetilde{G}$. The number of angles of G is $\sum_v d(v) = 2m$.

There is a subtlety with the notion of the dual and, hence, of the completion when the connectedness of G is too small. If G has a bridge then $\widetilde{G}$ has multiple edges. In general, however, the completion is at least as well behaved as G. The following implications hold:

- If G is connected and bridgeless $\implies$ $\widetilde{G}$ is 2-connected.
- If G is 2-connected $\implies$ $\widetilde{G}$ is 3-connected.

Completions of planar graphs have a nice characterization.

Proposition 1. *Let H be 2-connected, H is the completion of plane graph G iff the following three conditions hold:*

1. *H is planar.*
2. *All the faces of H are quadrangles, in particular H is bipartite.*
3. *In one of the two color classes of H all vertices have degree four.*

3.3. **Spanning trees.** We show that there is a bijection between the spanning trees of a planar graph $G = (V, E)$ and the α-orientations of the completion $\widetilde{G}$ of G for a certain α. Together with Theorem 1 this implies:

Theorem 2. *There is a distributive lattice of orientations of $\widetilde{G}$ which induces a distributive lattice on the spanning trees of a planar graph G.*

After having obtained this result we found that it was already known. Gilmer and Litherland [5] arrive at such a lattice on spanning trees in the context of knot theory. They also point out the equivalence to Kaufmann's *Clock Theorem.* Propp [12] describes a large class of distributive lattices related to orientations of graphs. If G is planar then the lattice of α-orientations of G is isomorphic to a Propp lattice of the dual G^*. Propp discovered lattices on spanning trees as a special case of his theory.

Let $T \subseteq E$ be the set of edges of a spanning tree of G. If T^* is the set of dual edges of non-tree edges (edges in $E \setminus T$), then T^* is the set of edges of a spanning tree of the dual graph G^*. This is the natural bijection between the spanning trees of G and G^*.

With a spanning tree T of G we associate an orientation of $\widetilde{G}$. First we select two special root vertices for $\widetilde{G}$, a primal-vertex v_r and a dual-vertex v_r^*. Now T and the corresponding dual tree T^* are thought of as directed trees in which every edge points towards the primal- respectively dual-root. The direction of edge $e = (u, w) \in T \cup T^*$ is passed on to the edges (u, v_e) and (v_e, w) in $\widetilde{G}$, where v_e is the edge-vertex of $\widetilde{G}$ corresponding to edge e. All the remaining edges of $\widetilde{G}$ are oriented so that they point away from their incident edge-vertex. Figure 3 illustrates the construction. The orientation thus obtained is an α_T-orientation

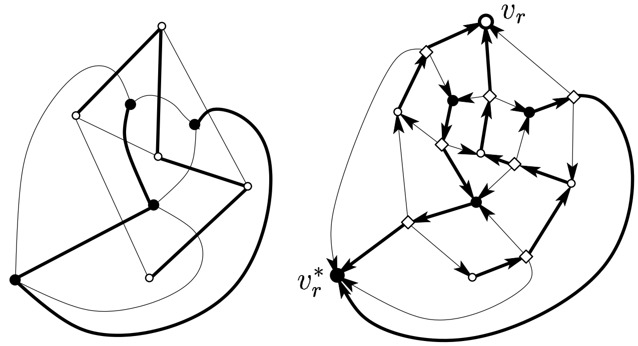

FIGURE 3. A pair of spanning trees for G and G^* and the corresponding orientation of the completion $\widetilde{G}$ with roots v_r and v_r^*.

for the following α_T:

- $\alpha_T(v_r) = 0$ and $\alpha_T(v_r^*) = 0$, i.e., the roots have outdegree zero.
- $\alpha_T(v_e) = 3$ for all edge-vertices v_e.
- $\alpha_T(v) = 1$ for all primal- and dual- non-root vertices v.

A pair of root vertices v_r and v_r^* is *legal* if both are incident to some face of $\widetilde{G}$.

Proposition 2. *The spanning trees of a planar graph G are in bijection to the α_T-orientations of $\widetilde{G}$ with a legal pair of root-vertices.*

Figure 4 shows the distributive lattice of the spanning trees of a graph with two different choices of the primal-root. The dual-root for both examples is the dual-vertex corresponding to the outer face.

It seems worthwile to understand the cover relation $T \prec T'$ between trees: The two trees only differ in one edge $T' = T - e + e'$ and there is a vertex $v \neq v_r$ such that e is the first edge of the $v \to v_r$ path in T and e' is the first edge of the $v \to v_r$ path in T'. Moreover, in the clockwise ordering of edges around v edge e' is the immediate successor of e and the angle between e and e' at v belongs to the interior of the unique cycle of $T + e'$ (this last condition is based on the choice of v_r^* as the dual-vertex corresponding to the unbounded face of G). The characterization is illustrated in Figure 5.

3.4. **Schnyder woods.** Let G be a plane graph and let a_1, a_2, a_3 be three different vertices in clockwise order from the outer face of G. The *suspension* G^σ of G is obtained by adding a half-edge that reaches into the outer face to each of the three special vertices a_i. The *closure* G^σ_∞ of a suspension G^σ is obtained

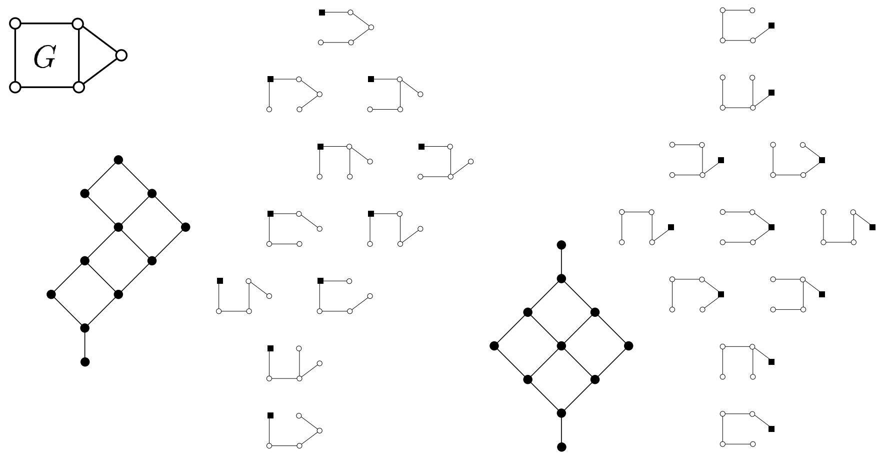

FIGURE 4. A graph and two distributive lattices for its spanning trees.

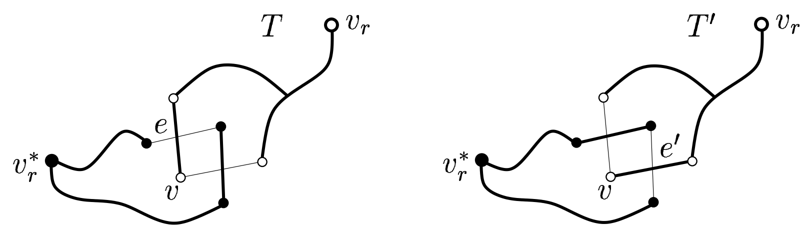

FIGURE 5. A typical flop between spanning trees $T \prec T'$ and their duals.

by adding a new vertex v_∞, this vertex is used as second endpoint for the three half-edges of G^σ.

Schnyder [14], [15] introduced edge orientations and equivalent angle labelings for planar triangulations. He used this structures for a remarkable characterization of planar graphs in terms of order dimension. The incidence order P_G of a graph $G = (V, E)$ is the order on $V \cup E$ with relations $v < e$ iff $v \in V$, $e \in E$ and $v \in e$. Schnyder proved: A graph G is planar $\iff$ the dimension of its incidence order is at most 3. Another important application of Schnyder's labelings is a proof that every planar n vertex graph admits a straight line drawing on the $(n-1) \times (n-1)$ grid.

De Fraysseix and de Mendez [4] prove a bijection between Schnyder labelings of a planar triangulation G and 3-orientations of G^σ_∞, i.e., α-orientations with $\alpha(v) = 3$ for every regular vertex and $\alpha(v_\infty) = 0$. Based on the bijection with 3-orientations de Mendez [9] and Brehm [1] have shown that the set of Schnyder labelings of a planar triangulation G has the structure of a distributive lattice.

In [2] the concept of Schnyder labelings was generalized to 3-connected planar graphs. It was also shown that like the original concept the generalization yields strong applications in the areas of dimension theory and graph drawing. The following definition of Schnyder woods is taken from [3] where it is also shown that Schnyder woods and Schnyder labelings are in bijection.

Let G^σ be the suspension of a 3-connected plane graph. A *Schnyder wood* rooted at a_1, a_2, a_3 is an orientation and labeling of the edges of G^σ with the labels $1, 2, 3$ (alternatively: red, green, blue) satisfying four rules. On the labels we assume a cyclic structure so that $i+1$ and $i-1$ is defined for all i.

(W1) Every edge e is oriented by one or two opposing directions. The directions of edges are labeled such that if e is bioriented the two directions have distinct labels.

(W2) The half-edge at a_i is directed outwards and labeled i.

(W3) Every vertex v has one outgoing edge in each label. The edges e_1, e_2, e_3 leaving v in labels 1,2,3 occur in clockwise order. Each edge entering v in label i enters v in the clockwise sector from e_{i+1} to e_{i-1}. (See Figure 6).

(W4) There is no interior face whose boundary is a directed cycle in one label.

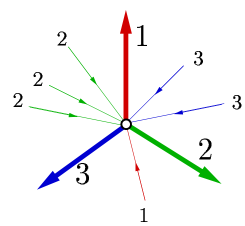

FIGURE 6. Edge orientations and labels at a vertex.

Unlike in the case of planar triangulations, the labeling of edges of a Schnyder wood can not be recovered from the underlying orientation. However, orientations of an appropriate primal dual completion of a suspended plane graph are in bijection to Schnyder woods (Proposition 3).

The first step of the proof consists of showing that Schnyder woods of a suspended plane graph are in bijection with Schyder woods of the (properly defined) dual. Figure 7 exemplifies the duality. Actually, the figure illustrates much more: With the primal and dual graphs and Schnyder woods it also shows a corresponding orthogonal surface. We include this figure for two reasons. The duality between primal and dual Schnyder woods becomes nicely visible on the surface. Moreover, it was in this context of geodesic embeddings of planar graphs on orthogonal surfaces that the duality was first observed by Miller [8]. For details on geodesic embeddings and the connections with Schnyder woods we refer to [8] and [3].

The *completion* $\widetilde{G^\sigma}$ of a plane suspension G^σ and its dual G^{σ^*} is obtained by superimposing G^σ and G^{σ^*} so that exactly the primal dual pairs of edges cross (the half edge at a_i has a crossing with the dual edge $\{b_j, b_k\}$, for $\{i, j, k\} = \{1, 2, 3\}$). In the completion each crossing is represented by a new edge-vertex.

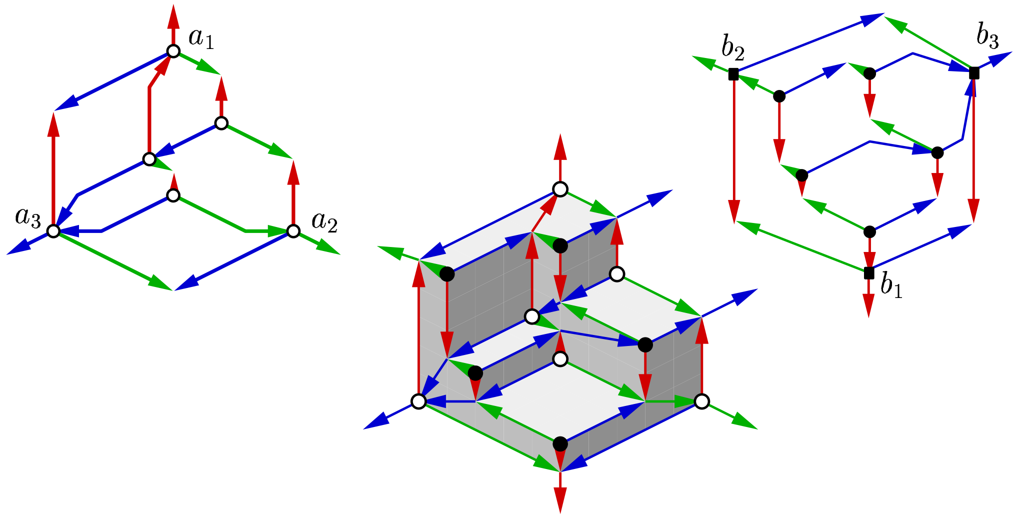

FIGURE 7. A suspended graph G^σ with a Schnyder wood, a corresponding embedding and the dual Schnyder wood.

The completion $\widetilde{G^\sigma}$ is planar and has six half-edges reaching into the unbounded face. Similar to the closure of a suspension we define the closure $\widetilde{G^\sigma_\infty}$ of $\widetilde{G^\sigma}$ by adding a new vertex v_∞ which is the second endpoint of the six half-edges.

A pair of corresponding Schnyder woods on G^σ and G^{σ^*} induces an orientation of $\widetilde{G^\sigma_\infty}$ which is an α-orientation for the following α_S:

- $\alpha_S(v) = 3$ for all primal- and dual-vertices v.
- $\alpha_S(v_e) = 1$ for all edge-vertices v_e.
- $\alpha_S(v_\infty) = 0$ for the special closure vertex v_∞.

Proposition 3. *The Schnyder woods of a planar suspension G^σ are in bijection with α_S-orientations of $\widetilde{G^\sigma_\infty}$.*

Combining Proposition 3 and Theorem 1 we obtain the main result of this section.

Theorem 3. *The set of Schnyder woods of a planar suspension G^σ form a distributive lattice.*

In the case of Schnyder woods a full characterization of all possible essential cycles seems to be a complex task. Unlike in the case of spanning trees or Eulerian orientations it is not enough to consider faces of $\widetilde{G^\sigma_\infty}$ as candidates for essential cycles. The next lemma shows that still in some sense the essential cycles cannot be too complicated.

Lemma 15. *Let G^σ be suspended plane graph. The possible length of essential cycles for α_S-orientations of $\widetilde{G^\sigma_\infty}$ are 4, 6, 8, 10 and 12.*

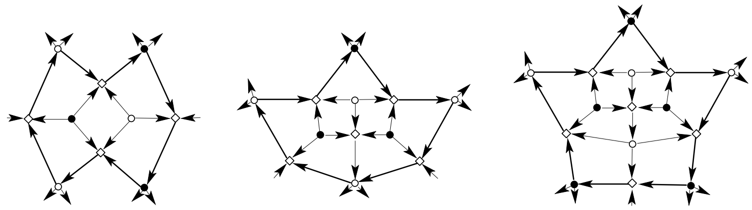

FIGURE 8. Bold edges show non-trivial essential cycles for α_S-orientations.

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