SANDPILE AVALANCHE DISTRIBUTION ON THE WHEEL.

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Abstract. This paper describes results on the distribution of avalanches on the wheel in the sandpile model. It presents how transducers applied to the language of recurrent configurations of the abelian sandpile model can help determining critical exponents of the underlying model. We present exact results for the simple wheel and give some clues for the multiple one where strange phenomena happen.
Résumé. Cet article présente quelques résultats sur la distribution du nombre d'éboulements dans le modèle du Tas de Sable abélien. Ce modèle issu de la physique est basé sur un automate cellulaire. Nous calculons dans un premier temps le langage rationnel associé aux configurations récurrentes du système sur la roue puis à l'aide d'un transducteur nous en déduisons la distribution. Nous présentons par ailleurs une approche dans le cas de la multi-roue.

1. Introduction

The standard abelian sandpile model (ASM) was introduced by Bak, Tang and Wiesenfeld [1] in 1987. This model, based on a cellular automaton, is the paradigm of a self-organized critical system. Its underlying Abelian structure was discovered by Dhar [2] and Creutz [3]. Other approaches to this model can be found in [4, 5, 6, 7].

We can briefly describe this model as follows : take a regular two-dimensionnal lattice and on each cell, put some grains. If the number of grains is greater or equal than four, then take four of the grains and put one on each of the neighbor's cell. We say that the vertex topples. If a grain falls out of the lattice then it is lost. This could be represented as falling into a special cell which could never be toppled. We call such a cell the *sink*. An addition of one grain can induce a high number of different topplings all over the lattice. Then we can plot the distribution of the size of the avalanche under the addition of a grain on a random cell of the lattice. This distribution has the same behaviour than self-organised critical systems.

A classical extension of this model consists in taking a connected multi graph without loops instead of a regular lattice. A particular vertex is decided to be the sink. A configuration on the corresponding sandpile is an application u from $\mathcal{V}^*$, the set of vertices different from the sink, to $\mathbb{N}$. A vertex v of $\mathcal{V}^*$ is also called a site. Then for any site i on a given configuration u, we used to say that site i has $u(i)$ particles. Moreover, site i is said to be unstable if it contains more (or as many) particles than its degree. Such a site can be toppled, following the sandpile-rule described in figure 1: it 'loses' d_i particles (d_i is the degree of site i), and 'gives' 1 particle or more (in case of multi-edges) to each of its neighbours. Starting from a configuration u we perform the topplings whenever it's possible. This process is always finite due to the connectivity of the graph. At the end we reach a configuration $\hat{u}$ where no site

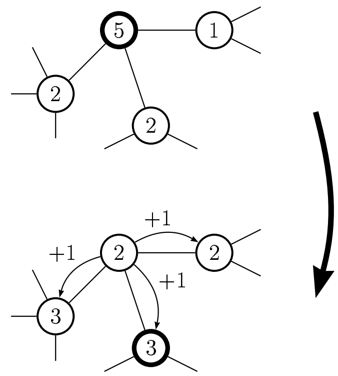

FIGURE 1. Sandpile-rule: toppling of an unstable site.

can topple. We call such a configuration a *stable* one. Furthermore, we can define the length of this sequence of topplings - called avalanche - : this is well defined as the order in which topplings are performed does not change the resulting stable configuration $\hat{u}$ as well as the number of times each vertex topple. Going from u to $\hat{u}$ is called *relaxation* of u.

Among the stable configurations, some are particular: the recurrent ones. They correspond to recurrent states of the following Markov chain. The states of this chain are the stable configurations and a transition consists in two different phases:

(1) The addition of one grain on a random site (with a uniform law)
(2) The relaxation of the resulting configuration

The set of recurrent configurations has a group structure (the sandpile group) with a natural addition, and every element of this group can be characterized by Dhar's criterium [8]. We denote $SP(\mathcal{G})$ the sandpile group associated to a graph $\mathcal{G}$. This definition is justified, because Cori and Rossin showed that the choice of the sink doesn't modify the group of the sandpile[7]. For many reasons, recurrent configurations are meaningful, and experiments are usually done on the sandpile group. Our work consists to analyze the distribution of the length of avalanches under the addition of a particle on a random configuration of the sandpile group. For a given recurrent configuration u and a given site i, we will denote by $\mathtt{L}(u, i)$ the size of the avalanche when we add a particle on site i in the configuration u. The motivation of our analysis comes from the observation of a strange phenomenon that occurs on the wheel for some values of its parameters. This strangeness deals with the observation of peaks in the avalanche distribution, where one can usually expect a power law function.

Definition of the wheel. The wheel can be described very shortly. It depends on two parameters n and k. When $k = 1$, we call it the simple wheel or sometimes the wheel as well. A simple wheel is a n-cycle with an extra vertex connected to all the others. This extra vertex is usually taken as the sink. A (n, k)-wheel, noted $\mathcal{R}(n, k)$, is a generalization of this graph. It can be defined by induction on k: basically it is a $(n, k-1)$-wheel with a n-cycle around, where every vertex of the last stage of the

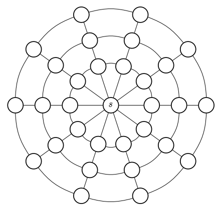

FIGURE 2. Example of a (10,3)-wheel.

$(n, k-1)$-wheel is connected to the opposite vertex in the cycle. Figure 2 gives an example of a (10,3)-wheel.

The experiments consist to choose a great number of times a random recurrent configuration, add a particle on a randomly chosen site and look at the length of the avalanche. In order to pick up a random recurrent configuration uniformely, we need to use the bijection discovered by Cori and Leborgne [9] and find a random spanning tree. We use Wilson loop-erased algorithm [10] for this part which gives good average complexity.

First, we will study the case of the simple wheel. For this graph, we show that recurrent configurations could be seen as words of a regular language. We show how to build an automaton associated to this language and then we determine the exact distribution of avalanche lengths.

On a second hand, we study the multiple wheel and give an explanation of the last peak that could be generalized for other ones.

2. Simple wheel

2.1. **Word representation of recurrent configurations.** In this section we consider the simple wheel, often refered as the wheel in the litterature. First we introduce a practical way to manipulate configurations on it. Let choose a random site v. We

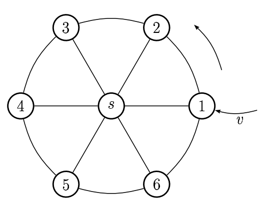

FIGURE 3. Simple wheel (case $n = 6$) and a numerotation of its vertices.

will refer to it as site 1. The other sites are defined by an orientation put on the plan (let say the trigonometric one for instance). Such a numerotation is shown on figure 3 for the 6-wheel. Thus we can associate a unique word of $\{0, 1, 2\}^n$ to each stable - recurrent - configuration. Given a stable configuration, we define the word w as $w = w_1 \dots w_n$ such that w_i is the number of particles on site i. We will denote by $\mathcal{L}$ the language corresponding to words associated to recurrent configurations on simple wheels of any size. In fact a word of $\mathcal{L}$ is very well characterized. Cori and Rossin [7] computed a valid, simple and non-ambiguous automaton (shown in figure 4) of this language thanks to Dhar's criterium [8].

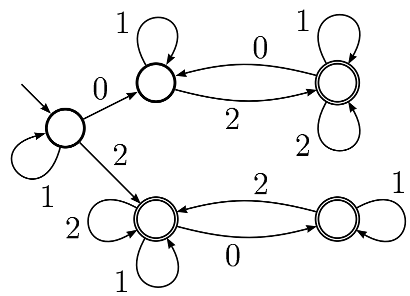

FIGURE 4. Automaton recognizing $\mathcal{L}$.

To know if a word belongs to the language recognized by the automaton, begin in the first state of the automaton -the one with an in-arrow-. Then, for each letter w_i of w follow the arrow labelled by w_i and if at the end of this process you arrive in an out-state - a double circled one- then the word w belongs to the language. On the other hand if you cannot read a letter - no arrow labelled by w_i- or if the final state is not an out-state then the word is not in the language.

It can be proved that the languages that could be recognized by a finite automaton are those which can be written in as a regular formula. See [11] for more detail results on automata.

In the sequel it is useful to consider a partition of $\mathcal{L}$. Like the automaton of Figure 4 is the union of two ones, we will consider for each of them, the associated language. We define $\mathcal{L}_2 = 1^*2(0+1+2)^* \cap \mathcal{L}$ and $\mathcal{L}_0 = 1^*0(0+1+2)^* \cap \mathcal{L}$. So by definition $\mathcal{L}_2 \cap \mathcal{L}_0 = \emptyset$ and $\mathcal{L}_2 \cup \mathcal{L}_0 = \mathcal{L}$. At last, we will also need a sub-language $\tilde{\mathcal{L}}_{2-2}$ of $\mathcal{L}_2$: $\tilde{\mathcal{L}}_{2-2} = 2^+(0+1+2)^*2^+ \cap \mathcal{L}_2 \setminus 2^*$. In other words w belongs to $\tilde{\mathcal{L}}_{2-2}$ if and only if it belongs to $\mathcal{L}_2$, it begins and ends by 2 and it is not in 2^*.

We will use the word representation rather than wheel configurations because properties of recurrent configurations can be read on associated words. Suppose that a recurrent configuration has an avalanche of size $m > 0$ when adding a particle on site i, then it is clear that $w_i = 2$ and that there exists a unique $k \geq 0$ such that $w_{i-k} = w_{i-k+1} = \ldots = w_i = \ldots = w_{i-k+m-1} = 2$ [1]. In other words, site i belongs to a greatest sub-sequence of 2 of length m. Thus counting these greatest sub-sequences of 2 is equivalent of finding the distribution of avalanche by size on the set of recurrent configurations.

2.2. **Building a transducer to count greatest sub-sequences of** 2. As an example, we will consider the case of $\mathcal{L}_2$.

The basic idea is to start with a non-ambiguous automaton which recognizes the language. We say that an automaton is non-ambiguous if there is only one way to read a word on it. Figure 5 points out a sub-automaton of figure 4.

[1]indices must be considered in $\mathbb{Z}/n\mathbb{Z}$

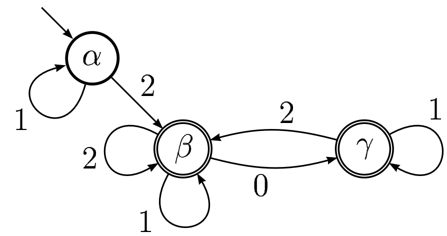

FIGURE 5. Non-ambiguous automaton recognizing $\mathcal{L}_2$.

In the sequel of this section, we will exhibit an algorithm which takes as input such an automaton and as output gives a transducer. When reading a word w of size n on the transducer, it outputs the polynomial $P_w(x) = \sum_{m \leq n} a_m x^m$ such that a_m is the number of greatest sub-sequences of 2 of size $m > 0$ in the circular word w. For a language $\mathcal{L}_k$ let $\mathcal{S}_k(x, y)$ be the sum of $P_w(x)y^{|w|}$ for all words w in $\mathcal{L}_k$. The parameter y counts the number of letters of the word.

A transducer is basically an extension of an automaton but the arrows are labelled by both a letter and an expression, and each time you read a letter you output the corresponding expression. But the transducers are not deterministic. So there are multiple paths to read a word. Then the expression produced when reading a word is the sum of the expressions output by the transducer when reading the word on each path. See e.g. figure 6 for an example.

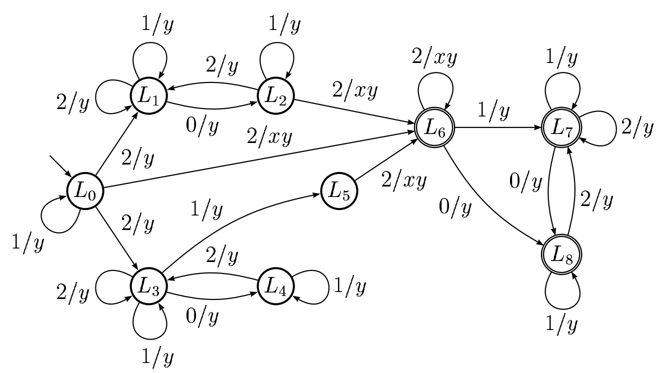

FIGURE 6. Transducer $\mathcal{T}^2$ to compute $\mathcal{S}_2(x, y)$.

In this transducer, read the word 2202. There are two different paths. The first one is $L_0L_1L_1L_2L_6$ and the second one $L_0L_6L_6L_8L_7$. The first path gives the monomial xy^4 and the second one x^2y^4. The power of x gives the length of a greatest sub-sequence of 2 and the power of y the number of letters of the word. Notice that this transducer is the one needed to compute $\mathcal{S}_2(x, y)$.

The algorithm to build the transducer from the automaton is the following: each time you have a transition labelled with 2 which can be the starting of a sequence of 2 then duplicate the next states in order to have two different paths. The first one which counts the sequence of 2 beginning at this point and the other one which will count further sequences. Take for example the language $2^+01^*2^+$. A non-ambiguous automaton recognizing this language is given in figure 7 as well as the associated transducer. Notice that the two loops labelled with 2 could not be the beginning of a sequence of 2.

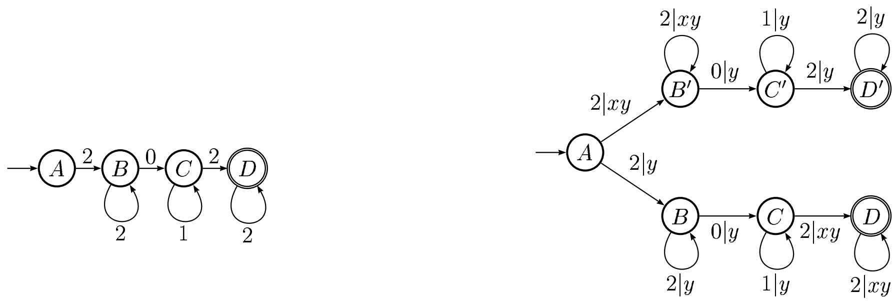

FIGURE 7. Simple non-ambiguous automaton for $2^+01^*2^+$ and the associated transducer.

So, take each edge of the automaton and for each one which can be the beginning of a sequence of 2 draw the associated transducer which counts only this sequence. Figures 8,9,10 show the three transducers built from the three 2-edges $\alpha - \beta$, $\beta - \beta$ and $\gamma - \beta$ of the automaton represented in figure 5.

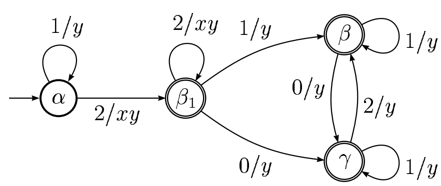

FIGURE 8. Transducer associated with edge $\alpha - \beta_1$

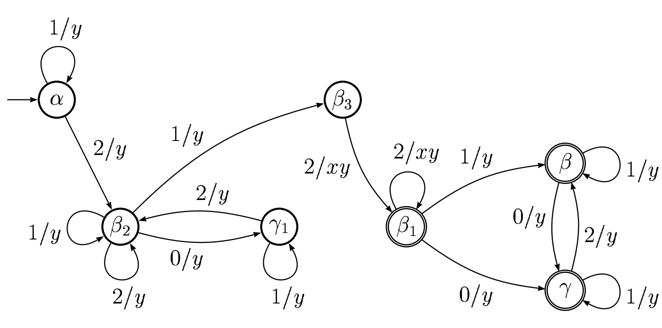

FIGURE 9. Transducer associated with $\beta_3 - \beta_1$

For each of these edges, we first duplicate some part of the automaton in order to isolate any greatest sub-sequence of 2 beginning with it, and to discriminate the

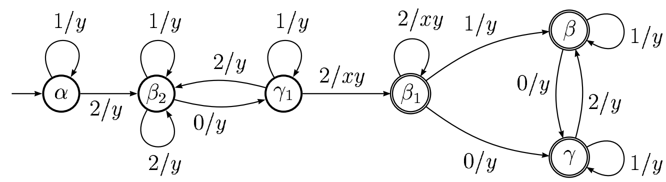

FIGURE 10. Transducer associated with edge $\gamma_1 - \beta_1$

graph before and after taking the edge. We have then created a 2-edge separator (the three edges pointing to β_1 in figures 8, 9,10 -$\alpha - \beta_1$,$\beta_3 - \beta_1$ and $\gamma_1 - \beta_1$-. Then we can label by $2/xy$ the 2-labelled sub-path beginning by this separator. We also have to make non-final all the states that are before it (states β_2, β_3 and γ_1 in figure 8). As a consequence, each accepted word has one of its greatest sub-sequence of 2 counted, because a greatest sub-sequence of 2 has to be counted to reach a final state. For the other edges, we add $/y$ to their label, to count the letters.

Then, we can merge the transducers, and we get the expected result. For example, the transducer $\mathcal{T}^2$ in figure 6 is the merged transducer of the three represented on figures 8,9,10. We have to pay attention that no greatest sub-sequence of 2 is counted twice, because the starting automaton (figure 5 in our example) is non-ambiguous. During the process, this property stays true on pointed-word (word with one of its greatest sub-sequence of 2 pointed). We can say that for such a word, there is only one accepted path, ie a path leading to a final state for the word, such that the pointed greatest sub-sequence of 2 is read by the $(2/xy)$-labelled edges. In the case where the starting automaton is ambiguous, then some greatest sub-sequences of 2 can be counted twice or more.

2.3. **Counting the greatest sub-sequences of 2 in a word of $\mathcal{L}$.** We divide in two the problem and build transducers for $\mathcal{L}_2$ and $\mathcal{L}_0$ instead of working with a bigger one for $\mathcal{L}$.

Figure 6 points out a transducer which recognizes $\mathcal{L}_2$. In fact, this transducer does not give us always the expected result: there are some side effects. For example, if $w = 22011211102212$ we will get $P^2_w(x,y) = (2x^2 + 2x)y^{14}$ instead of $P_w(x,y) = (x^3+x^2+x)y^{14}$. The side effects concern words of $\tilde{\mathcal{L}}_{2-2}$. These words begin and end by a greatest sub-sequence of 2 , and the transducer of Figure 6 counts the first one and the last one separatively instead of merging them. A good way to solve this problem is to consider two other transducers $\mathcal{T}^+$ and $\mathcal{T}^-$ defined by: if $w = 2^b u_1 \dots u_k 2^e$ with $k \geq 1$ and $u_1, u_k \neq 2$ is in $\tilde{\mathcal{L}}_{2-2}$ (ie $w \in \mathcal{L}_2$ and $e, b \neq 0$), then $\mathcal{T}^+(w) = x^{b+e}y^n$ and $\mathcal{T}^-(w) = (x^b + x^e)y^n$. Moreover $\mathcal{T}^+(w) = \mathcal{T}^-(w) = 0$ for $w \in \mathcal{L} - \mathcal{L}_2$. Then for all w in $\mathcal{L}_2$, we will have $P_w(x,y) = P^2_w(x,y) + P^+_w(x,y) - P^-_w(x,y)$ with obvious notations. Corresponding series (when summing on $\mathcal{L}_2$ or $\mathcal{L}$) will be denoted by $\mathcal{S}_2(x,y)$, $\mathcal{S}_+(x,y)$ and $\mathcal{S}_-(x,y)$. Figure 11 shows a transducer to compute $\mathcal{S}_+(x,y)$. Figure 12 represents a transducer to compute $\mathcal{S}_0(x,y)$ associated to $\mathcal{L}_0$: $\mathcal{S}_0(x,y) = \sum_{w \in \mathcal{L}_0} P_w(x)y^{|w|}$. We can remark that with $\mathcal{L}_0$, we have no side effects (no words can begin by 2).

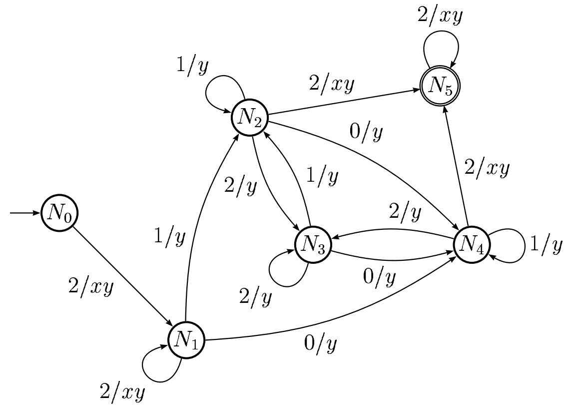

FIGURE 11. Transducer $\mathcal{T}^+$ to compute $\mathcal{S}_+(x, y)$.

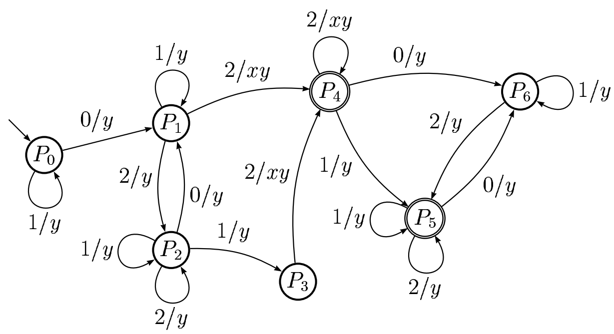

FIGURE 12. Transducer $\mathcal{T}^0$ to compute $\mathcal{S}_0(x, y)$.

With all these transducers, we can compute the claimed series - see [11] for details about computing them -. We get the following results:

$$\begin{aligned}
\mathcal{S}_2(x,y) &= \tfrac{xy(1-y)^3}{(1-xy)(1-3y+y^2)^2} \\
\mathcal{S}_+(x,y) &= \tfrac{x^2y^3(1-y)(2-y)}{(1-xy)^2(1-3y+y^2)} \\
\mathcal{S}_-(x,y) &= \tfrac{2xy^3(2-y)}{(1-xy)(1-3y+y^2)} \\
\mathcal{S}_0(x,y) &= \tfrac{xy^2(1-y)^3}{(1-xy)(1-3y+y^2)^2}
\end{aligned}$$

The exact solution $\mathcal{S}(x, y)$ is then:

$$\mathcal{S}(x,y) = \mathcal{S}_2(x,y) + \mathcal{S}_+(x,y) - \mathcal{S}_-(x,y) + \mathcal{S}_0(x,y)$$

The number of greatest sub-sequences of 2 of size $m > 0$ in words of $\mathcal{L}$ of size n is $[x^m y^n]\mathcal{S}(x, y)$. But as we will see, only an asymptotical result will be sufficient for our purpose. Besides we can remark that $\mathcal{S}_0(x, y) = y\ \mathcal{S}_2(x, y)$. There is a natural bijection between words in $\mathcal{L}_2$ of size n and words in $\mathcal{L}_0$ of size $n + 1$ with the same greatest sub-sequences of 2 , considering the first letter and the last one not being connected. The bijection is the following: let $w = w_1 \dots w_n$ be in $\mathcal{L}_2$, then either w ends by 21^* or by 01^*. In the first case we define $\Psi(w) = 0w_n \dots w_1$ and on the other case $\Psi(w) = 1w_n \dots w_1$. For both cases, we have $\Psi(w)$ in $\{u \in \mathcal{L}_0, |u| = n+1\}$. Moreover, the fact that Ψ is one-to-one is obvious. The fact it is onto is also straightforward. It results that Ψ is a bijection that also keeps the greatest sub-sequences of 2 . That explains why $\mathcal{S}_0(x,y) = y\ \mathcal{S}_2(x,y)$. We can say that $\mathcal{T}^2$ is not minimal. Effectively, we can buid from $\mathcal{T}^0$ a transducer equivalent to $\mathcal{T}^2$ with strictly less states. However, even if the transducers we have built are not minimal, they are quite clear and easy to compute. That is why we kept them.

2.4. **Asymptotic analysis of the series and comparison with experimentations.** Each of the series is a rational fraction. We have to find the pole of minimal radius. It corresponds to the little zero of the polynomial $1 - 3y + y^2$. Its two zeros are Φ^2 and Φ^{-2}, where $\Phi = (1+\sqrt{5})/2$ is the golden number. Thus we have:

$$[y^n]\frac{1}{1-3y+y^2} = C\Phi^{2n} + \mathcal{O}(\Phi^{-2n})$$

Applied to our series we get:

$$[x^m y^n]\mathcal{S}_2(x,y) = \Phi\left(\frac{n+1-m}{5}\right)\Phi^{2(n-m)} + \mathcal{O}(\Phi^{-2(n-m)})$$

$$[x^m y^n]\mathcal{S}_+(x,y) = m\left(\frac{1}{\sqrt{5}}\right)\Phi^{2(n-m)} + \mathcal{O}(\Phi^{-2(n-m)})$$

$$[x^m y^n]\mathcal{S}_-(x,y) = \left(\frac{2}{\Phi}\right)\Phi^{2(n-m)} + \mathcal{O}(\Phi^{-2(n-m)})$$

$$[x^m y^n]\mathcal{S}_0(x,y) = \left(\frac{n-m}{5\Phi}\right)\Phi^{2(n-m)} + \mathcal{O}(\Phi^{-2(n-m)})$$

So for n great, we have:

$$[x^m y^n]\mathcal{S}(x,y) \sim \left(\frac{n}{\sqrt{5}}\right)\Phi^{2(n-m)}$$

With this result, we can predict the results of the experimentations done. Let's note $\mathcal{R}(x,y)$ the serie corresponding to the experimentations. Let fix n and let take a word (configuration) w of $\mathcal{L}$ of size n. If $P_w(x) = \sum_{m=0}^{n} a_m x^m$ is the polynomial which counts the greatest sub-sequences of 2 of w, and if $Q_w(x) = \sum_{m=0}^{n} l_m x^m$ is the one counting the length of the n avalanches that happen when putting a grain on site 1, then on site 2 on so on till n, then for $m > 0$ we have $ma_m = l_m$, ie:

$$Q_w(x) = xP'_w(x) + C \qquad (1)$$

Effectively, if site i has 2 particles in w, then site i is at the middle of a greatest sub-sequence of 2 of size strictly positive. Let $m > 0$ be the length of this greatest sub-sequence of 2 . For each site of this greatest sub-sequence of 2 , the fact to add a particle produces an avalanche of size m. If there are a_m greatest sub-sequences of 2 of size m, them after the experiment, we have seen ma_m avalanches of size m. Hence $ma_m = l_m$. For n great enough, [7] shows that the number of recurrent configurations on a simple n-wheel ($|\mathcal{L} \cap \{0,1,2\}^n|$) is equivalent to Φ^{2n}. Hence from the definition of $Q_w(x)$, we have the following relation with $\mathcal{R}(x,y)$:

$$\mathcal{R}(x,y) = \frac{1}{n\Phi^{2n}} \sum_{w \in \mathcal{L}} Q_w(x) y^{|w|}$$

The same remark goes for $P_w(x)$ and $\mathcal{S}(x,y)$:

$$\mathcal{S}(x,y) = \sum_{w \in \mathcal{L}} P_w(x) y^{|w|}$$

Thus from equation 1, we get:

$$[y^n]\mathcal{R}(x,y) = \frac{x\Phi^{-2n}}{n} \frac{\partial \mathcal{S}(x,y)}{\partial x} + C(n)$$

where $C(n)$ is the proportion of avalanches of size 0 for the n-wheel. For $m > 0$ it gives us an asymptotical equivalent of the proportion of avalanches of size m:

$$[x^m y^n]\mathcal{R}(x,y) \sim \left(\frac{m}{\sqrt{5}}\right) \Phi^{-2m} \tag{2}$$

To determine the proportion of avalanches of size 0, we use the normalization cri-

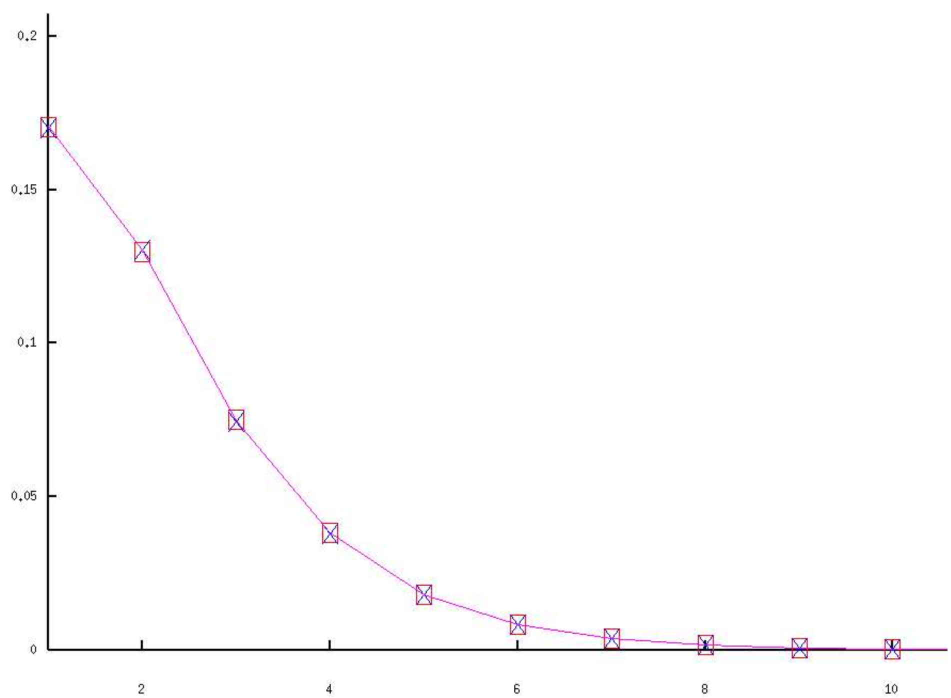

FIGURE 13. Distribution of the length of avalanches on the n-wheel: expected result (line) and values for $n = 100$ (square) and $n = 1000$ (cross).

terium. As $\mathcal{R}(x,y)$ is the generative serie of a distribution, we have the relation:

$$\forall n\ ,\ [y^n]\mathcal{R}(1,y) = 1$$

With equation 2 it gives us $C(n) = 1 - 1/\sqrt{5}$ for any n. In conclusion we have:

$$[x^m y^n]\mathcal{R}(x,y) \sim \begin{cases} \left(\frac{m}{\sqrt{5}}\right)\Phi^{-2m} & \text{if} \quad m > 0 \\ 1 - \frac{1}{\sqrt{5}} & \text{if} \quad m = 0 \end{cases}$$

The first remark to be done from the calculus is that this distribution is independant of n. In fact, it is quite coherent and realistic with the results of the experimentation. Figure 13 shows the expected distribution on a line, and the experimental results for $n = 100$ and $n = 1000$ over 10^6 computations. We present only the values for $m > 0$, because the value for $m = 0$ is much higher. However, in this case also, the predicted value $(1 - 1/\sqrt{5})$ is very accurate.

3. Case of the multiple wheel

The study of the distribution of the avalanches on the general (n,k)-wheel is not as easy as in the precedent case. First of all, we have no exact value for the number of recurrent configurations of the (n,k)-wheel, when $k > 1$. However, experiments show that $exp[\alpha(k+\gamma)(n+\gamma)]$ with $\gamma = -1 + (2\log\Phi)/\alpha$ and $\alpha \sim 1.1674$, is a good approximation of $|SP(\mathcal{R}(n,k))|$. The motivation of this analysis comes from the observation of some strange phenomenon: appearance of peaks. Nevertheless this phenomenon seems to appear for some particular values of the parameters. Figure 14 shows the distribution on the $(3,10)$-wheel. In this case, besides the first peak (case $m = 0$ not shown on figure 14), we can observe many other peaks of similar heights. This phenomenon was first observed in [12]. The author shows that there are k peaks, and their abscisses are: $x_1 = kn, x_2 = kn + (k-1)n, \ldots, x_k = nk(k+1)/2$. In particular, x_k is the maximal value of the length of an avalanche on the (n,k)-wheel. So first we are going to study the last peak $(m = x_k)$.

3.1. **Analysis of the last peak.** As we said, the last peak corresponds to avalanches of maximal sizes. We will denote by l_{max} (dependant of n and k) this size: $l_{max} = nk(k+1)/2$. In particular, l_{max} is the length of the avalanche when we add a particle on a site of the last stage, while the configuration is the maximal one r_{max} (every site v different from the sink has $d_v - 1$ particles on it, where d_v is the degree of v). Effectively, when we add a particle on a site i of stage κ [2], we have: $\mathtt{L}(r_{max}, i) = n\kappa(2k+1-\kappa)/2$. Besides, we have to recall that a natural partial order can be put on the sandpile group and that r_{max} is the maximal element. Moreover, for any site i, if we have two configurations u_1 and u_2 such that $u_1 \geq u_2$ then $\mathtt{L}(u_1, i) \geq \mathtt{L}(u_2, i)$. In fact, this property is true on each site: if a site j has been toppled t times during the avalanche of (u_1, i) then the same site j has been toppled at most t times during the avalanche of (u_2, i). In particular, it implies that an element (u, i) admits an avalanche of size l_{max} only if site i belongs to the last stage $(\kappa = k)$. In fact, if we

[2] $\kappa = 1$ corresponding to the inner stage

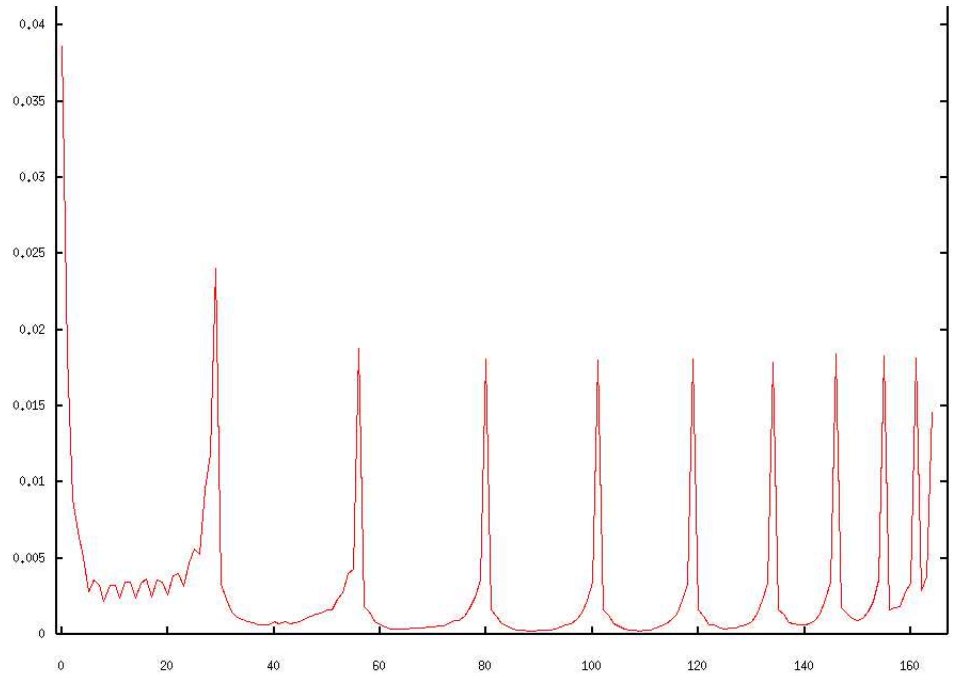

FIGURE 14. Appearance of peaks in the distribution of avalanches on the $(3, 10)$-wheel. The first peak $(m = 0)$ which represents almost 0.5 is not shown.

restraint the choice of the site in our experimentations such that it always belongs to the last stage, we can see that the last peak appears entirely, as shown on figure 15. Then, we can determine the configuration on the last stage. It has to be very similar to r_{max}. In fact, it must contain at most one site different from i with only 1 particle, and the others containing 2 particles. A good way to see this is to express the fact that the configuration obtained after the $nk(k+1)/2$ topplings has to be recurrent if the element belongs to the last peak. With the same consideration, we could have informations on the induced configuration u' on $\mathcal{R}(n, k-1)$, corresonding to the $k-1$ inner stages. If the last stage contains a site with only 1 particle, then the configuration u' belongs to $SP(\mathcal{R}(n, k-1))$. On the contrary, if no site of the last stage contains strictly less than 2 particles, then the configuration u' of $\mathcal{R}(n, k-1)$ is either recurrent or equals to a recurrent configuration with one particle less on a site j of the last stage of $\mathcal{R}(n, k-1)$ and which is not a neighbour of site i. Putting this two remarks together, we can find a major value $M(n, k)$ and a minor value $m(n, k)$ of the proportion $\mathcal{P}$ of elements (u, i) whose length of avalanche equals l_{max}. We have:

$$\mathcal{P} \geq \frac{n(n-1)|SP(\mathcal{R}(n,k-1))| + n|SP(\mathcal{R}(n,k-1))|}{nk|SP(\mathcal{R}(n,k))|}$$

$$\mathcal{P} \leq \frac{n(n-1)|SP(\mathcal{R}(n,k-1))| + n(1+n-1)|SP(\mathcal{R}(n,k-1))|}{nk|SP(\mathcal{R}(n,k))|}$$

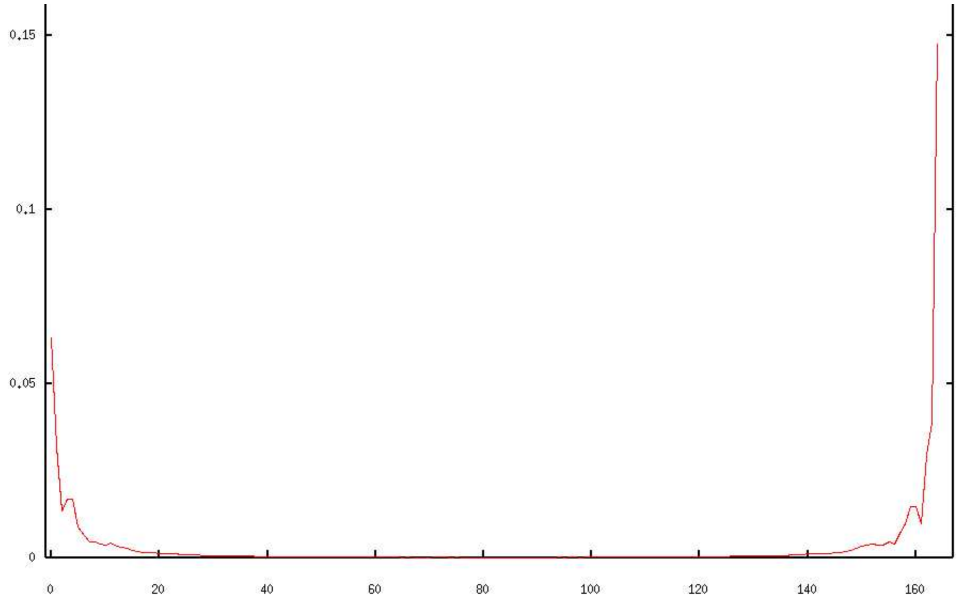

FIGURE 15. Random site i is always taken among sites of the last stage. The last peak for the $(3,10)$-wheel appears entirely (rate coefficient of $1/k$, here $1/10$).

Thus we can take for $m(n,k)$ and $M(n,k)$ the following values:

$$m(n,k) = \frac{n}{k}\frac{|SP(\mathcal{R}(n,k-1))|}{|SP(\mathcal{R}(n,k))|}$$

$$M(n,k) = \frac{2n}{k}\frac{|SP(\mathcal{R}(n,k-1))|}{|SP(\mathcal{R}(n,k))|}$$

With the approximation $|SP(\mathcal{R}(n,k))| \sim exp[\alpha(k+\gamma)(n+\gamma)]$ we get:

$$m(n,k) = \frac{n}{k}e^{-\alpha(n+\gamma)} \tag{3}$$

$$M(n,k) = \frac{2n}{k}e^{-\alpha(n+\gamma)} \tag{4}$$

In particular, if k increases then l_{max} grows like k^2, but the proportion of avalanches of size l_{max} decreases only like $1/k$. In what concerns the parameter n, the behaviour is much more common: the decreasing is exponential. We can thus infer that peaks are better to be observed with small values of n and possibly greater values of k. The value of $m(n,k)$ makes us understand why the distribution could have a great value for l_{max}.

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