SCHUBERT POLYNOMIALS AND QUIVER FORMULAS

ANDERS BUCH, ANDREW KRESCH, HARRY TAMVAKIS, AND ALEXANDER YONG

ABSTRACT. Fulton's universal Schubert polynomials [F3] represent degeneracy loci for morphisms of vector bundles with rank conditions coming from a permutation. The quiver formula of Buch-Fulton [BF] expresses these polynomials as an integer linear combination of products of Schur determinants. We present a positive, nonrecursive combinatorial formula for the coefficients. We apply our result to obtain new expansions for the Schubert polynomials of Lascoux and Schützenberger [LS1] and explicit Giambelli formulas in the classical and quantum cohomology ring of any partial flag variety.

RÉSUMÉ. Les polynômes de Schubert universels de Fulton représentent des lieux de dégénérescence d'une séquence de fibrés vectorielles avec des conditions sur le rang des fonctions venant d'une permutation. La formule "quiver" de Buch-Fulton exprime ces polynômes comme combinaison linéaire à coefficients entiers, de déterminants de Schur. Nous présentons une formule combinatoire positive et non récurrente pour les coefficients. Nous appliquons notre résultat pour obtenir les polynômes de Schubert de Lascoux et Schützenberger, et des formules de Giambelli dans l'anneau de cohomologie classique et quantique des variétés de drapeaux.

1. INTRODUCTION

The work of Buch and Fulton [BF] established a formula for a general kind of degeneracy locus associated to an oriented quiver of type A. The main ingredients in this formula are Schur determinants and certain integers, the *quiver coefficients*, which generalize the classical Littlewood-Richardson coefficients. Our aim in this paper[1] is to prove a positive combinatorial formula for the quiver coefficients when the rank conditions defining the degeneracy locus are given by a permutation. In particular, this gives new expansions for Fulton's universal Schubert polynomials [F3] and the Schubert polynomials of Lascoux and Schützenberger [LS1].

Let $\mathfrak{X}$ be a smooth complex algebraic variety and let

$$G_1 \to \cdots \to G_{n-1} \to G_n \to F_n \to F_{n-1} \to \cdots \to F_1 \tag{1}$$

Date: April 27, 2003.

2000 *Mathematics Subject Classification.* 05E15; 14M15.

The authors were supported in part by NSF Grant DMS-0070479 (Buch), an NSF Postdoctoral Research Fellowship (Kresch), and NSF Grant DMS-0296023 (Tamvakis).

[1]An expanded version, including complete proofs, may be found on the FPSAC conference CD.

be a sequence of vector bundles and morphisms over $\mathfrak{X}$, such that G_i and F_i have rank i for each i. For every permutation w in the symmetric group S_{n+1} there is a degeneracy locus

$$\Omega_w(G_\bullet \to F_\bullet) = \{x \in \mathfrak{X} \mid \operatorname{rank}(G_q(x) \to F_p(x)) \leqslant r_w(p,q) \text{ for all } 1 \leqslant p, q \leqslant n\}\,,$$

where $r_w(p,q)$ is the number of $i \leqslant p$ such that $w(i) \leqslant q$. The universal double Schubert polynomial $\mathfrak{S}_w(c;d)$ of Fulton gives a formula for this locus; this is a polynomial in the variables $c_i(j)$ and $d_i(j)$ for $1 \leqslant i \leqslant j \leqslant n$. When the codimension of $\Omega_w(G_\bullet \to F_\bullet)$ is equal to the length of w, its class $[\Omega_w]$ in the cohomology (or Chow ring) of $\mathfrak{X}$ is obtained by evaluating $\mathfrak{S}_w(c;d)$ at the Chern classes $c_i(p)$ and $d_i(q)$ of the bundles F_p and G_q, respectively.

The quiver formula given in [BF] specializes to a formula for universal double Schubert polynomials:

$$\mathfrak{S}_w(c;d) = \sum_\lambda c_{w,\lambda}^{(n)}\, s_{\lambda^1}(d(2)-d(1)) \cdots s_{\lambda^n}(c(n)-d(n)) \cdots s_{\lambda^{2n-1}}(c(1)-c(2))\,.$$

Here the sum is over sequences of $2n-1$ partitions $\lambda = (\lambda^1, \ldots, \lambda^{2n-1})$ and each s_{λ^i} is a Schur determinant in the difference of the two alphabets in its argument. The quiver coefficients $c_{w,\lambda}^{(n)}$ can be computed by an inductive algorithm, and are conjectured to be nonnegative [BF].

Let $\operatorname{col}(T)$ denote the column word of a semistandard Young tableau T, the word obtained by reading the entries of the columns of the tableau from bottom to top and left to right. The following theorem is our main result.

Theorem 1. *Suppose that $w \in \mathcal{S}_{n+1}$ and $\lambda = (\lambda^1, \ldots, \lambda^{2n-1})$ is a sequence of partitions. Then $c_{w,\lambda}^{(n)}$ equals the number of sequences of semistandard tableaux $(T_1, \ldots, T_{2n-1})$ such that the shape of T_i is conjugate to λ^i, the entries of T_i are at most $\min(i, 2n-i)$, and $\operatorname{col}(T_1) \cdots \operatorname{col}(T_{2n-1})$ is a reduced word for w.*

As a consequence of Theorem 1, we obtain formulas for the Schubert polynomials of Lascoux and Schützenberger, expressing them as linear combinations of products of Schur polynomials in disjoint sets of variables. The coefficients in these expansions are all quiver coefficients $c_{w,\lambda}^{(n)}$; this is true in particular for the expansion of a Schubert polynomial as a linear combination of monomials (see, e.g., [BJS, FS]).

The *Stanley symmetric functions* or stable Schubert polynomials [St] play a central role in this paper. Our main result generalizes the formula of Fomin and Greene [FG] for the expansion of these symmetric functions in the Schur basis, as well as the connection between quiver coefficients and Stanley symmetric functions obtained in [B]. In fact, we show that the universal double Schubert polynomials can be expressed as a multiplicity-free sum of products of Stanley symmetric functions (Theorem 3).

It should be noted that the formula for Schubert polynomials suggested in this paper is different from the one given in [BF]. For example, the formula from [BF, §2.3] does not make it clear that the monomial coefficients of Schubert polynomials are quiver coefficients, or even that these monomial coefficients are nonnegative. We remark however that the arguments used in this article do not imply the quiver formula of [BF] in the case of universal Schubert polynomials, but rather rely on some results of loc. cit.

Knutson, Miller, and Shimozono have recently announced that they can prove that the general quiver coefficients defined in [BF] are non-negative, using different methods. On the other hand, the techniques of the present paper may also be used to obtain an analogous treatment of Grothendieck polynomials and quiver formulas for the structure sheaves of degeneracy loci in K-theory. This application is presented in [BKTY].

We review the universal Schubert polynomials and Stanley symmetric functions in Section 2; in addition, we prove some required properties. In Section 3 we introduce quiver varieties and we prove Theorem 1. In the following section we apply our results to the case of ordinary Schubert polynomials.

The authors thank Sergey Fomin, Peter Magyar, and Alexander Postnikov for useful discussions.

2. Preliminaries

2.1. **Universal Schubert polynomials.** We begin by recalling the definition of the double Schubert polynomials of Lascoux and Schützenberger [LS1, L1]. Let $X = (x_1, x_2, \ldots)$ and $Y = (y_1, y_2, \ldots)$ be two sequences of commuting independent variables. Given a permutation $w \in S_n$, the double Schubert polynomial $\mathfrak{S}_w(X;Y)$ is defined recursively as follows. If $w = w_0$ is the longest permutation in S_n then we set

$$\mathfrak{S}_{w_0}(X;Y) = \prod_{i+j \leqslant n} (x_i - y_j)\,.$$

Otherwise we can find a simple transposition $s_i = (i, i+1) \in S_n$ such that $\ell(ws_i) = \ell(w) + 1$. Here $\ell(w)$ denotes the length of w, which is the smallest number ℓ for which w can be written as a product of ℓ simple transpositions. We then define

$$\mathfrak{S}_w(X;Y) = \partial_i(\mathfrak{S}_{ws_i}(X;Y))$$

where ∂_i is the divided difference operator given by

$$\partial_i(f) = \frac{f(x_1, \ldots, x_i, x_{i+1}, \ldots, x_n) - f(x_1, \ldots, x_{i+1}, x_i, \ldots, x_n)}{x_i - x_{i+1}}\,.$$

The (single) Schubert polynomial is defined by $\mathfrak{S}_w(X) = \mathfrak{S}_w(X;0)$.

Suppose now that w is a permutation in S_{n+1}. If $u_1, \ldots, u_r$ are permutations, we will write $u_1 \cdots u_r = w$ if $\ell(u_1) + \cdots + \ell(u_r) = \ell(w)$ and the product of $u_1, \ldots, u_r$ is equal to w. In this case we say $u_1 \cdots u_r$ is a *reduced factorization* of w.

The *double universal Schubert polynomial* $\mathfrak{S}_w(c;d)$ of [F3] is a polynomial in the variables $c_i(j)$ and $d_i(j)$ for $1 \leqslant i \leqslant j \leqslant n$. For convenience we set $c_0(j) = d_0(j) = 1$ for all j and $c_i(j) = d_i(j) = 0$ if $i < 0$ or $i > j$. The classical Schubert polynomial $\mathfrak{S}_w(X)$ can be written uniquely in the form

$$\mathfrak{S}_w(X) = \sum a_{i_1,\ldots,i_n}\, e_{i_1}(x_1)\, e_{i_2}(x_1,x_2)\cdots e_{i_n}(x_1,\ldots,x_n)$$

where the sum is over all sequences $(i_1,\ldots,i_n)$ with each $i_\alpha \leqslant \alpha$ and $\sum i_\alpha = \ell(w)$ [LS3]. The coefficients $a_{i_1,\ldots,i_n}$ are uniquely determined integers depending on w. Define the single universal Schubert polynomial for w by

$$\mathfrak{S}_w(c) = \sum a_{i_1,\ldots,i_n}\, c_{i_1}(1)\, c_{i_2}(2)\cdots c_{i_n}(n)$$

and the double universal Schubert polynomial by

$$\mathfrak{S}_w(c;d) = \sum_{u\cdot v=w} (-1)^{\ell(u)} \mathfrak{S}_{u^{-1}}(d)\mathfrak{S}_v(c)\,.$$

Since the ordinary Schubert polynomial $\mathfrak{S}_w(X)$ does not depend on which symmetric group w belongs to, the same is true for $\mathfrak{S}_w(c;d)$.

As explained in the introduction, the double universal Schubert polynomials describe the degeneracy loci $\Omega_w(G_\bullet \to F_\bullet)$ of morphisms between vector bundles on a smooth algebraic variety $\mathfrak{X}$. The precise statement is the following result.

Theorem 2 (Fulton [F3]). *If the codimension of $\Omega_w(G_\bullet \to F_\bullet)$ is equal to $\ell(w)$ (or if this locus is empty) then the class of $\Omega_w(G_\bullet \to F_\bullet)$ in the cohomology ring of $\mathfrak{X}$ is obtained from $\mathfrak{S}_w(c;d)$ by evaluating at the Chern classes of the bundles, i.e., setting $c_i(j) = c_i(F_j)$ and $d_i(j) = c_i(G_j)$.*

2.2. **Symmetric functions.** For each integer partition $\alpha = (\alpha_1 \geqslant \cdots \geqslant \alpha_p \geqslant 0)$, let $|\alpha| = \sum \alpha_i$ and let α' denote the conjugate of α. Let $c = \{c_1, c_2, \ldots\}$ and $d = \{d_1, d_2, \ldots\}$ be ordered sets of independent variables. Define the Schur determinant

$$s_\alpha(c-d) = \det(h_{\alpha_i+j-i})_{p\times p}$$

where the elements h_k are determined by the identity of formal power series

$$\sum_{k\in\mathbb{Z}} h_k t^k = \frac{1 - d_1 t + d_2 t^2 - \cdots}{1 - c_1 t + c_2 t^2 - \cdots}\,.$$

In particular, $h_0 = 1$ and $h_k = 0$ for $k < 0$. The *supersymmetric Schur functions* $s_\alpha(X/Y)$ are obtained by setting $c_i = e_i(X)$ and $d_i = e_i(Y)$ for all i, and the usual Schur polynomials are given by the specializations

$$s_\alpha(X/Y)|_{Y=0} = s_\alpha(X) \qquad \text{and} \qquad s_\alpha(X/Y)|_{X=0} = (-1)^{|\alpha|} s_{\alpha'}(Y)\,.$$

If E and F are two vector bundles with total Chern classes $c(E)$ and $c(F)$, respectively, we will denote $s_\alpha(c(E)-c(F))$ by $s_\alpha(E-F)$. For any three bundles E, F, and G, there is a basic combinatorial identity [M2, §1.5]:

$$s_\alpha(G-E) = \sum N_{\beta\gamma}^{\alpha}\, s_\beta(F-E)\, s_\gamma(G-F), \tag{2}$$

where the sum is over partitions β and γ with $|\beta| + |\gamma| = |\alpha|$, and $N^{\alpha}_{\beta\gamma}$ is a Littlewood-Richardson coefficient.

Let Λ denote the ring of symmetric functions (as in [M2]). For each permutation $w \in S_n$ there is a stable Schubert polynomial or Stanley symmetric function $F_w \in \Lambda$ which is uniquely determined by the property that

$$F_w(x_1, \dots, x_k) = \mathfrak{S}_{1^m \times w}(x_1, \dots, x_k)$$

for all $m \geqslant k$. [2] Here $1^m \times w \in S_{n+m}$ is the permutation which is the identity on $\{1, \dots, m\}$ and which maps j to $w(j-m)+m$ for $j > m$ (see [M1, (7.18)]). When F_w is written in the basis of Schur functions, one has

$$F_w = \sum_{\alpha : |\alpha| = \ell(w)} d_{w\alpha}\, s_\alpha$$

for some nonnegative integers $d_{w\alpha}$ [EG, LS2]. Fomin and Greene show that the coefficient $d_{w\alpha}$ equals the number of semistandard tableaux T of shape α' such that the column word of T is a reduced word for w [FG, Thm. 1.2]. On the other hand, Buch [B, Cor. 4.1] proved that the $d_{w\alpha}$ are special cases of quiver coefficients, and we will use this connection in the sequel.

3. Quiver varieties

3.1. **Definitions.** Let

$$E_\bullet \;:\; E_1 \to E_2 \to \cdots \to E_n$$

be a sequence of vector bundles and bundle maps over a non-singular variety $\mathfrak{X}$. Given rank conditions $r = \{r_{ij}\}$ for $1 \leqslant i < j \leqslant n$ there is a *quiver variety* given by

$$\Omega_r(E_\bullet) = \{x \in \mathfrak{X} \mid \operatorname{rank}(E_i(x) \to E_j(x)) \leqslant r_{ij}\ \forall i < j\}\,.$$

For convenience, we set $r_{ii} = \operatorname{rank} E_i$ for all i, and we demand that the rank conditions satisfy $r_{ij} \geqslant \max(r_{i-1,j}, r_{i,j+1})$ and $r_{ij} + r_{i-1,j+1} \geqslant r_{i-1,j} + r_{i,j+1}$ for all $i \le j$. In this case, the expected codimension of $\Omega_r(E_\bullet)$ is the number $d(r) = \sum_{i<j}(r_{i,j-1} - r_{ij})(r_{i+1,j} - r_{ij})$. The main result of [BF] states that when the quiver variety $\Omega_r(E_\bullet)$ has this codimension, its cohomology class is given by

$$[\Omega_r(E_\bullet)] = \sum_\lambda c_\lambda(r)\, s_{\lambda^1}(E_2 - E_1) \cdots s_{\lambda^{n-1}}(E_n - E_{n-1})\,. \tag{3}$$

Here the sum is over all sequences of partitions $\lambda = (\lambda^1, \dots, \lambda^{n-1})$ such that $\sum |\lambda^i| = d(r)$, and the coefficients $c_\lambda(r)$ are integers computed by a combinatorial algorithm which we will not reproduce here. These coefficients are uniquely determined by the condition that (3) is true for all varieties $\mathfrak{X}$ and sequences $E_\bullet$, as well as the condition that $c_\lambda(r) = c_\lambda(r')$, where $r' = \{r'_{ij}\}$ is the set of rank conditions given by $r'_{ij} = r_{ij} + 1$ for all $i \leqslant j$.

[2] In Stanley's notation, the function $F_{w^{-1}}$ is assigned to w.

Suppose the index p is such that all rank conditions $\operatorname{rank}(E_i(x) \to E_p(x)) \leqslant r_{ip}$ and $\operatorname{rank}(E_p(x) \to E_j(x)) \leqslant r_{pj}$ may be deduced from other rank conditions. Following [BF, §4], we will then say that the bundle E_p is *inessential*. Omitting an inessential bundle E_p from $E_\bullet$ produces a sequence

$$E'_\bullet \; : \; E_1 \to \cdots \to E_{p-1} \to E_{p+1} \to \cdots \to E_n,$$

where the map from E_{p-1} to E_{p+1} is the composition $E_{p-1} \to E_p \to E_{p+1}$. If r' denotes the restriction of the rank conditions to $E'_\bullet$, we have that $\Omega_{r'}(E'_\bullet) = \Omega_r(E_\bullet)$. We can use (2) to expand any factor $s_\alpha(E_{p+1} - E_{p-1})$ occurring in the quiver formula for $\Omega_{r'}(E'_\bullet)$ into a sum of products of the form $s_\beta(E_p - E_{p-1}) s_\gamma(E_{p+1} - E_p)$, and thus arrive at the quiver formula (3) for $\Omega_r(E_\bullet)$.

The loci associated with universal Schubert polynomials are special cases of quiver varieties. Given $w \in S_{n+1}$ we define rank conditions $r^{(n)} = \{r^{(n)}_{ij}\}$ for $1 \leqslant i \leqslant j \leqslant 2n$ by

$$r^{(n)}_{ij} = \begin{cases} r_w(2n+1-j, i) & \text{if } i \leqslant n < j \\ i & \text{if } j \leqslant n \\ 2n+1-j & \text{if } i \geqslant n+1. \end{cases}$$

Then $\Omega_w(G_\bullet \to F_\bullet)$ is identical to the quiver variety $\Omega_{r^{(n)}}(G_\bullet \to F_\bullet)$, and furthermore we have $\ell(w) = d(r^{(n)})$. If we let $c^{(n)}_{w,\lambda} = c_\lambda(r^{(n)})$ denote the quiver coefficients corresponding to this locus, it follows that

$$\mathfrak{S}_w(c; d) = \sum_\lambda c^{(n)}_{w,\lambda} \, s_{\lambda^1}(d(2)-d(1)) \cdots s_{\lambda^n}(c(n)-d(n)) \cdots s_{\lambda^{2n-1}}(c(1)-c(2)) \,. \tag{4}$$

3.2. **Proof of Theorem 1.** It will be convenient to work with the element $P^{(n)}_w \in \Lambda^{\otimes 2n-1}$ defined by

$$P^{(n)}_w = \sum_\lambda c^{(n)}_{w,\lambda} \, s_{\lambda^1} \otimes \cdots \otimes s_{\lambda^{2n-1}} \,.$$

Theorem 1 is a consequence of Fomin and Greene's formula for stable Schubert polynomials combined with the following result.

Theorem 3. *For $w \in S_{n+1}$ we have*

$$P^{(n)}_w = \sum_{u_1 \ldots u_{2n-1} = w} F_{u_1} \otimes \cdots \otimes F_{u_{2n-1}}$$

where the sum is over all reduced factorizations $w = u_1 \cdots u_{2n-1}$ such that $u_i \in S_{\min(i,2n-i)+1}$ for each i.

Proof. Since $r_w(p,q)+m = r_{1^m\times w}(p+m,q+m)$ for $m \geq 0$, it follows that the coefficients $c_{w,\lambda}^{(n)}$ are uniquely defined by the condition that

$$(5)\quad \mathfrak{S}_{1^m\times w}(c;d) = \sum_{\lambda} c_{w,\lambda}^{(n)}\, s_{\lambda^1}(d(2+m)-d(1+m))\cdots \\ s_{\lambda^n}(c(n+m)-d(n+m))\cdots s_{\lambda^{2n-1}}(c(1+m)-c(2+m))$$

for all $m \geqslant 0$ (see also [B, §4]).

Given any two integers $p \leqslant q$ we let $P_w^{(n)}[p,q]$ denote the sum of the terms of $P_w^{(n)}$ for which λ^i is empty when $i<p$ or $i>q$:

$$P_w^{(n)}[p,q] = \sum_{\lambda:\lambda^i=\emptyset \text{ for } i\notin[p,q]} c_{w,\lambda}^{(n)}\, s_{\lambda^1}\otimes\cdots\otimes s_{\lambda^{2n-1}}\,.$$

Lemma 1. *For any* $1 < i \leqslant 2n-1$ *we have*

$$(6)\qquad P_w^{(n)} = \sum_{u\cdot v=w} P_u^{(n)}[1,i-1]\cdot P_v^{(n)}[i,2n-1]\,.$$

Lemma 2. *For* $1 \leqslant i \leqslant 2n-1$ *we have*

$$P_w^{(n)}[i,i] = \begin{cases} 1^{\otimes i-1}\otimes F_w \otimes 1^{\otimes 2n-1-i} & \textit{if } w\in S_{m+1},\ m=\min(i,2n-i)\\ 0 & \textit{otherwise.}\end{cases}$$

Theorem 3 follows immediately from lemmas 1 and 2. □

Example 1. For the permutation $w = s_2s_1 = 312$ in S_3, the sequences of tableaux which satisfy the conditions of Theorem 1 are

$$(\ \emptyset\ ,\ \begin{array}{|c|}\hline 1\\ \hline 2\\ \hline\end{array}\ ,\ \emptyset\ ) \quad\text{and}\quad (\ \emptyset\ ,\ \boxed{2}\ ,\ \boxed{1}\ )\,.$$

It follows that

$$\begin{aligned}\mathfrak{S}_{312}(c;d) &= s_2(c(2)-d(2)) + s_1(c(2)-d(2))s_1(c(1)-c(2))\\ &= c_1(1)c_1(2)-c_1(1)d_1(2)-c_2(2)+d_2(2)\,.\end{aligned}$$

In [BF], a conjectural combinatorial rule for general quiver coefficients $c_\lambda(r)$ was given. Although this rule was also stated in terms of sequences of semistandard tableaux satisfying certain conditions, it is different from Theorem 1 in the case of universal Schubert polynomials. It would be interesting to find a bijection that establishes the equivalence of these two rules.

3.3. **Skipping bundles.** A permutation w has a descent position at i if $w(i) > w(i+1)$. We say that a sequence $\{a_k\} : a_1 < \cdots < a_p$ of integers is *compatible with* w if all descent positions of w are contained in $\{a_k\}$. Suppose that $w \in S_{n+1}$ and let $1 \leqslant a_1 < a_2 < \cdots < a_p \leqslant n$ and $1 \leqslant b_1 < b_2 < \cdots < b_q \leqslant n$ be two sequences compatible with w and w^{-1}, respectively.

We let $E_\bullet$ denote the subsequence

$$G_{b_1} \to G_{b_2} \to \cdots \to G_{b_q} \to F_{a_p} \to \cdots \to F_{a_2} \to F_{a_1} \,,$$

and define rank conditions $\tilde{r}^{(n)} = \{\tilde{r}_{ij}^{(n)}\}$ for $1 \leqslant i \leqslant j \leqslant p+q$ by

$$\tilde{r}_{ij}^{(n)} = \begin{cases} r_w(a_{p+q+1-j}, b_i) & \text{if } i \leqslant q < j \\ b_i & \text{if } j \leqslant q \\ a_{p+q+1-j} & \text{if } i \geqslant q+1 \end{cases}$$

Then the expected codimension of the locus $\Omega_{\tilde{r}^{(n)}}(E_\bullet)$ is equal to $\ell(w)$. However, in general this locus may contain $\Omega_w(G_\bullet \to F_\bullet)$ as a proper closed subset. We will need the following criterion for equality (see also the remarks in [F3, §3] and Exercise 10 of [F2, §10].)

Lemma 3. *Suppose that the map $G_{i-1} \to G_i$ is injective for $i \notin \{b_k\}$ and the map $F_i \to F_{i-1}$ is surjective for $i \notin \{a_k\}$. Then $\Omega_{\tilde{r}^{(n)}}(E_\bullet) = \Omega_w(G_\bullet \to F_\bullet)$ as subschemes of $\mathfrak{X}$.*

Corollary 1. *Let $w \in S_{n+1}$ and $\{a_k\}$ and $\{b_k\}$ be as above. Then we have*

$$[\Omega_{\tilde{r}^{(n)}}(E_\bullet)] = \sum_{\mu} \tilde{c}_{w,\mu}^{(n)} \, s_{\mu^1}(G_{b_2} - G_{b_1}) \cdots s_{\mu^q}(F_{a_p} - G_{b_q}) \cdots s_{\mu^{p+q-1}}(F_{a_1} - F_{a_2})$$

with coefficients $\tilde{c}_{w,\mu}^{(n)} = c_{w,\lambda}^{(n)}$ where the sequence $\lambda = (\lambda^1, \ldots, \lambda^{2n-1})$ is given by

$$\lambda^i = \begin{cases} \mu^k & \textit{if } i = b_k \\ \mu^{p+q-k} & \textit{if } i = 2n - a_k \\ \emptyset & \textit{otherwise.} \end{cases}$$

4. Schubert polynomials

4.1. **Degeneracy loci.** In this section, we will interpret the previous results for ordinary double Schubert polynomials. Let V be a vector bundle of rank n and let

$$G_1 \subset G_2 \subset \cdots \subset G_{n-1} \subset V \twoheadrightarrow F_{n-1} \twoheadrightarrow \cdots \twoheadrightarrow F_2 \twoheadrightarrow F_1$$

be a complete flag followed by a dual complete flag of V. If $w \in S_n$ then Fulton has proved [F1] that

$$[\Omega_w(G_\bullet \to F_\bullet)] = \mathfrak{S}_w(x_1, \ldots, x_n; y_1, \ldots, y_n)$$

where $x_i = c_1(\ker(F_i \to F_{i-1}))$, $y_i = c_1(G_i/G_{i-1})$, and $\mathfrak{S}_w(X;Y)$ is the double Schubert polynomial of Lascoux and Schützenberger.

Set $G_i' = V/G_i$ and $F_i' = \ker(V \to F_i)$. Then we have a sequence

$$F_{n-1}' \subset \cdots \subset F_1' \subset V \twoheadrightarrow G_1' \twoheadrightarrow \cdots \twoheadrightarrow G_{n-1}'$$

and it is easy to check that $\Omega_w(G_\bullet \to F_\bullet) = \Omega_{w_0 w^{-1} w_0}(F_\bullet' \to G_\bullet')$ as subschemes of $\mathfrak{X}$, where w_0 is the longest permutation in S_n.

Let $1 \leqslant a_1 < \cdots < a_p \leqslant n-1$ and $0 \leqslant b_1 < \cdots < b_q \leqslant n-1$ be two sequences compatible with w and w^{-1}, respectively. Then by applying section 3.3 to the subsequence $F'_{a_p} \to \cdots \to F'_{a_1} \to G'_{b_1} \to \cdots \to G'_{b_q}$ we obtain

$$[\Omega_w(G_\bullet \to F_\bullet)] = \sum_\mu \tilde{c}^{(n-1)}_{w_0 w^{-1} w_0,\mu}\, s_{\mu^1}(F'_{a_{p-1}} - F'_{a_p}) \cdots s_{\mu^p}(G'_{b_1} - F'_{a_1}) \cdots s_{\mu^{p+q-1}}(G'_{b_q} - G'_{b_{q-1}})\,.$$

Set $a_0 = b_0 = 0$. If we let $X_i = \{x_{a_{i-1}+1}, \ldots, x_{a_i}\}$ denote the Chern roots of $\ker(F_{a_i} \to F_{a_{i-1}})$ and $Y_i = \{y_{b_{i-1}+1}, \ldots, y_{b_i}\}$ be the Chern roots of $G_{b_i}/G_{b_{i-1}}$ then the previous equality can be written as

$$\mathfrak{S}_w(X;Y) = \sum_\mu \tilde{c}^{(n-1)}_{w_0 w^{-1} w_0,\mu}\, s_{\mu^1}(X_p) \cdots s_{\mu^p}(X_1/Y_1) \cdots s_{\mu^{p+q-1}}(0/Y_q)\,. \tag{8}$$

This equation is true in the cohomology ring $H^*(\mathfrak{X};\mathbb{Z})$, in which there are relations between the variables x_i and y_i (including e.g. the relations $e_j(x_1, \ldots, x_n) = c_j(V) = e_j(y_1, \ldots, y_n)$ for $1 \leqslant j \leqslant n$). We claim, however, that (8) holds as an identity of polynomials in independent variables. For this, notice that the identity is independent of n, i.e. the coefficient $\tilde{c}^{(n-1)}_{w_0 w^{-1} w_0,\mu}$ does not change when n is replaced with $n+1$ and w_0 with the longest element in S_{n+1}. If we choose n sufficiently large, we can construct a variety $\mathfrak{X}$ on which (8) is true, and where all monomials in the variables x_i and y_i of total degree at most $\ell(w)$ are linearly independent, which establishes the claim.

4.2. **Splitting Schubert polynomials.** We continue by reformulating equation (8) to obtain a more natural expression for double Schubert polynomials.

It follows from (8) together with the main result of [B] that $F_{w_0 w^{-1} w_0} = F_w$. Therefore, the coefficient $d_{w\alpha}$ of the Schur expansion of F_w is equal to the number of semistandard tableaux of shape α' such that the column word is a reduced word for $w_0 w^{-1} w_0$. Notice that if $e = (e_1, e_2, \ldots, e_\ell)$ is a reduced word for $w_0 w^{-1} w_0$ then $\widetilde{e} = (n+1-e_\ell, \ldots, n+1-e_1)$ is a reduced word for w. Furthermore, if e is the column word of a tableau of shape α', then $\widetilde{e}$ is the column word of a skew tableau whose shape is the 180 degree rotation of α'. If we denote this rotated shape by $\widetilde{\alpha}$ then we conclude that $d_{w\alpha}$ is also equal to the number of skew tableaux of shape $\widetilde{\alpha}$ such that the column word is a reduced word for w. Theorem 3 now implies the following variation of our main theorem:

Theorem 1′. *If $w \in S_{n+1}$ then the coefficient $c^{(n)}_{w,\lambda}$ equals the number of sequences of semistandard skew tableaux $(T_1, \ldots, T_{2n-1})$ such that T_i has shape $\widetilde{\lambda}^i$, the entries of T_i are at most $\min(i, 2n-i)$, and $\operatorname{col}(T_1) \cdots \operatorname{col}(T_{2n-1})$ is a reduced word for w.*

We say that a sequence of tableaux $(T_1, \ldots, T_r)$ is *strictly bounded below* by an integer sequence $(a_1, \ldots, a_r)$ if the entries of T_i are strictly greater than a_i, for each i.

Theorem 4. *Let $w \in S_n$ and let $1 \leqslant a_1 < \cdots < a_p$ and $0 \leqslant b_1 < \cdots < b_q$ be two sequences compatible with w and w^{-1}, respectively. Then we have*

$$\mathfrak{S}_w(X;Y) = \sum_{\lambda} c_\lambda \, s_{\lambda^1}(0/Y_q) \cdots s_{\lambda^q}(X_1/Y_1) \cdots s_{\lambda^{p+q-1}}(X_p) \tag{9}$$

where $X_i = \{x_{a_{i-1}+1}, \ldots, x_{a_i}\}$ and $Y_i = \{y_{b_{i-1}+1}, \ldots, y_{b_i}\}$ and the sum is over all sequences of partitions $\lambda = (\lambda^1, \ldots, \lambda^{p+q-1})$. Each c_λ is a quiver coefficient, equal to the number of sequences of semistandard tableaux $(T_1, \ldots, T_{p+q-1})$ strictly bounded below by $(b_{q-1}, \ldots, b_1, 0, a_1, a_2, \ldots, a_{p-1})$, such that the shape of T_i is conjugate to λ^i and $\mathrm{col}(T_1) \cdots \mathrm{col}(T_{p+q-1})$ is a reduced word for w.

Notice that if one takes $b_1 = 0$ in Theorem 4 then the set of variables Y_1 is empty, so equation (9) contains only signed products of single Schur polynomials. Observe also that a factor $s_{\lambda^i}(0/Y_k)$ in (9) will vanish if λ^i has more than $b_k - b_{k-1}$ columns and that $s_{\lambda^i}(X_k)$ vanishes if λ^i has more than $a_k - a_{k-1}$ rows. Therefore equation (9) uses only a subset of the quiver coefficients of Theorem 4.

Example 2. Let $w = 321$ be the longest element in S_3, and choose the sequences $\{a_i\} = \{b_i\} = \{1 < 2\}$. Then the four sequences of tableaux satisfying the conditions of Theorem 4 are

$$\left(\,\boxed{2}\,,\,\boxed{1}\,,\,\boxed{2}\,\right), \quad \left(\,\boxed{2}\,,\,\boxed{1\,|\,2}\,,\,\emptyset\,\right), \quad \left(\,\emptyset\,,\,\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}\,,\,\boxed{2}\,\right), \quad \text{and} \quad \left(\,\emptyset\,,\,\begin{array}{|c|c|}\hline 1 & 2\\\hline 2 \\\cline{1-1}\end{array}\,,\,\emptyset\,\right),$$

all of which give nonvanishing terms and correspond to the reduced word $s_2 s_1 s_2$. We thus have

$$\begin{aligned}
&\mathfrak{S}_{321}(X;Y) = \\
&s_1(0/y_2)s_1(x_1/y_1)s_1(x_2) + s_1(0/y_2)s_{1,1}(x_1/y_1) + s_2(x_1/y_1)s_1(x_2) + s_{2,1}(x_1/y_1) \\
&= -y_2(x_1 - y_1)x_2 - y_2(y_1^2 - x_1y_1) + (x_1^2 - x_1y_1)x_2 + (x_1y_1^2 - x_1^2y_1) \\
&= (x_1 - y_1)(x_1 - y_2)(x_2 - y_1)\,.
\end{aligned}$$

Corollary 2. *Suppose that $w \in S_n$ is a permutation compatible with the sequence $a_1 < \cdots < a_p$. Then we have*

$$\mathfrak{S}_w(X) = \sum_{\lambda} c_\lambda \, s_{\lambda^1}(X_1) \cdots s_{\lambda^p}(X_p)$$

where $X_i = \{x_{a_{i-1}+1}, \ldots, x_{a_i}\}$ and the sum is over all sequences of partitions $\lambda = (\lambda^1, \ldots, \lambda^p)$. Each c_λ is a quiver coefficient, equal to the number of sequences of semistandard tableaux $(T_1, \ldots, T_p)$ strictly bounded below by $(0, a_1, a_2, \ldots, a_{p-1})$, such that the shape of T_i is conjugate to λ^i and $\mathrm{col}(T_1) \cdots \mathrm{col}(T_p)$ is a reduced word for w.

Example 3. Consider the permutation $w = s_1 s_2 s_1 s_3 s_4 s_3 = 32541$ in S_5, with descent positions at 1, 3, and 4. The two sequences of tableaux satisfying the conditions of Corollary 2 which give nonvanishing terms are

$$\left(\begin{array}{|c|}\hline 1\\\hline 2\\\hline 4\\\hline\end{array}\, , \begin{array}{|c|c|}\hline 2&3\\\hline\end{array}\, , \begin{array}{|c|}\hline 4\\\hline\end{array}\right) \quad\text{and}\quad \left(\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}\, , \begin{array}{|c|c|}\hline 2&3\\\hline 4\\\cline{1-1}\end{array}\, , \begin{array}{|c|}\hline 4\\\hline\end{array}\right).$$

The reduced words for w corresponding to these sequences are $s_4s_2s_1s_2s_3s_4$ and $s_2s_1s_4s_2s_3s_4$, respectively. It follows that

$$\begin{aligned}\mathfrak{S}_{32541}(X) &= s_3(x_1)s_{1,1}(x_2,x_3)s_1(x_4) + s_2(x_1)s_{2,1}(x_2,x_3)s_1(x_4)\\ &= x_1^3\cdot x_2x_3\cdot x_4 + x_1^2(x_2^2x_3 + x_3^2x_2)x_4.\end{aligned}$$

The special case of Corollary 2 with $a_k = k$ gives a formula for the coefficient of each monomial in $\mathfrak{S}_w(X)$, which is equivalent to that of [BJS, Thm. 1.1]. We deduce that these monomial coefficients are quiver coefficients. The same conclusion holds for double Schubert polynomials:

Corollary 3. *Let $w \in S_n$ and let $x^u y^v = x_1^{u_1}\cdots x_{n-1}^{u_{n-1}} y_1^{v_1}\cdots y_{n-1}^{v_{n-1}}$ be a monomial of total degree $\ell(w)$. Set $g_i = \sum_{k=n-i}^{n-1} v_k$ and $f_i = g_{n-1} + \sum_{k=1}^{i} u_k$. Then the coefficient of $x^u y^v$ in the double Schubert polynomial $\mathfrak{S}_w(X;Y)$ is equal to $(-1)^{g_{n-1}}$ times the number of reduced words $(e_1,\ldots,e_{\ell(w)})$ for w such that $n-i \leqslant e_{g_{i-1}+1} < \cdots < e_{g_i}$ and $e_{f_{i-1}+1} > \cdots > e_{f_i} \geqslant i$ for all $1 \leqslant i \leqslant n-1$.*

Remark. We note that alternative positive expressions for the c_λ may be deduced from [LS2, 1.5] and [BS, Cor. 1.2].

References

[BS] N. Bergeron and F. Sottile : *Schubert polynomials, the Bruhat order, and the geometry of flag manifolds*, Duke Math. J. **95** (1998), no. 2, 373–423.

[BJS] S. Billey, W. Jockusch and R. P. Stanley : *Some combinatorial properties of Schubert polynomials*, J. Algebraic Combin. **2** (1993), no. 4, 345–374.

[Bo] A. Borel : *Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lie compacts*, Ann. of Math. **57** (1953), 115–207.

[B] A. S. Buch : *Stanley symmetric functions and quiver varieties*, J. Algebra **235** (2001), no. 1, 243–260.

[BF] A. S. Buch and W. Fulton : *Chern class formulas for quiver varieties*, Invent. Math. **135** (1999), no. 3, 665–687.

[BKTY] A. S. Buch, A. Kresch, H. Tamvakis and A. Yong : *Grothendieck polynomials and quiver formulas*, in preparation.

[C-F] I. Ciocan-Fontanine : *On quantum cohomology rings of partial flag varieties*, Duke Math. J. **98** (1999), no. 3, 485–524.

[EG] M. Edelman and C. Greene : *Balanced tableaux*, Adv. Math. **63** (1987), 42-99.

[FGP] S. Fomin, S. Gelfand and A. Postnikov : *Quantum Schubert polynomials*, J. Amer. Math. Soc. **10** (1997), 565–596.

[FG] S. Fomin and C. Greene : *Noncommutative Schur functions and their applications*, Discrete Math. **193** (1998), no. 1-3, 179–200.

[FK] S. Fomin and A. N. Kirillov : *The Yang-Baxter equation, symmetric functions, and Schubert polynomials*, Discrete Math. **153** (1996), no. 1-3, 123–143.

[FS] S. Fomin and R. P. Stanley : *Schubert polynomials and the nilCoxeter algebra*, Adv. Math. **103** (1994), 196–207.

[F1] W. Fulton : *Flags, Schubert polynomials, degeneracy loci, and determinantal formulas*, Duke Math. J. **65** (1992), no. 3, 381-420.

[F2] W. Fulton : *Young Tableaux*, L.M.S. Student Texts **35**, Cambridge University Press, Cambridge, 1997.

[F3] W. Fulton : *Universal Schubert polynomials*, Duke Math. J. **96** (1999), no. 3, 575–594.

[K] A. N. Kirillov : *Cauchy identities for universal Schubert polynomials*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **283** (2001), 123–139.

[KP] W. Kraskiewicz and P. Pragacz : *Foncteurs de Schubert*, C. R. Acad. Sci. Paris Sér. I Math. **304** (1987), 209–211.

[L1] A. Lascoux : *Classes de Chern des variétés de drapeaux*, C. R. Acad. Sci. Paris Sér. I Math. **295** (1982), no. 5, 393–398.

[L2] A. Lascoux : *Notes on interpolation in one and several variables*, available at http://phalanstere.univ-mlv.fr/~al/

[LS1] A. Lascoux and M.-P. Schützenberger : *Polynômes de Schubert*, C. R. Acad. Sci. Paris Sér. I Math. **294** (1982), 447–450.

[LS2] A. Lascoux and M.-P. Schützenberger : *Structure de Hopf de l'anneau de cohomologie et de l'anneau de Grothendieck d'une variété de drapeaux*, C. R. Acad. Sci. Paris Sér. I Math. **295** (1982), 629–633.

[LS3] A. Lascoux and M.-P. Schützenberger : *Funtorialité des polynomes de Schubert*, Contemporary Math **88** (1989), 585–598.

[M1] I. G. Macdonald : *Notes on Schubert polynomials*, Publ. LACIM **6**, Univ. de Québec à Montréal, Montréal, 1991.

[M2] I. G. Macdonald : *Symmetric Functions and Hall Polynomials*, Second edition, Clarendon Press, Oxford, 1995.

[So] F. Sottile : *Pieri's formula for flag manifolds and Schubert polynomials*, Ann. Inst. Fourier **46** (1996), 89-110.

[St] R. Stanley : *On the number of reduced decompositions of elements of Coxeter groups*, European J. Combin. **5** (1984), 359–372.

Matematisk Institut, Aarhus Universitet, Ny Munkegade, 8000 Århus C, Denmark

E-mail address: abuch@imf.au.dk

Department of Mathematics, University of Pennsylvania, 209 South 33rd Street, Philadelphia, PA 19104-6395, USA

E-mail address: kresch@math.upenn.edu

Department of Mathematics, Brandeis University - MS 050, P. O. Box 9110, Waltham, MA 02454-9110, USA

E-mail address: harryt@brandeis.edu

Department of Mathematics, University of Michigan, 525 East University Ave., Ann Arbor, MI 48109-1109, USA

E-mail address: ayong@umich.edu