# THE ORDER DIMENSION OF BRUHAT ORDER (PRELIMINARY REPORT)

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ABSTRACT. We investigate the order dimension of the Bruhat (or strong) ordering of infinite Coxeter groups, and in particular the affine group $\tilde{A}_n$. We give a quadratic lower bound and a cubic upper bound on the order dimension of the Bruhat ordering of $\tilde{A}_n$ and conjecture a sharper cubic upper bound. We demonstrate that the order dimension of the Bruhat order is infinite for a large class of Coxeter groups.

RÉSUMÉ. Nous étudions la dimension d'ordre de l'ordre de Bruhat (ou fort) sur les groupes infinis de Coxeter. Dans le cas du groupe affine $\tilde{A}_n$, nous donnons une borne inférieure quadratique et une borne supérieure cubique sur la dimension d'ordre de l'ordre de Bruhat, et conjecturons une borne supérieure cubique plus exacte . Nous démontrons également que la dimension d'ordre de l'ordre de Bruhat est infinie pour une grande classe de groupes de Coxeter.

## 1. Main Results

We study the order dimension of the Bruhat (or strong) ordering on infinite Coxeter groups. In particular for the affine group $\tilde{A}_n$, we have the following:

**Theorem 1.** *The order dimension of the Bruhat ordering of the Coxeter group $\tilde{A}_n$ satisfies the following bounds:*

$$n(n+1) \leq \dim(\tilde{A}_n) \leq n(n+1)^2.$$

**Conjecture 2.** *The Bruhat ordering of the Coxeter group $\tilde{A}_n$ satisfies the following upper bound:*

$$\dim(\tilde{A}_n) \leq (n+1)\left\lfloor \frac{(n+1)^2}{4} \right\rfloor.$$

More generally:

**Theorem 3.** *Let $\tilde{W}$ be an affine Coxeter group with Weyl group $W$. Let ${}^J\tilde{W}^K$ be a minuscule two-sided quotient of $\tilde{W}$. Then Bruhat order on ${}^J\tilde{W}^K$ is isomorphic to a connected component of the standard order on dominant weights for a root system associated to $W$.*

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1991 *Mathematics Subject Classification.* Primary: 20F55; Secondary 06A07.

*Key words and phrases.* affine, Bruhat, Coxeter, dissector, join-irreducible, order dimension.

Nathan Reading was partially supported by NSF grant DMS–0202430 and John Stembridge was partially supported by NSF Grant DMS–0070685.

The quotient ${}^J\tilde{W}^K$ is minuscule if both $\tilde{W}_J$ and $\tilde{W}_K$ are isomorphic to $W$. Each component of the standard order on dominant weights is a subposet of $\mathbb{Z}^n$ where $n$ is the rank of $W$ (see [17]). When $\tilde{W}$ is $\tilde{A}_n$, every quotient with respect to a maximal parabolic subgroup is minuscule, and the rank of $W = A_n$ is $n$. It is known that $u \leq v$ in Bruhat order if and only if $P(u) \leq P(v)$ for each projection $P$ onto a maximal double quotient. Since there are $(n+1)^2$ possible double quotients with respect to maximal parabolic subgroups, we obtain the upper bound of Theorem 1.

The lower bound in Theorem 1 and the upper bound in Conjecture 2 derive from the following theorem which is proven in the finite case in [14]. The proof for finitary posets (such as Bruhat order on an infinite Coxeter group) is similar.

**Theorem 4.** *If $P$ is a finitary poset such that* $\mathrm{width}(\mathrm{Dis}(P))$ *is finite or countable, then* $\mathrm{width}(\mathrm{Dis}(P)) \leq \dim(P) \leq \mathrm{width}(\mathrm{Irr}(P))$.

The posets $\mathrm{Dis}(P)$ and $\mathrm{Irr}(P)$ are the subposets of $P$ consisting respectively of dissectors and join-irreducibles (see Section 2). The width of a poset $P$ is the cardinality of a largest antichain in $P$, or equivalently the size of a smallest decomposition of $P$ into chains. Finitary posets are posets in which every lower principal order ideal is finite.

Bruhat orders are finitary, so Theorem 4 can be applied to give bounds on $\dim(\tilde{A}_n)$. We prove Theorem 1 by exhibiting an antichain of $n(n+1)$ dissectors in $\tilde{A}_n$. Computer calculations suggest that $n(n+1)$ is in fact the width of $\mathrm{Dis}(\tilde{A}_n)$, so the lower bound cannot be strengthened using Theorem 4. Conjecture 2 is the result of an analysis of the join-irreducibles in $\tilde{A}_n$. It appears from extensive computer study that any join-irreducible can be written in a particularly nice form. Assuming the nice form, $\mathrm{Dis}(\tilde{A}_n)$ can be decomposed into chains to give the upper bound. The sticking point in the proof of Conjecture 2 is showing that elements of $\tilde{A}_n$ which do not have the nice form are not join-irreducible. We hope to resolve this problem soon using Theorem 3.

One might wonder whether all Coxeter groups have finite order dimension. Using Theorem 4, we show that the Bruhat order on a universal (or "free") Coxeter group has infinite order dimension. There are also many less extreme examples of Coxeter groups whose Bruhat order has infinite dimension. One example is the group whose Coxeter graph has three vertices and edge labels 3, 3 and 4.

The organization of this paper is as follows: Preliminary definitions and results are found in Section 2, followed in Sections 3 and 4 by background information on order quotients and Bruhat order respectively. Beginning in Section 5, we specialize to the affine Coxeter group $\tilde{A}_n$. Section 5 gives background on the realization of $\tilde{A}_n$ by affine permutations. Section 6 contains the proof of the lower bound of Theorem 1, and Conjecture 2 is discussed in Section 7. The proof of Theorem 3 is not given in this preliminary report.

## 2. Preliminaries

We begin by establishing some notation, definitions, and some general tools. An *order ideal* in a poset $P$ is a set $I$ such that $x \in I$ and $y \leq x$ implies $y \in I$. Given $x \in P$, define

$$\begin{aligned} D(x) &:= \{y \in P : y < x\} \\ U(x) &:= \{y \in P : y > x\} \\ D[x] &:= \{y \in P : y \leq x\} \\ U[x] &:= \{y \in P : y \geq x\}. \end{aligned}$$

An order ideal of the form $D[x]$ for some $x \in P$ is called a *principal order ideal.* A poset $P$ will be called *finitary* if every principal order ideal has a finite number of elements. If $P$ is a lattice, this definition agrees with the definition of a finitary lattice given in [15]. Every poset considered in this paper will be assumed to be finitary.

The *order dimension* $\dim(P)$ of a finitary poset $P$ is the smallest cardinal $d$ such that $P$ is the intersection of $d$ linear extensions of $P$. Equivalently the order dimension is the smallest $d$ so that $P$ can be embedded as a subposet of $\mathbb{R}^d$ with componentwise partial order. In this paper, we will not consider any posets whose order dimension is more than countably infinite. The "standard example" of a poset of dimension $n$ is the set of subsets of $[n] := \{1, 2, \ldots n\}$ of cardinality 1 or $n-1$, ordered by inclusion.

Given $x$ and $y$, if $U[x] \cap U[y]$ has a unique minimal element, this element is called the *join* of $x$ and $y$ and is written $x \vee^P y$ or simply $x \vee y$. If $D[x] \cap D[y]$ has a unique maximal element, it is called the *meet* of $x$ and $y$, $x \wedge_P y$ or $x \wedge y$. Given a set $S \subseteq P$, if $\cap_{x \in S} U[x]$ has a unique minimal element, it is called $\vee S$. The join $\vee\emptyset$ is $\hat{0}$ if $P$ has a unique minimal element $\hat{0}$, and otherwise $\vee\emptyset$ does not exist. If $\cap_{x \in S} D[x]$ has a unique maximal element, it is called $\wedge S$. The meet $\wedge\emptyset$ exists if and only if a unique maximal element $\hat{1}$ exists, in which case they coincide. The notation, $x \vee y = a$ means "$x$ and $y$ have a join, which is $a$," and similarly for other statements about joins and meets. A poset is called a *lattice* if every finite set has a join and a meet.

An element $a$ of a poset $P$ is *join-irreducible* if there is no set $X \subseteq P$ with $a \notin X$ and $a = \vee X$. Notice that when $P$ is finitary, we can rephrase this: "$a$ is join irreducible if there is no **finite** set $X \subseteq P$ with $a \notin X$ and $a = \vee X$." If $P$ has a unique minimal element $\hat{0}$, then $\hat{0}$ is $\vee\emptyset$ and thus is not join-irreducible. In a lattice, $a$ is join-irreducible if and only if it covers exactly one element. Such elements are also join-irreducible in non-lattices, but an element $a$ which covers distinct elements $\{x_i\}$ is join-irreducible if $\{x_i\}$ has an upper bound incomparable to $a$. A minimal element of a non-lattice is also join-irreducible, if it is not $\hat{0}$. It is easily checked that if $x \in P$ is not join-irreducible, then $x = \vee D(x)$. The subposet of $P$ induced by the join-irreducible elements is denoted $\mathrm{Irr}(P)$. An element $a$ of a poset $P$ is *meet-irreducible* if there is no set $X \subseteq P$ with $a \notin X$ and $a = \wedge X$. For $x \in P$, let $I_x$ denote $D[x] \cap \mathrm{Irr}(P)$, the set of join-irreducibles weakly below $x$ in $P$. The following proposition is proved in [14] for finite posets. The proof holds for finitary posets without alteration.

**Proposition 5.** *Let $P$ be a finitary poset, and let $x \in P$. Then $x = \vee I_x$.*

A poset is *directed* if for any $x, y \in P$, there is some $z \in P$ with $z \geq x$ and $z \geq y$. An element $x$ in a finitary poset $P$ is called a(n) *(upper) dissector* of $P$ if $P - U[x]$ is directed. Call $x$ a *strong dissector* if $P - U[x] = D[\beta(x)]$ for some $\beta(x) \in P$. In other words, $P$ can be dissected as a disjoint union of the principal order filter generated by $x$ and the principal order ideal generated by $\beta(x)$. A strong dissector is in particular a dissector, and if $P$ is finite the two notions are equivalent. The subposet of dissectors of $P$ is called $\mathrm{Dis}(P)$. In the lattice case the definition of dissector coincides with the notion of a prime element. An element $x$ of a lattice $L$ is called *prime* if whenever $x \leq \vee Y$ for some $Y \subseteq L$, then there exists $y \in Y$ with $x \leq y$. The following easy proposition, proven in [14] for finite posets, holds for finitary posets by the same proof.

**Proposition 6.** *If $x$ is a dissector of $P$ then $x$ is join-irreducible.*

The converse is not true in general, and the reader can find a 5-element lattice to serve as a counterexample. A poset $P$ in which every join-irreducible is a dissector is called a *dissective* poset. In [9] this property of a finite poset is called "clivage."

*Proof of Theorem 4.* Let $C_1, C_2, \ldots, C_w$ be a chain decomposition of $\mathrm{Irr}(P)$. For each $m \in [w]$, and $x \in P$, let $f_m(x) := |I_x \cap C_m|$. By Proposition 5, $x \leq y$ if and only if $I_x \subseteq I_y$ if and only if $f_m(x) \leq f_m(y)$ for all $m \in [w]$. Thus $x \mapsto (f_1(x), f_2(x), \ldots, f_w(x))$ is an embedding of $P$ into $\mathbb{N}^w$.

For the lower bound, consider a finite antichain $A$ in $\mathrm{Dis}(P)$. For each $a \in A$, define $b(a)$ to be an upper bound in $P - U_P[a]$ for the set $A - \{a\}$. By applying (a finite number of times) the property that $a$ is a dissector, we can be assured of the existence of such an element. Define $\beta(A) := \{b(a) : a \in A\}$. The subposet of $P$ induced by $A \cup \beta(A)$ is a "standard example" of size $|A|$. Thus $\dim(P) \geq \dim(A \cup \beta(A)) = |A|$. If the width of $\mathrm{Dis}(P)$ is finite, choose $A$ to be a largest antichain. If the width is countable, then consider a sequence of antichains whose cardinality approaches infinity. □

One half of this proof is exactly as in the finite case [14]. The other half is a straightforward modification of the finite case. Notice that this proof actually constructs an embedding of $P$ into $\mathbb{N}^w$, where $w = \mathrm{width}(\mathrm{Irr}(P))$.

**Corollary 7.** *If $P$ is a finitary dissective poset such that* $\mathrm{width}(\mathrm{Irr}(P))$ *is finite or countable, then* $\dim(P) = \mathrm{width}(\mathrm{Irr}(P))$.

The dissective property is a generalization of the distributive property, in the following sense:

**Proposition 8.** *A finitary lattice $L$ is distributive if and only if it is dissective.*

Proposition 8 is well known [6, 11] in the finite case, and the proof in the finitary case is a straightforward generalization.

The Bruhat order on the finite Coxeter groups of types A, B and H is known to be dissective [14]. The Bruhat order on $\tilde{A}_1$ is easily seen to be dissective and the Bruhat order on $\tilde{A}_2$ is dissective if the conjectured characterization of join-irreducibles holds. This is reflected in the fact that the upper and lower bounds of Theorem 1 and Conjecture 2 agree for $n = 1$ and $n = 2$. For $n > 2$, the Bruhat order on $\tilde{A}_n$ is not dissective.

## 3. Order-Quotients

In this section, we define poset congruences and order quotients and relate them to join-irreducibles and dissectors. The results in this section are generalizations to the finite case of results from [14]. For more information on poset congruences and order quotients see [3, 12, 14].

**Definition 9.** *Let $P$ be a finitary poset with an equivalence relation $\Theta$ defined on the elements of $P$. Given $a \in P$, let $[a]_\Theta$ denote the equivalence class of $a$ under $\Theta$. The equivalence relation $\Theta$ is a* congruence *if:*

(a) *Every equivalence class has a unique minimal element.*
(b) *The projection $\pi_\downarrow : P \to P$, mapping each element $a$ of $P$ to the minimal element in $[a]_\Theta$, is order-preserving.*
(c) *Whenever $\pi_\downarrow(a) \le b$ there exists a $t \in [b]_\Theta$ such that $a \le t$ and $b \le t$.*

Chajda and Snášel [3] give a more general version of the above definition and show that their definition is equivalent to lattice congruence in case $P$ is a lattice.

There is a natural partial order on the congruence classes, with $[a]_\Theta \le [b]_\Theta$ if and only if there exists $x \in [a]_\Theta$ and $y \in [b]_\Theta$ such that $x \le_P y$. The set of equivalence classes under this partial order is $P/\Theta$, the *quotient* of $P$ with respect to $\Theta$. When $P$ is finitary, is convenient to identify $P/\Theta$ with the induced subposet $Q := \pi_\downarrow(P)$, as is typically done for example with quotients of Bruhat order. Such a subposet $Q$ is called an *order-quotient* of $P$.

For a congruence $\Theta$, there is a simple connection between $\mathrm{Dis}(P/\Theta)$ and $\mathrm{Dis}(P)$, as well as a similar connection between $\mathrm{Irr}(P/\Theta)$ and $\mathrm{Irr}(P)$. The following two propositions were proven in [14] in the finite case.

**Proposition 10.** *Suppose $Q$ is an order-quotient of a finitary poset $P$. If $x = \vee^Q Y$ for some $Y \subseteq Q$, then $x = \vee^P Y$. If $x = \vee^P Y$ for some $Y \subseteq P$, then $\pi_\downarrow(x) = \vee^Q \pi_\downarrow(Y)$.*

*Proof.* Suppose $x = \vee^Q Y$ for $Y \subseteq Q$ and suppose $z \in P$ has $z \ge y$ for every $y \in Y$. Then $\pi_\downarrow(z) \ge \pi_\downarrow(y) = y$ for every $y \in Y$. Therefore $z \ge \pi_\downarrow(z) \ge x$. Thus $x = \vee^P Y$.

Suppose $x = \vee^P Y$ for $Y \subseteq P$, and suppose that for some $z \in Q$, $z \ge \pi_\downarrow(y)$ for every $y \in Y$. By condition (c) in Definition 9, for each $y \in Y$, there exists a $z_y \in [z]_\Theta$ such that $z_y \geq z$ and $t_y \geq y$. Since each $z_y$ has $z_y \geq \pi_\downarrow(z) = z$, by iterating condition (c), we obtain an element $z'$, congruent to $z$, which is an upper bound for the set $\{z_y : y \in Y\}$. (Since $P$ is finitary, $Y$ must be a finite set). Then $z' \geq y$ for every $y \in Y$, and so $z' \geq x$. Thus also $\pi_\downarrow(z') \geq \pi_\downarrow(x)$, but $\pi_\downarrow(z') = z$, and so $\pi_\downarrow(x) = \vee^Q \pi_\downarrow(Y)$. $\square$

**Proposition 11.** *Suppose $Q$ is an order-quotient of a finitary poset $P$ and let $x \in Q$. Then $x$ is join-irreducible in $Q$ if and only if it is join-irreducible in $P$, and $x$ is a dissector of $Q$ if and only if it is a dissector of $P$. In other words,*

$$\mathrm{Irr}(Q) = \mathrm{Irr}(P) \cap Q \text{ and,} \tag{1}$$

$$\mathrm{Dis}(Q) = \mathrm{Dis}(P) \cap Q. \tag{2}$$

*Proof.* Suppose $x \in Q$ is join-irreducible in $Q$, and suppose $x = \vee^P Y$ for some $Y \subseteq P$. Then by Proposition 10, $x = \pi_\downarrow(x) = \vee^Q \pi_\downarrow(Y)$. Since $x$ is join-irreducible in $Q$, we have $x \in \pi_\downarrow(Y)$, and thus there exists $x' \in Y$ with $\pi_\downarrow(x') = x$ and in particular $x \leq x'$. But since $x = \vee^P Y$, we have $x' \leq x$ and so $x = x' \in Y$. Conversely, suppose $x \in Q$ is join-irreducible in $P$, and suppose $x = \vee^Q Y$ for some $Y \subseteq Q$. Then by Lemma 10, $x = \vee^P Y$, so $x \in Y$. Thus $x$ is join-irreducible in $Q$.

Suppose $x \in Q$ is a dissector of $Q$, and let $y, z \in P - U_P[x]$. We need to find an upper bound in $P - U_P[x]$ for $y$ and $z$. Since $y \not\geq x$, $\pi_\downarrow(y) \not\geq x$, and similarly $\pi_\downarrow(z) \not\geq x$. Because $x$ is a dissector in $Q$, there is some $b \in Q - U_Q[x]$ with $b \geq \pi_\downarrow(y)$ and $b \geq \pi_\downarrow(z)$. By condition (c), there is an element $b'$, congruent to $b$ with $b' \geq y$ and $b' \geq b$. Again, by condition (c), there is an element $b''$ congruent to $b'$ with $b'' \geq z$ and $b'' \geq b'$. Thus $b''$ is an upper bound for $y$ and $z$, and since $b''$ is congruent to $b$, it is not in $U_P[x]$: if we did have $b'' \geq x$, we would have $b = \pi_\downarrow(b'') \geq \pi_\downarrow(x) = x$.

Conversely, suppose $x \in Q$ is a dissector of $P$, and let $y, z \in Q - U_Q[x]$. Thus also $y, z \in P - U_P[x]$, so there is some $b \in P - U_P[x]$ such that $b \geq y$ and $b \geq z$. Then $\pi_\downarrow(b) \geq \pi_\downarrow(y) = y$ and $\pi_\downarrow(b) \geq \pi_\downarrow(z) = z$. Since $\pi_\downarrow(b) \leq b$ and $b \not\geq x$, necessarily $\pi_\downarrow(b) \not\geq x$. We have found an upper bound $\pi_\downarrow(b)$ for $y$ and $z$ in $U_Q[x]$. Thus $x$ is a dissector in $P$. $\square$

## 4. Bruhat Order on a Coxeter Group

In this section we present background information on Coxeter groups and on the Bruhat order. For more details, and for proofs of results quoted here, see [8].

A *Coxeter group* is a group $W$ given by generators $S$, and relations $s^2 = 1$ for all $s \in S$ and the *Coxeter relations* $(st)^{m(s,t)} = 1$ for all $s \neq t \in S$. Each $m(s,t)$ is an integer $> 1$, or is $\infty$ (in which case, no relation of the form $(st)^m = 1$ is imposed). Important examples of Coxeter groups include the finite and affine Weyl groups. In this paper, we consider the affine Weyl group $\tilde{A}_{n-1}$ with $S = \{s_1, s_2, \ldots, s_n\}$, $m(s_1, s_n) = 3$, $m(s_i, s_{i+1}) = 3$ for $i \in [n-1]$ and $m = 2$ otherwise. To simplify notation, subscripts will be interpreted mod $n$, so that for example, $s_{n+1} = s_1$.

Each element of a Coxeter group $W$ can be written (in many different ways) as a word with letters in $S$. A word $a$ for an element $w$ is called *reduced* if the length (number of letters) of $a$ is minimal among words representing $w$. The length of a reduced word for $w$ is called the *length* $l(w)$ of $w$.

The Bruhat order on a Coxeter group can be defined in several ways. One way is by the *Subword Property.* Given $u, w \in W$, say that $u \leq w$ if some reduced word for $w$ contains as a subword some reduced word for $u$ (in which case any reduced word for $w$ contains a reduced word for $u$). It is immediate that Bruhat order is a finitary poset.

The following proposition follows immediately from the subword characterization of Bruhat order:

**Proposition 12.** *Suppose $u \leq x$, $v \leq y$, $l(xy) = l(x) + l(y)$ and $l(uv) = l(u) + l(v)$. Then $uv \leq xy$.* □

The "Lifting Property" of Bruhat order is also easily proven using the Subword Property.

**Proposition 13.** *If $u, w \in W$ and $s \in S$ have $w > ws$ and $u > us$, then the following are equivalent:*

(i) $w \geq u$
(ii) $w \geq us$
(iii) $ws \geq us$

An equivalent definition of Bruhat order is as follows: A *reflection* is any element conjugate to a Coxeter generator. For any reflection $t$ and any element $u$, if $l(u) < l(ut)$ then $u \leq ut$ and Bruhat order is the transitive closure of such relations.

When $J$ is any subset of $S$, the subgroup of $W$ generated by $J$ is another Coxeter group, called the *parabolic subgroup* $W_J$. Since we will often need $J$ to be $S - \{s\}$ for some $s \in S$, we define $\langle s \rangle := S - \{s\}$. It is known that for any $w \in W$ and $J, K \subseteq S$, the double coset $W_J w W_K$ has a unique Bruhat minimal element ${}^J w^K$, and $w$ can be factored (non-uniquely) as $w_J \cdot {}^J w^K \cdot w_K$, where $w_J \in W_J$ and $w_K \in W_K$, such that $l(w) = l(w_J) + l({}^J w^K) + l(w_K)$. The subset ${}^J W^K$ consisting of the minimal coset representatives is called a *two-sided quotient* of $W$.

The more widely used one-sided quotients can be obtained by letting $J = \emptyset$ or $K = \emptyset$, in which case we will write the quotient as $W^K$ or ${}^J W$. In the case of one-sided quotients, the factorization $w = w^K w_K$ is unique, and furthermore, if $x \in W^K$ and $y \in W_K$ then $l(xy) = l(x) + l(y)$. The analogous fact for two-sided quotients does not hold.

**Proposition 14.** *The quotient ${}^J W^K$ is an order-quotient of $W$.*

*Proof.* We must verify the conditions of Definition 9. As mentioned above, condition (a) is known. The proof of condition (b) when $W$ is finite can be found in [14, Proposition 31], and the same proof goes through in general. To verify condition (c), let $x, y \in W$ have ${}^J x^K \leq y$ and make a particular choice of $x_J$, $x_K$, $y_J$ and $y_K$ as follows: Write $x = x_J {}^J x$ so that $x_J \in W_J$, ${}^J x \in {}^J W$ and $l(x) = l(x_J) + l({}^J x)$. Write ${}^J x = ({}^J x)^K ({}^J x)_K$ so that $({}^J x)^K \in W^K$, $({}^J x)_K \in W_K$ and $l({}^J x) = l(({}^J x)^K) + l(({}^J x)_K)$. It is also known that $({}^J x)^K = {}^J x^K$, so we will write $x = x_J {}^J x = x_J {}^J x^K x_K$. Similarly, write $y = y_J {}^J y = y_J {}^J y^K y_K$.

Bruhat order is directed, so choose $z_K$ to be some bound for $x_K$ and $y_K$ in $W_K$. Let $z := {}^J y^K z_K$. Because ${}^J y^K \in {}^J W^K$ and $z_K \in W_K$, we have $l(z) = l({}^J y^K) + l(z_K)$, so by Proposition 12, $z \geq {}^J x^K x_K = {}^J x$ and $z \geq {}^J y^K y_K = {}^J y$. Write $z = z_J {}^J z$ so that ${}^J z \in {}^J W$ and $z_J \in W_J$ and $l(z) = l(z_J) + l({}^J z)$. By condition (b), ${}^J z \geq {}^J x$ and ${}^J z \geq {}^J y$. Choose $v_J$ to be some upper bound for $x_J$ and $y_J$ in $W_J$ and let $v := v_J {}^J z$. As before, by Proposition 12, $v \geq x_J {}^J = x$ and $v \geq y_J {}^J y = y$. It remains to show that ${}^J v^K = {}^J y^K$. Since $v = v_J {}^J z = v_J (z_J)^{-1} z = v_J (z_J)^{-1} {}^J y^K z_K$, we have $v \in W_J {}^J y K W_K$, so by uniqueness of minimal coset representatives, ${}^J v^K = {}^J y^K$. □

Proposition 14 appears in [14] under the assumption that $W$ is finite.

In [7, 9] it is shown that join-irreducibles in the Bruhat order are always *bi-grassmannians*. That is, any join-irreducible $x$ in $W$ is contained in ${}^{\langle s \rangle} W^{\langle t \rangle}$ for some (necessarily unique) choice of $s, t \in S$. Equivalently, there is a unique $s \in S$ such that $sx < x$ and a unique $t \in S$ such that $xt < x$. Thus Proposition 11 can be used to simplify the task of finding join-irreducibles and dissectors in $W$.

**Proposition 15.** *For a finitary Coxeter group $W$ under the Bruhat order:*

(i) $\mathrm{Irr}(W) = \cup_{s,t \in S} \mathrm{Irr}({}^{\langle s \rangle} W^{\langle t \rangle})$ *and*
(ii) $\mathrm{Dis}(W) = \cup_{s,t \in S} \mathrm{Dis}({}^{\langle s \rangle} W^{\langle t \rangle})$.

Assertion (i) is due to Geck and Kim [7] in the finite case.

The following proposition is useful in finding dissectors in Bruhat order on infinite Coxeter groups.

**Proposition 16.** *If $x \in W^{\langle s \rangle}$ and $x \neq 1$, then*

$$W - U[x] = \bigcup_{y \in W - U(xs)} y W_{\langle s \rangle}.$$

Note the use of square brackets in $U[x]$ and round brackets in $U(xs)$.

*Proof.* Suppose for the sake of contradiction that there exists an element $z$ of the right hand side with $z \geq x$, and choose $z$ to be of minimal length among such elements. Thus $z$ is in one of the cosets on the right hand side, so let $y$ be the minimal coset representative, and write $z = yw$ for some $w \in W_{\langle s \rangle}$. If $w = 1$ then $y = z$, so $y \geq x$, contradicting the fact that $y \not\geq xs$. If $w \neq 1$ then choose $t$ such that $wt < w$. We have $t \neq s$ and since $x \in W^{\langle s \rangle}$, we have $z > zt$ and $xt > x$, so by lifting $zt \geq x$. Since $zt \in y W_{\langle s \rangle}$, this is a contradiction of the minimality of $z$.

Conversely, suppose $z$ is not an element of the right hand side. In other words, when we write $z = z^{\langle s \rangle} z_{\langle s \rangle}$ as in Proposition 14, we have $z^{\langle s \rangle} > xs$. Since $x > xs$

and $z^{\langle s\rangle} > z^{\langle s\rangle}s$, by lifting $z^{\langle s\rangle} \geq x$, and therefore $z \geq x$, or in other words, $z$ is not an element of the left hand side. ☐

**Proposition 17.** *For a Coxeter group $W$, the following are equivalent:*

(i) *$W_J$ is finite for all $J \subsetneq S$.*
(ii) *For all $x \in W$ the set $W - U[x]$ is finite.*

*Proof.* For any $J \subsetneq S$ and $s \in (S - J)$, we have $W_J \subseteq W - U[x]$, and therefore (ii) implies (i). Conversely, suppose $W_J$ is finite for all $J \subsetneq S$, let $x \in W$ and proceed by induction on $l(x)$. The case $l(x) = 0$ is trivial so suppose $l(x) \geq 1$. If $x$ is not join-irreducible, then $x = \vee D(x)$, so $U[x] = \bigcap_{a \in D(x)} U[a]$. Thus $W - U[x] = \bigcup_{a \in D(x)}(W - U[a])$ and each term in this finite union is finite by induction. If $x$ is join-irreducible, then in particular by Proposition 15, $x \in W^{<s>}$ for some $s$. Now Proposition 16 writes $W - D[x]$ as a union of sets each of which is finite by induction. Also by induction, the union is over a finite number of terms. ☐

The affine Coxeter groups and compact hyperbolic Coxeter groups satisfy the conditions of Proposition 17 (see [8] for definitions). Note that if $W$ satisfies the conditions of Proposition 17, $x \in W$ is a dissector if and only if it is a strong dissector.

A nontrivial element $x \in W$ is called *rigid* if it admits exactly one reduced word. The following can be proved by induction using Proposition 16.

**Proposition 18.** *If $x$ is rigid then it is a dissector.*

*Proof.* By induction on $l(x)$. If $l(x) = 1$, then $x = s$ for some $s \in S$ and $W - U[x] = W_{<s>}$, which is directed. If $l(x) > 1$, then let $s$ be the last letter of $x$. Then $xs$ is rigid, so by induction $W - U[xs]$ is directed. By Proposition 16, $W - U[x] = \bigcup_{y \in W-U(xs)} yW_{<s>}$. Let $u$ and $v$ be elements of $\bigcup_{y \in W-U(xs)} yW_{<s>}$. Since $W - U[xs]$ is directed, there is an element $w \in W - U[xs]$ with $w \geq u$ and $w \geq v$. So also $w^{<s>} \geq u^{<s>}$ and $w^{<s>} \geq v^{<s>}$. Since $W_{<}s>$ is directed, there is an element $z \in W_{<s>}$ with $z \geq u_{<s>}$ and $z \geq v_{<s>}$. Thus by Proposition 12, $w^{<s>}z$ is an upper bound for $u$ and $v$ in $\bigcup_{y \in W-U(xs)} yW_{<s>}$. ☐

Proposition 18 allows us to give examples of Coxeter groups with infinite order dimension. The *universal Coxeter group* $U_n$ with generators $S = \{s_1, s_2, \ldots s_n\}$ has $m(s,t) = \infty$ for each $s, t \in S$. Every nontrivial element of $U_n$ is rigid, thus $\mathrm{Dis}(U_n) = U_n - \{1\}$, and the order dimension of $U_n$ is equal to its width, which is easily seen to be infinite. A less extreme example is the Coxeter group $W$ with Coxeter generators $S = \{s_1, s_2, s_3, s_4\}$ with $m(s_1, s_2) = 2$ and $m(s,t) = 3$ for all other $s, t \in S$. Any path in the Coxeter graph is a rigid element and it is easily verified that there is no upper bound on the size of a set of paths all having the same length. An even smaller example is the Coxeter group on three elements with $m(s_1, s_2) = 4$ and $m(s_1, s_3) = m(s_2, s_3) = 3$. The last two examples are hyperbolic Coxeter groups. The authors are not aware of any affine Coxeter group whose Bruhat order has infinite dimension.

## 5. Affine Permutations

In this section we give a combinatorial description of the affine Coxeter group $\tilde{A}_{n-1}$, which is due to Lusztig [10], and a criterion due to Björner and Brenti, for making Bruhat comparisons. A similar criterion was given by H. Eriksson in [5].

Let $\tilde{S}_n$ be the set of *affine permutations*, that is, permutations $x$ of $\mathbb{Z}$ with the following properties:

$$x(i+n) = x(i) + n, \tag{3}$$

for all $i \in \mathbb{Z}$, and

$$\sum_{i=1}^{n} x(i) = \binom{n+1}{2}. \tag{4}$$

An affine permutation is uniquely identified by its *window*, consisting of the values $x(1), x(2), \ldots, x(n)$, and we will refer to specific affine permutations by writing the window values in square brackets, separated by commas. The set $tS_n$ forms a group under composition, and it generated by $S = \{s_1, s_2, \ldots, s_n\}$, with

$$s_j = [1, 2, \ldots, j-1, j+1, j, j+2, \ldots, n]$$

for $j \in [n-1]$ and

$$s_n = [0, 2, 3, \ldots, n-1, n+1].$$

Putting $s_{n+1} = s_1$, we have $m(s_j, s_{j+1}) = 3$ for all $j \in [n]$, and all the other pairwise orders are 2. There are no other relations in the affine permutation group $\tilde{S}_n$, so $\tilde{S}_n$ is isomorphic to the Coxeter group $\tilde{A}_{n-1}$.

The length $l(x)$ for $x \in \tilde{S}_n$ is

$$|\{(i,j) \in [n] \times \mathbb{Z} : i < j, x(i) > x(j)\}|.$$

The reflections in $\tilde{S}_n$ are infinite products of transpositions

$$t_{i,j} := \prod_{r \in \mathbb{Z}} (i + rn, j + rn)$$

for $i, j \in \mathbb{Z}$ and $i \not\equiv j (\mathrm{mod}\ n)$. Thus if $t_{i,j}$ is a reflection with $i < j$ and $x \in \tilde{S}_n$ has $x(i) < x(j)$, then $x \leq xt_{i,j}$ in Bruhat order. All other Bruhat relations are obtained by transitivity.

Björner and Brenti [2] gave a criterion for making Bruhat comparisons on $\tilde{S}_n$, similar to the Tableau Criterion on certain finite groups. For $x \in \tilde{S}_n$ and $i, j \in \mathbb{Z}$, define

$$x[i,j] := \#\{k \leq i : x(k) \geq j\}.$$

Then $u \leq v$ in Bruhat order if and only if $u[i,j] \leq v[i,j]$ for all $i, j \in \mathbb{Z}$. They also show that it is enough to check $i \in [n]$ and that for each $u$ and $v$, there is only a finite number of values of $j$ which must be checked. To make this criterion resemble more closely the tableau criterion for finite type A, we define an infinite tableau $T_{a,b}(u)$ as follows. For each $a \in \mathbb{Z}$ and $b \leq a$, let $T_{a,b}(u)$ be the entry at position $b$ in the increasing rearrangement of the set $\{u(i) : i \leq a\}$. That is, rearrange the set and place the rearranged values so that they occupy the integer positions of $(-\infty, a]$. The easy proof of the following proposition is omitted.

**Proposition 19.** *Let $u, v \in \tilde{S}_n$. Then $u[i,j] \leq v[i,j]$ for all $i, j \in \mathbb{Z}$ if and only if $T_{a,b}(u) \leq T_{a,b}(v)$ for all $a \in \mathbb{Z}$ and $b \leq a$.*

We now make note of some properties of the infinite tableau $T_{a,b}(u)$. Properties (i) to (iv) below follow immediately from the definitions of $\tilde{S}_n$ and $T_{a,b}(u)$. Property (v) follows from the fact that the identity permutation is minimal in $\tilde{S}_n$. We give proofs of Properties (vi) and (vii).

**Proposition 20.** *Let $u \in \tilde{S}_n$, $a \in \mathbb{Z}$ and $b \in (-\infty, a]$ and write $T_{a,b}$ for $T_{a,b}(u)$. Then*

(i) $T_{a,b-1} < T_{a,b}$.
(ii) $T_{a+1,b} \leq T_{a,b} \leq T_{a+1,b+1}$.
(iii) $T_{a+n,b+n} = T_{a,b} + n$.
(iv) *If $j$ occurs as an entry in row $a$ of $T_{a,b}$ then $j - n$ also occurs in row $a$.*
(v) $T_{a,b}(x) \geq b$.
(vi) *If $T_{a,b} = T_{a,b-n} + n$ then $T_{a,b} = b$.*
(vii) *For each fixed $a$ there is a $B$ such that $T_{a,b} = b$ for every $b \leq B$.*

*Proof.* To prove Property (vi), suppose $T_{a,b} = T_{a,b-n} + n$. Then by (iii), $T_{a-n,b-n} = T_{a,b-n}$. Therefore elements in the set $\{u(i) : i \leq a\} - \{u(i) : i \leq a-n\}$ all occur to the right of column $b$ in $T$. Thus

$$\begin{aligned} \sum_{i=a-n+1}^{a} u(i) &= \sum_{j=b}^{a} T_{a,j} - \sum_{j=b}^{a-n} T_{a-n,j} \\ &= \sum_{j=b}^{a} T_{a,j} - \sum_{j=b}^{a-n} (T_{a,j+n} - n) \\ &= n(a-n-b+1) + \sum_{j=b}^{b+n-1} T_{a,j} \\ &= n(a-n-b+1) + n(T_{a,b} - 1) + \binom{n+1}{2} \end{aligned}$$

On the other hand, combining Equations (3) and (4) says that

$$\sum_{i=a-n+1}^{a} u(i) = n(a-n) + \binom{n+1}{2}.$$

Thus $T_{a,b} = b$. Property (vi) implies in particular that $T_{a,b} - b$ decreases as $b$ decreases, unless $T_{a,b} = b$. Therefore we have (vii). $\square$

A function $T : \{a, b \in \mathbb{Z} : b \leq a\} \to \mathbb{Z}$ will be called an *infinite tableau*. An infinite tableau $T_{a,b}$ satisfying the conditions of Proposition 20 will be called an *affine monotone triangle*. In this paper we will represent an affine monotone triangle $T_{a,b}$ as an array of $n$ rows corresponding to $a \in [n]$, with a vertical line at the left of the array to indicate that all entries to the left of the line have $T_{a,b} = b$. Entries with $T_{a,b} = b$ will be called *trivial*. So, for example, when $n = 3$, the permutation $[3, -2, 5]$ has

$$T([3,-2,5]) = \left|\begin{array}{ccccccc} -3 & -1 & 0 & 2 & 3 & & \\ -3 & -2 & -1 & 0 & 2 & 3 & \\ -3 & -2 & -1 & 0 & 2 & 3 & 5 \end{array}\right. .$$

## 6. Dissectors in $\tilde{A}_{n-1}$

In this section we prove the lower bound in Theorem 1, by exhibiting an antichain of dissectors. We will need two ways of describing the dissectors. The first description helps in the proof that they indeed are dissectors, while the other description is useful in determining the order relations among them.

For $a \in [n]$, and $b, c \in \mathbb{Z}$ with $b \leq a$ and $b \leq c$, define $J_{a,b,c}$ to be the unique Bruhat minimal element in the set $\{x \in \tilde{S}_n : T_{a,b}(x) \geq c\}$. Notice that if $b = c$ then $J_{a,b,c}$ is the identity. There are choices of $a$, $b$ and $c$ for which $J_{a,b,c}$ does not exist. For example, in $\tilde{S}_3$, the affine permutations $[3, -2, 5]$ and $[5, 0, 1]$ are both minimal among permutations with $T_{1,0} \geq 2$. Similarly, define $M_{a,b,c}$ to be the unique Bruhat maximal element in the set $\{x \in \tilde{S}_n : T_{a,b}(x) < c\}$, if such an element exists. It is apparent that if $J_{a,b,c}$ and $M_{a,b,c}$ both exist for a triple $(a, b, c)$, then $J_{a,b,c}$ is a dissector with $\beta(J_{a,b,c}) = M_{a,b,c}$. A similar approach to finding dissectors in certain finite Coxeter groups was taken in [9] and [14].

The second description of dissectors is as left-justified rectangles in the array:

$$\begin{array}{ccccc} s_1 & s_0 & s_{-1} & \cdots & s_{-n+3} \\ s_2 & s_1 & s_0 & \cdots & s_{-n+4} \\ s_3 & s_2 & s_1 & \cdots & s_{-n+5} \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \end{array}, \tag{5}$$

with infinitely many rows of length $n-1$, where the $i$ in $s_i$ is to be interpreted mod $n$. Rectangles are interpreted as elements of $\tilde{A}_{n-1}$ by reading the characters in the usual direction for reading written English. So for example, the rectangle

$$\begin{array}{cccc} s_1 & s_0 & s_{-1} & s_{-2} \\ s_2 & s_1 & s_0 & s_{-1} \\ s_3 & s_2 & s_1 & s_0 \end{array}$$

stands for the word $s_1 s_n s_{n-1} s_{n-2} s_2 s_1 s_n s_{n-1} s_3 s_2 s_1 s_n$. The rectangle which is $i$ columns wide and $k$ rows long, and whose top left corner is $s_j$ will be referred to as $R_{i,j,k}$.

We will make use of the following lemmas:

**Lemma 21.** *If $i + k \leq n$ then $J_{j+k-i,j-i+1,j+1}$ exists and is equal to $R_{i,j,k}$.*

**Lemma 22.** *$M_{a,b,c}$ exists whenever $b \leq a$ and $b < c$.*

**Lemma 23.** *The set $\{R_{i,j,k} : i + k = n\}$ is an antichain in Bruhat order.*

In light of Lemma 22, whenever $J_{a,b,c}$ exists it is a dissector, so in particular the rectangles $R_{i,j,k}$ are dissectors. Thus Lemma 23 exhibits an antichain in $\mathrm{Dis}(\tilde{A}_{n-1})$ with $n(n-1)$ elements, thus by Theorem 4, the order dimension of the Bruhat order on $\tilde{A}_{n-1}$ is at least $n(n-1)$. This is the lower bound in Theorem 1.

Note that when $i = 1$ the element $R_{i,j,k}$ is rigid for any $j$ and $k$, and therefore is a dissector by Proposition 18. These rectangles are called *cyclic words* because they correspond to cyclic paths in the Coxeter graph for $\tilde{A}_{n-1}$. There are also cyclic words in the opposite direction. The cyclic words and the rectangles with $i + k \leq n$ appear to be the only dissectors in $\tilde{S}_n$.

*Proof of Lemma 21.* Because of the cyclic symmetry of $\tilde{A}_{n-1}$, for each fixed $i$ and $k$, checking Lemma 21 for one particular $j$ is enough. Specifically, the map $\gamma$ which sends $s_i$ to $s_{i+1}$ for each $i \in [n]$ everywhere in $\tilde{A}_{n-1}$ corresponds to moving the window one position to the left and then adding one to each entry in the window. The corresponding map on tableaux is $\gamma(T)_{a,b} = T_{a-1,b-1} + 1$. Thus if $J_{a,b,c}$ exists, $\gamma(J_{a,b,c}) = J_{a+1,b+1,c+1}$. On the other hand $\gamma(R_{i,j,k}) = R_{i,j+1,k}$.

For convenience, we will consider the case when $j = i - k + n$. In effect this fixes the bottom-right element of the rectangle to be $s_n$. To simplify the notation for $R_{i,i-k+n,k}$ and its associated tableau, the notation $x$——$y$ will stand for a row of integers increasing by ones from $x$ to $y$, so that for example 2——6 is 2 3 4 5 6. It is easily seen by induction on $k$ that for $i + k \leq n$, the rectangle $R_{i,i-k+n,k}$ is the affine permutation whose window is

$$(1-k)\text{——}(i-k)(i+1)\text{——}(n-k)(n-k+i+1)\text{——}(n+i)$$

$$R_{i,i-k+n,k}(m) = \begin{cases} m-k & \text{if } m \leq i \\ m & \text{if } i+1 \leq m \leq n-k \\ m+i & \text{if } m \geq n-k+1. \end{cases}.$$

So, for example for $n = 7$,

$$R_{2,6,3} = [-2, -1, 3, 4, 7, 8, 9].$$

We now show that $T(R_{i,i-k+n,k})$ is minimal under componentwise comparison among all affine monotone triangles whose $(n, n-k+1)$ entry is at least $n+i-k+1$.

It is awkward to represent $T(R_{i,i-k+n,k})$ in its full generality, but a continuation of our example should illustrate our argument. When $n = 7$,

$$T(R_{2,6,3}) = \left|\begin{array}{cccccccccc}
-2 & & 0 & 1 & 2 & & & & & \\
-2 & -1 & 0 & 1 & 2 & & & & & \\
-2 & -1 & 0 & 1 & 2 & 3 & & & & \\
-2 & -1 & 0 & 1 & 2 & 3 & 4 & & & \\
-2 & -1 & 0 & 1 & 2 & 3 & 4 & 7 & & \\
-2 & -1 & 0 & 1 & 2 & 3 & 4 & 7 & 8 & \\
-2 & -1 & 0 & 1 & 2 & 3 & 4 & 7 & 8 & 9
\end{array}\right.$$

The row $a = n$ of $T(R_{i,i-k+n,k})$ is

$$|(1-k)\text{———}(n-k)\ \ (n-k+i+1)\text{———}(n+i).$$

By properties (i) and (v), each of these entries is minimal among affine monotone triangles whose $(n, n-k+1)$ entry is at least $n+i-k+1$. The rows $a = n-1$ down to $a = i$ are obtained from the row $a = n$ by repeatedly deleting the largest element of the row, so the entries in these rows are minimal by Property (ii). Now, notice that $T_{0,b}$ for $b \in [-k+1, 0]$ is $(-k+i+1)\text{———}i$, which agrees with $T_{i,b}$ for $b \in [-k+i+1, i]$. Thus by Property (ii), minimal entries in row $a = i-1$ can be obtained by deleting the entry at $(i, -k+i)$ from row $a = i$. For each row $a$ with $1 \le a \le i$, the entry at $(a, -k+a)$ is deleted from row $a$ to obtain row $a-1$. Since these are the largest entries that can be deleted without violating Property (ii), the entries in this row are minimal, and we have completed the proof that $T(R_{i,i-k+n,k}) = J_{n,n-k-1,n+i-k+1}$.

□

*Proof of Lemma 22.* By cyclic symmetry, we need only consider the case where $a = n$. We claim that $M_{n,b,c}$ is the affine permutation whose window is

$$(n^2 - nb + c - 1)\ (c-2)\text{———}(c-n+1)\ (c+bn-cn).$$

So, for $n = 7$, we have

$$M_{7,5,7} = [20, 5, 4, 3, 2, 1, -7],$$

and the associated affine monotone triangle is

$$\left|\begin{array}{ccccccccccccccc}
-6 & -5 & -4 & -3 & -2 & -1 & 6 & 13 & 20 & & & & & & \\
-6 & -5 & -4 & -3 & -2 & -1 & 5 & 6 & 13 & 20 & & & & & \\
-6 & -5 & -4 & -3 & -2 & -1 & 4 & 5 & 6 & 13 & 20 & & & & \\
-6 & -5 & -4 & -3 & -2 & -1 & 3 & 4 & 5 & 6 & 13 & 20 & & & \\
-6 & -5 & -4 & -3 & -2 & -1 & 2 & 3 & 4 & 5 & 6 & 13 & 20 & & \\
-6 & -5 & -4 & -3 & -2 & -1 & 1 & 2 & 3 & 4 & 5 & 6 & 13 & 20 & \\
-7 & -6 & -5 & -4 & -3 & -2 & -1 & 1 & 2 & 3 & 4 & 5 & 6 & 13 & 20
\end{array}\right.$$

We now show that the affine monotone triangle $T(M_{n,b,c})$ really is maximal among affine monotone triangles whose $(n, b)$ entry is at most $c - 1$. Let $L := c - cn + bn + 1$. Then row $a = n$ of $T(M_{a,b,c})$ is

$$|L\text{——}(L+n-2)(L+n)\text{——}(L+2n-2)\cdots(c-n+1)\text{——}(c-1)(n+c-1)\cdots(n^2-nb+c-1).$$

The entries to the right of position $(n, b)$ are maximal by Property (iv). The entries in positions $(n, b - n + 2)$ to $(n, b - 1)$ are maximal by Property (i). Now suppose that some affine monotone triangle $T$ has $T_{n,b} \le c - 1$. Property (vi) says that $T_{n,b-n} \ne T_{n,b} - n$, and Property (iv) says that the value $T_{n,b} - n$ must occur somewhere in the row. Thus we cannot have $T_{n,b-n+1} = T_{n,b} - n + 1$, or in other words, we must have $T_{n,b-n+1} \le T_{n,b} - n \le c - n - 1$. So the entry in position $(n, b-n+1)$ of $T(M_{a,b,c})$ is maximal. We can continue moving left in the row, using Properties (i), (iv) and (vi) in the same manner to show that all the entries in this row are maximal, until eventually, by Property (vii), the remaining entries in the row are trivial. In particular, $T_{n,L+n-1} \le L + n - 1$, and therefore by Property (iii), $T_{0,L-1} \le L - 1$. So the best we can possibly do to make row $a = n - 1$ from row $a = n$ is to delete the entry $L$ and thus the entries in row $a = n-1$ are also maximal. Applying Property (iii) to the upper bounds we have proven in row $a = n$ gives upper bounds on row $a = 0$ and these upper bounds agree with the upper bounds we have proven on row $a = n - 1$ whenever we are weakly to the right of column $b - n$. Thus by Property (ii), the best we can do is to delete entries in the column $b - n + 1$ to make new rows. Specifically, each remaining row $a$ with $1 \le a \le n-2$ is also obtained from row $a+1$ by deleting the entry in positions $(a+1, b-n+1)$. These entries are $c-n+1, c-n+2, \ldots, c-2$. Thus the remaining rows of $T(M_{a,b,c})$ are also maximal. □

To prove Lemma 23, notice that $R_{i,j,k}$ is a *fully commutative* element. Fully commutative elements [16] are elements $w$ such that any two reduced words for $w$ are related by commuting generators. In $R_{i,j,k}$, between any two occurrences of a generator $s$, there occur two distinct generators $t$ and $t'$ with $m(s, t) = m(s, t') = 3$. This is enough to insure that the rectangles $R_{i,j,k}$ are words for fully commutative elements [1]. The following proposition is immediate from the definition of full commutativity.

**Proposition 24.** *Let $w$ be a fully commutative element of $W$, let $s_1 s_2 \cdots s_k$ be a reduced word for $w$, and let $s_{i_1} s_{i_2} \cdots s_{i_j}$ be a subword such that for every $m \in [j - 1]$, the generators $s_{i_m}$ and $s_{i_{m+1}}$ do not commute. Then $s_{i_1} s_{i_2} \cdots s_{i_j}$ occurs as a subword of every reduced word for $w$.*

A subrectangle of $R_{i,j,k}$ is a rectangle that can be obtained by deleting columns from the left and/or right of $R_{i,j,k}$ and/or deleting rows from the top and/or bottom of $R_{i,j,k}$. The following proposition is an affine version of [14, Proposition 38].

**Proposition 25.** *If $i' + k' \le n$ then $R_{i,j,k} \le R_{i',j',k'}$ if and only if $R_{i,j,k}$ is a sub-rectangle of $R_{i',j',k'}$.*

*Proof.* The "if" direction follows immediately from the subword property.

Suppose $R_{i,j,k} \leq R_{i',j',k'}$, and let $a$ be the word obtained from $R_{i',j',k'}$ by reading across rows as described above. The subword property requires that some reduced word for $R_{i,j,k}$ be a subword of $a$. But $R_{i,j,k}$ has a subword

$$s_j s_{j-1} \cdots s_{j-i+2} s_{j-i+1} s_{j-i+2} \cdots s_{j-i_k-1} s_{j-i+k}$$

which satisfies the hypotheses of Proposition 24. Therefore, the subword of $a$ which is a reduced word for $R_{i,j,k}$ must itself contain the same subword. For $a$ to contain the letters $s_j s_{j-1} \cdots s_{j-i+2} s_{j-i+1}$ in that order, in particular, it must contain the letter $s_{j-1}$ somewhere after an occurrence of $s_j$. Thus, because $a$ comes from a rectangle, there is either an occurrence of $s_{j-1}$ immediately to the right of some occurrence of $s_j$, or there is an occurrence of $s_{j-1}$ in the position $x$ columns left and $n-1-x$ rows below some occurrence of $s_j$, for some $x \in [n-2]$. The latter possibility is excluded by the hypothesis that $i' + k' \leq n$. Proceeding in this manner, we find that some row in $R_{i',j',k'}$ contains $s_j s_{j-1} \cdots s_{j-i+2} s_{j-i+1}$. For the letters $s_{j-i+2} s_{j-i+1} s_{j-i+2} \cdots s_{j-i_k-1} s_{j-i+k}$ to occur after that row, in that order, there must be at least $k-1$ more rows. $\square$

Lemma 23 follows from this subrectangle condition. In $\{R_{i,j,k} : i + k = n\}$, subrectangle relations are impossible when the dimensions of the rectangles disagree. Two rectangles of the same dimensions but different top-left entries are also not related by the sub-rectangle order.

**Remark 26.** By Lemma 22, the set of meet-irreducibles is contained in the set

$$\{M_{a,b,c} : a \in [n], b \leq a, b < c\}$$

because any other element $x$ can be written

$$x = \wedge\{M_{a,b,T_{a,b}(x)} : a \in [n], T_{a,b} > b\}.$$

By Property (vii) of Proposition 20, this is the meet of a finite set. One can prove a version of Theorem 4 which bounds the order dimension of a finitary set below the width of the subposet of meet-irreducibles. Thus one might hope to get a bound on $\dim(\tilde{A}_{n-1})$ as the width of the set of $M_{a,b,c}$'s. However, computer tests suggest that the width of the poset of meet-irreducibles is not finite.

## 7. Join-Irreducibles in $\tilde{A}_{n-1}$

In this section we give a conjectured characterization of the join-irreducibles in $\tilde{A}_{n-1}$ and, assuming the characterization, a decomposition of $\mathrm{Irr}(\tilde{A}_{n-1})$ into $n\left\lfloor \frac{n^2}{4} \right\rfloor$ chains, in support of Conjecture 2.

Any element of $\tilde{A}_{n-1}$ can be written as a left-justified subarray of the array (5), where the subarray is interpreted as a word by reading across rows, from top to bottom. The subarray is called reduced if the corresponding word is reduced. If $x$ is bigrassmannian, then any left-justified reduced subarray for $x$ has the same top-left generator. We can lexicographically order the set of left-justified arrays with some fixed top-left generator $s_j$, starting at the top row and moving down, such that a larger row length precedes a smaller row length. The *code* for a bigrassmannian $x$ is the lexicographically first reduced subarray standing for $x$, written as $(j : n_1, n_2, \ldots, n_l)$, meaning that the top-left generator is $s_j$, and the length of the $r^{\text{th}}$ row is $n_r$. Here, when we say the $r^{\text{th}}$ row, we are calling the first non-empty row "row 1." The code has a finite number of non-zero entries, and we assume that $n_l \neq 0$, but $n_r = 0$ for all $r > l$. A similar representation of join-irreducibles by codes was made for finite Coxeter groups in [7, 9], and the results in this section have a similar feel to the results in [7, 9]. The following conjecture is supported by substantial computer evidence.

**Conjecture 27.** *The code for a join-irreducible element of $\tilde{A}_{n-1}$ has the form $(j : i, i, \ldots, i, m, m, \ldots, m)$ such that $i > m$ and the number of $m$'s is either zero or $n - i$.*

Some progress towards proving Conjecture 27 can be made by considering the fact that join-irreducibles in a Coxeter group must be bigrassmannians, but there are bigrassmannians in $\tilde{A}_{n-1}$ which do not satisfy the condition in Conjecture 27.

**Proposition 28.** *Let $(j : n_1, n_2, \ldots, n_l)$ be the code of a bigrassmannian element of $\tilde{A}_{n-1}$. Then for all $r$ with $1 \leq r \leq l-1$ we have $n_r \geq n_{r+1}$. Furthermore, if $n_r > n_{r+1}$ for some $r$ with $1 \leq r \leq l-1$, then $l \geq r + n - n_r$.*

We conclude the section by considering the subposet $\Gamma$ of $\tilde{A}_{n-1}$ consisting of elements whose codes fit the description given in Conjecture 27. Given $u, v \in \Gamma$, consider the subarrays of the array (5) described by the codes of $u$ and $v$. By the subword description of Bruhat order, if the array for $u$ is contained in the array for $v$ then $u \leq v$ in Bruhat order. The converse is not true. We use containment of arrays to give a chain decomposition.

For $0 \leq m < i \leq n-1$, $1 \leq j \leq n$ and $k \geq 1$, let $\Gamma_{i,j,k,m}$ be the element of $\tilde{A}_{n-1}$ whose code is $(j : i, i, \ldots, i, m, m, \ldots, m)$ with the number of $i$'s equal to $k$ and the number of $m$'s equal to $n - i$. We cover the possibility that there are zero $m$'s in the code by allowing $m$ to be zero. The goal is to decompose $\Gamma$ into chains under the partial order of containment of arrays.

Consider the set $\Gamma_{i,j} := \{\Gamma_{i,j,k,m} : 0 \leq m < i, k \geq 1\}$ for each $i$ and $j$. There are two ways to break $\Gamma_{i,j}$ into chains. We can decompose $\Gamma_{i,j}$ into $i$ chains, with one chain for each $m$ with $0 \leq m < i$. The covering relations in these chains are $\Gamma_{i,j,k+1,m} \gtrdot \Gamma_{i,j,k,m}$. Alternately, we can decompose $\Gamma_{i,j}$ into $n - i$ chains. The covering relations in these chains are $\Gamma_{i,j,k,m} \gtrdot \Gamma_{i,j,k,m-1}$ if $m > 1$ and $\Gamma_{i,j,k,0} \gtrdot \Gamma i, j, k - n + i, i - 1$ if $k > n - i$. The bottoms of these chains are the $\Gamma_{i,j,k,0}$ with $1 \leq k \leq n - i$. In particular, each $\Gamma_{i,j}$ is a union of $\min(i, n - i)$ chains. We obtain a decomposition of $\mathrm{Irr}(\mathcal{A}_{n-1})$ into chains, and the total number of chains is

$$n \sum_{i=1}^{n-1} \min(i, n-i) = n \left\lfloor \frac{n^2}{4} \right\rfloor.$$

In particular, the width of $\Gamma$ is at most $n\left\lfloor\frac{n^2}{4}\right\rfloor$, so Conjecture 27 implies Conjecture 2. Computer calculations suggest that the width of $\mathrm{Irr}(\mathcal{A}_{n-1})$ is indeed $n\left\lfloor\frac{n^2}{4}\right\rfloor$.

## Acknowledgments

The authors wish to thank Vic Reiner for helpful conversations.

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