The sequence of linear algebraic systems generated by Zeilberger's algorithm

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Abstract

Zeilberger's algorithm, also known as the method of creative telescoping, generates a sequence of systems of linear algebraic equations until a system with a special property appears, provided that such a system exists. We show how the values computed during the investigation of the (J-1)-th system can be used to accelerate the investigation of the J-th system.

Résumé

L'algorithme de Zeilberger, parfois aussi appelé "méthode de télescopage créatif", génère successivement des systèmes d'équations algébriques linéaires jusqu'à ce que le système possède une propriété voulue (en supposant qu'un tel système existe). Nous montrons comment les valeurs calculées pour générer le (J-1)-ème système peuvent être utilisées pour accélérer la génération du J-ème système.

1 Introduction

Zeilberger's algorithm (we name it hereafter as $\mathcal{Z}$ for short) has a wide range of applications which include verification of combinatorial identities, finding closed forms of definite sums of hypergeometric terms, and asymptotic estimates [8, 7].

Let $F(n,k)$ be a hypergeometric term (or a *term* for short) whose certificates $F(n+1,k)/F(n,k)$ and $F(n,k+1)/F(n,k)$ are rational functions of n and k over a field K of characteristic 0. $\mathcal{Z}$ tries to find

$$A_0(n), \ldots, A_J(n) \in K(n), \quad A_J(n) \neq 0, \tag{1}$$

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such that for a term $S(n,k)$,

$$A_J(n)F(n+J,k)+\cdots+A_1(n)F(n+1,k)+A_0(n)F(n,k)=S(n,k+1)-S(n,k). \tag{2}$$

The value of $J\in\mathbb{N}$ such that (1) and (2) hold must be as small as possible.

If those J and $A_0,\ldots,A_J$ do not exist for a given term $F(n,k)$ then $\mathcal{Z}$ does not terminate. In [1, 2] a necessary and sufficient condition for the termination of $\mathcal{Z}$ on a given term is presented.

$\mathcal{Z}$ uses an item-by-item examination on the values of J. It starts with 0 and keeps on incrementing J until it is successful in finding the $A_0,\ldots,A_J$ such that (2) holds. (A direct algorithm is known only for the case when $F(n,k)$ is a rational function [5, 6].) For a particular value of J under investigation, $\mathcal{Z}$ constructs a system of linear algebraic equations whose coefficients are in $K(n)$, and the right-hand side contains the parameters $A_0,\ldots,A_J$. $\mathcal{Z}$ then checks if it is possible to find (1) such that the system is compatible (see [8, 7] for details). This check is expensive if the value of J is large. Even when we know a non-trivial lower bound J_0 for J (see [3]), we can still waste a lot of time on the fruitless examination at steps $J_0, J_0+1,\ldots,J-1$.

The examination done at each step is independent of that at other steps. However, the systems on consecutive steps bear similarities, and it would be logical to try to take advantage of this. It is shown in this paper that after we considered the system corresponding to step $(J-1)$ and found that the system is not compatible, we can use some intermediate results of this step in order to simplify the system corresponding to step J (here "simplify" means the elimination of the parameters $A_0,\ldots,A_{J-1}$ in a number of equations of the J-th system).

Throughout the paper, K is a field of characteristic zero, $\mathbb{N}$ is the set of nonnegative integers. The symbols E_n, E_k denote the shift operators w.r.t. n and k, respectively, defined by $E_nF(n,k)=F(n+1,k)$, $E_kF(n,k)=F(n,k+1)$.

2 J-parameterized systems

2.1 J-solvability

Let

$$Mx=u \tag{3}$$

be a system of linear algebraic equations where M is a $\nu\times\kappa$ matrix whose entries are in a field Λ, and u is a column vector whose ν entries are in a Λ-linear space U. We call a column $y\in U^\kappa$ a *simplificator* of (3) if the first entry of the column $u-My$ is zero, and assign the *height* of a simplificator y the number of all initial entries of $u-My$ that are equal to zero. Evidently, each simplificator of height ν is a solution of (3).

For any $J\geq 0$, we use the notation $\Gamma_{J,\Lambda}$ for the Λ-space of linear forms in $A_0,\ldots,A_J$

$$R_0A_0+\cdots+R_JA_J,\quad R_0,\ldots,R_J\in\Lambda. \tag{4}$$

If entries of u are in $\Gamma_{J,\Lambda}$, then we call system (3) a *J-parameterized* system.

Let (3) be a J-parameterized system, $J \geq 0$. The J-*increment* of this system is the number σ of all initial components of u which do not depend on $A_0, \ldots, A_{J-1}$.

We call a J-parameterized system J-*solvable* if there exist $A_0^*, \ldots, A_J^* \in \Lambda$ such that

- $A_J^* \neq 0$,
- if $A_0 = A_0^*, \ldots, A_J = A_J^*$, then (3) is compatible .

2.2 J-solvability recognition, irregular equations, and simplificators

Suppose the recognition of the J-solvability of system (3) is carried out by an elimination process. During this process we can get an equation of the form

$$0 = \tilde{R}_0 A_0 + \cdots + \tilde{R}_J A_J. \tag{5}$$

Such an equation is called

- *trivial* if $\tilde{R}_0 = \cdots = \tilde{R}_J = 0$;
- *irregular* if ($\tilde{R}_0 = \cdots = \tilde{R}_{J-1} = 0$ and $\tilde{R}_J \neq 0$) or if ($\tilde{R}_1 = \cdots = \tilde{R}_J = 0$ and $\tilde{R}_0 \neq 0$) ;
- *regular* otherwise .

During the elimination process we build up a trapezoidal system W of regular equations each of which is of the form (5).

At the current step of the elimination process, suppose that W already had λ regular equations, and the current step produces another regular equation. Then we eliminate the λ unknowns in this equation. Only if it remains regular, we include it into W and increase λ by 1. If this equation becomes irregular, then the original system is not J-solvable.

Suppose we have carried out a few stages of this elimination process.

Proposition 2.1 *The original system is a fortiori not J-solvable if one of the following two events happens at the current step of the elimination process:*

(a) *an irregular equation appears ;*

(b) *W contains λ equations, $\lambda < J$, and we recognize that it is impossible to get $J - \lambda$ additional regular equations for the remaining of the elimination process .*

Proof:

(a) by definition of J-solvability ;

(b) due to the fact that the $A_0, \ldots, A_J$ should be uniquely defined up to a factor from $K(n)$. ∎

It is quite often possible to construct a simplificator of a J-parameterized system during its J-solvability recognition. The value of constructing such a simplificator is explained in the next subsection.

Although the equations might change their orderings during the elimination process, we assign to each equation a label which is the number of this equation in the original system, and hence are still able to keep track of its position during the process. This elimination process results in systems V and W. The system W was introduced right before Proposition 2.1; the system V consists of the equations which are obtained during the elimination process, and which are not of the form (5).

If W is compatible with $A_J \neq 0$, $A_0 \neq 0$, then the original system is J-solvable. Otherwise, it is not J-solvable. In the latter case, we can try to construct a simplificator of the system as follows.

(i) Find the maximal N such that the equations labeled by $1, \ldots, N$ are in V;

(ii) for all $i = 1, \ldots, N$, the unknown x_i was eliminated by an equation labeled by a j, $1 \leq j \leq N$.

As a consequence, we have a system V', which is a subsystem of V, of equations labeled by $1, \ldots, N$. V' is in trapezoidal form, and by using V' we can easily represent $x_1, \ldots, x_N$ via $x_{N+1}, \ldots, x_\kappa, A_0, \ldots, A_J$. Set $x_{N+1} = \cdots = x_\kappa = 0$ in this representation. Evidently, this way we get the vector $(x_1, \ldots, x_\kappa)^T$ that is a simplificator of height $\geq N$ of the original system.

2.3 Systems generated by $\mathcal{Z}$

$\mathcal{Z}$ generates a sequence of systems

$$M_J x = u_J, \quad J = 0, 1, \ldots \tag{6}$$

where the field Λ in Section 2.1 is $K(n)$, and the J-th system is J-parameterized. The main goal of the algorithm is to find the minimal $J \geq 0$ such that the system $M_J x = u_J$ is J-solvable, and to find the corresponding values of $A_0, \ldots, A_J \in K(n)$. (The vector solution x helps determine the term $S(n, k)$ in (2).)

In Section 3 we will prove the following theorem.

Theorem 2.2 *Suppose it was recognized that the system from the sequence (6) with $i = J-1$ is not $(J-1)$-solvable and additionally, suppose that a simplificator y_{J-1} of height H_{J-1} for this system was computed. Then the vector y_{J-1} can be transformed into a vector y'_{J-1} of a suitable dimension such that if σ_J is the J-increment of the system $M_J x = u_J$ then the J-increment of the system*

$$M_J x = u_J - M_J\, y'_{J-1} \tag{7}$$

is equal to $\sigma_J + H_{J-1}$.

An increase of the J-increment of a system from the sequence (6) by using the system (7) reduces the work on the system since it simplifies the right-hand side of a number of equations of the system (the coefficients of $A_0, \ldots, A_{J-1}$ vanishes). If the system (7) is not J-solvable, then we find a simplificator for it and transform the system $M_{J+1} x = u_{J+1}$, increasing its J+1-increment as a consequence. The process is repeated until we reach a $J \in \mathbb{N}$ such that the corresponding J-parameterized systems (6) is J-solvable, provided that such a J exists.

3 Step-by-step examination in $\mathcal{Z}$

3.1 A review on the item-by-item examination

For a term $F(n,k)$ and for a particular value of $J \in \mathbb{N}$, set

$$T_J(n,k) = A_J(n)F(n+J,k) + \cdots + A_1(n)F(n+1,k) + A_0(n)F(n,k) \tag{8}$$

where the $A_i(n) \in K(n)$ are unknowns. Since F is a term, T_J is also a term [8]. $\mathcal{Z}$ now attempts to compute the A_i in (8) and a term S such that (2) holds. This is done by using a variant of Gosper's algorithm [4]. Given the term T_J in (8), the algorithm determines if there exists a term S_J such that

$$T_J = (E_k - 1)\, S_J, \tag{9}$$

and computes S_J if such a term exists. Gosper's algorithm first transforms (9) into the problem of computing a rational solution $x(k)$ with coefficients which are elements of $K(n)$ of

$$\frac{T_J(n,k+1)}{T_J(n,k)} x(k+1) - x(k) = 1. \tag{10}$$

The algorithm then transforms (10) into the problem of computing a polynomial solution of a first-order linear recurrence equation with polynomial coefficients and polynomial right hand side (12). The procedure can be summarized as follows.

1. Compute a PNF_k of the k-certificate $T_J(n,k+1)/T_J(n,k)$. This results in a triple (a_J, b_J, c_J), a_J, b_J, $c_J \in K(n)[k] \setminus \{0\}$ such that

$$\frac{T_J(n,k+1)}{T_J(n,k)} = \frac{a_J}{b_J} \cdot \frac{E_k c_J}{c_J}, \;\; \gcd(a_J, E_k^h b_J) = 1 \text{ for all } h \in \mathbb{N}. \tag{11}$$

   See [7] for a description of such a construction.

2. Find a polynomial solution $y(k)$ of the linear recurrence

$$a_J(k)\, y(k+1) - b_J(k-1)\, y(k) = c_J(k) \tag{12}$$

   provided that such a solution exists.

If it does, then set

$$L_J = A_J(n)\, E_n^J + \cdots + A_1(n)\, E_n + A_0(n), \tag{13}$$

$$S_J = \frac{b_J(k-1)\, y(k)}{c_J(k)}\, T_J. \tag{14}$$

The computed $\mathcal{Z}$-pair (L_J, S_J) defined in (13) and (14) is the output from $\mathcal{Z}$. The recurrence operator L_J is called a telescoper for the input term F.

The search for a polynomial solution $y(k)$ of (12) can be done using the method of undetermined coefficients. First one computes an upper bound d for the degree of the polynomial $y(k)$. Then one substitutes a generic polynomial of degree d for $y(k)$ into (12), equates the coefficients of like powers in k. This results in a J-parameterized system. The problem is now reduced to determining if this J-parameterized system is J-solvable. If it is, then compute a solution of the system. Note that this enables one to compute not only a polynomial solution $y(k)$ in (12), but also the unknowns A_i in (8).

3.2 Equalities for first-order operators

Let v, w, p, q be polynomials in k. We consider the following two types of operators:

$$G_{\frac{v(k)}{w(k)}} = \frac{v(k)}{w(k)}E_k - 1, \tag{15}$$

$$G_{v(k),w(k)} = v(k)E_k - w(k-1). \tag{16}$$

Let $(a,\, b,\, c)$ be a PNF_k of

$$\frac{v(k)}{w(k)} = \frac{T_J(n,k+1)}{T_J(n,k)} \in K(n,k).$$

For any $y(k) \in K[n,k]$, set $x(k) = (b(k-1)\,y(k))/c(k)$. Then by (15), the relation $G_{\frac{v(k)}{w(k)}}x(k) = 1$ is equivalent to (12). Additionally, it follows from (16) that

$$G_{a(k),b(k)}y(k) = a(k)\,y(k+1) - b(k-1)$$

which is the left hand side of (12). For this reason, the operator $G_{a,b}$ is called the *Gosper's form* of the operator $G_{\frac{v}{w}}$.

It is simple to check that the following relations are valid:

$$G_{\frac{v(k)}{w(k)}} = G_{v(k),w(k)} \circ \frac{1}{w(k-1)}, \tag{17}$$

$$G_{\frac{v(k)p(k+1)}{w(k)p(k)}} = \frac{1}{p(k)}G_{v(k),w(k)} \circ \frac{p(k)}{w(k-1)} = \frac{1}{p(k)}G_{\frac{v(k)}{w(k)}} \circ p(k), \tag{18}$$

$$G_{\frac{v(k)q(k)}{w(k)q(k+1)}} = q(k)G_{v(k),w(k)} \circ \frac{1}{q(k)w(k-1)} = q(k)G_{\frac{v(k)}{w(k)}} \circ \frac{1}{q(k)}. \tag{19}$$

By (17) and (18), the equality

$$G_{\frac{v(k)}{w(k)}} = \frac{1}{c(k)}G_{a(k),b(k)} \circ \frac{c(k)}{b(k-1)} \tag{20}$$

is also valid.

3.3 Gosper's equations in $\mathcal{Z}$

The goal of this section is to establish relations between $G_{a_J(k),b_J(k)}$ and $G_{a_{J+1}(k),b_{J+1}(k)}$ (Proposition 3.1).

At step J of the item-by-item examination, $\mathcal{Z}$ tries to compute a telescoper L_J of the form (13) for a given term $F(n,k)$. The k-certificate of the term $T_J(n,k) = L_J\,F$ can be written in the form

$$\frac{v_J(n,k)}{w_J(n,k)} = \frac{\varphi_J(n,k)}{\psi_J(n,k)}\,\frac{p_J(A_0,\ldots,A_J,n,k+1)}{p_J(A_0,\ldots,A_J,n,k)} \tag{21}$$

where $\varphi_J(n,k)$ and $\psi_J(n,k)$ do not depend on $A_0,\ldots,A_J$; $p_J \in \Gamma_{J,K(n,k)}$; $v_J,\, w_J \in K[n,k]$.

Let $s_1(n,k)$, $s_2(n,k)$ be relatively prime polynomials such that

$$\frac{F(n,k)}{F(n-1,k)} = \frac{s_1(n,k)}{s_2(n,k)}.$$

Then we can derive the following recurrences:

$$\begin{aligned}
p_{J+1}(A_0,\ldots,A_{J+1},n,k) &= p_J(A_0,\ldots,A_J,n,k)\,s_2(n+J+1,k)+ \\
&\quad A_{J+1}\prod_{i=1}^{J+1} s_1(n+i,k), && (22)\\
\varphi_{J+1}(n,k) &= \varphi_J(n,k)\,s_2(n+J+1,k), && (23)\\
\psi_{J+1}(n,k) &= \psi_J(n,k)\,s_2(n+J+1,k+1). && (24)
\end{aligned}$$

(They are similar to (6.3.3)–(6.3.8) in [7].) Let

$$\frac{a_J(k)}{b_J(k)}\,\frac{\xi_J(k+1)}{\xi_J(k)}$$

be a PNF$_k$ of $\frac{\varphi_J}{\psi_J}$. It follows from (23) and (24) that

$$\begin{aligned}
\frac{\varphi_{J+1}(n,k)}{\psi_{J+1}(n,k)} &= \frac{\varphi_J(n,k)}{\psi_J(n,k)}\,\frac{s_2(n+J+1,k)}{s_2(n+J+1,k+1)}\\
&= \frac{a_J(k)\xi_J(k+1)}{b_J(k)\xi_J(k)}\,\frac{s_2(n+J+1,k)}{s_2(n+J+1,k+1)}. && (25)
\end{aligned}$$

It then follows that[1]

$$\begin{aligned}
G_{\frac{\varphi_{J+1}(k)}{\psi_{J+1}(k)}} &\overset{(19)}{=} s_2(n+J+1,k)\,G_{\frac{a_J(k)\xi_J(k+1)}{b_J(k)\xi_J(k)}}\circ\frac{1}{s_2(n+J+1,k)}\\
&\overset{(18)}{=} \frac{s_2(n+J+1,k)}{\xi_J(k)}G_{\frac{a_J(k)}{b_J(k)}}\circ\frac{\xi_J(k)}{s_2(n+J+1,k)} && (26)\\
&\overset{(17)}{=} \frac{s_2(n+J+1,k)}{\xi_J(k)}G_{a_J(k),b_J(k)}\circ\frac{\xi_J(k)}{s_2(n+J+1,k)b_J(k-1)}.
\end{aligned}$$

Additionally,

$$G_{\frac{\varphi_{J+1}(k)p_{J+1}(k+1)}{\psi_{J+1}(k)p_{J+1}(k)}} \overset{(18)}{=} \frac{1}{p_{J+1}(k)}G_{\frac{\varphi_{J+1}(k)}{\psi_{J+1}(k)}}\circ p_{J+1}(k). \qquad (27)$$

Consequently, the substitution of the right side of $\overset{(17)}{=}$ in (26) into (27) yields

$$G_{\frac{\varphi_{J+1}(k)p_{J+1}(k+1)}{\psi_{J+1}(k)p_{J+1}(k)}} = \frac{s_2(n{+}J{+}1,k)}{p_{J+1}(k)\xi_J(k)}G_{a_J(k),b_J(k)}\circ\frac{p_{J+1}(k)\xi_J(k)}{s_2(n{+}J{+}1,k)b_J(k{-}1)}. \qquad (28)$$

Up to this point, we have represented

$$G_{\frac{\varphi_{J+1}(k)p_{J+1}(k+1)}{\psi_{J+1}(k)p_{J+1}(k)}} \qquad (29)$$

in terms of $G_{a_J(k),b_J(k)}$. In order to establish relations between $G_{a_J(k),b_J(k)}$ and $G_{a_{J+1}(k),b_{J+1}(k)}$, we now represent (29) in terms of $G_{a_{J+1}(k),b_{J+1}(k)}$ where $a_{J+1}(k), b_{J+1}(k)$ are such that

$$\frac{a_{J+1}(k)}{b_{J+1}(k)}\,\frac{c_{J+1}(k+1)}{c_{J+1}(k)}$$

[1]The relation $A \overset{(N)}{=} B$ means that the right hand side B is derived from the left hand side A using relation (N).

is a PNF$_k$ of $v_{J+1}(k)/w_{J+1}(k)$. Following notation in (21), we have

$$\frac{v_{J+1}(k)}{w_{J+1}(k)} = \frac{\varphi_{J+1}(n,k)}{\psi_{J+1}(n,k)}\,\frac{p_{J+1}(A_0,\ldots,A_{J+1},n,k+1)}{p_{J+1}(A_0,\ldots,A_{J+1},n,k)},$$

and a PNF$_k$ of $\varphi_{J+1}(k)/\psi_{J+1}(k)$:

$$\frac{a_{J+1}(k)}{b_{J+1}(k)}\,\frac{\xi_{J+1}(k+1)}{\xi_{J+1}(k)}. \tag{30}$$

We have

$$\begin{aligned}
G_{\frac{\varphi_{J+1}(k)p_{J+1}(k+1)}{\psi_{J+1}(k)p_{J+1}(k)}} &\overset{(27)}{=} \frac{1}{p_{J+1}(k)}G_{\frac{\varphi_{J+1}(k)}{\psi_{J+1}(k)}} \circ p_{J+1}(k) \\
&\overset{(30)}{=} \frac{1}{p_{J+1}(k)}G_{\frac{a_{J+1}(k)\xi_{J+1}(k+1)}{b_{J+1}(k)\xi_{J+1}(k)}} \circ p_{J+1}(k) \\
&\overset{(18)}{=} \frac{1}{p_{J+1}(k)\xi_{J+1}(k)}G_{\frac{a_{J+1}(k)}{b_{J+1}(k)}} \circ p_{J+1}(k)\xi_{J+1}(k).
\end{aligned}$$

Hence, by (17)

$$G_{\frac{\varphi_{J+1}(k)p_{J+1}(k+1)}{\psi_{J+1}(k)p_{J+1}(k)}} = \frac{1}{p_{J+1}(k)\xi_{J+1}(k)}G_{a_{J+1}(k),b_{J+1}(k)} \circ \frac{p_{J+1}(k)\xi_{J+1}(k)}{b_{J+1}(k-1)}. \tag{31}$$

Proposition 3.1 *The operators $G_{a_J(k),b_J(k)}$ and $G_{a_{J+1}(k),b_{J+1}(k)}$ for $J \in \mathbb{N}$ are related by the following recurrences:*

$$G_{a_{J+1}(k),b_{J+1}(k)} = \frac{s_2(n+J+1,k)\xi_{J+1}(k)}{\xi_J(k)}G_{a_J(k),b_J(k)} \circ \frac{\xi_J(k)}{s_2(n+J+1,k)\xi_{J+1}(k)}, \tag{32}$$

$$G_{a_J(k),b_J(k)} = \frac{\xi_J(k)}{s_2(n+J+1,k)\xi_{J+1}(k)}G_{a_{J+1}(k),b_{J+1}(k)} \circ \frac{s_2(n+J+1,k)\xi_{J+1}(k)}{\xi_J(k)}. \tag{33}$$

Proof: By a comparison between (28) and (31). ∎

The recurrences (32) and (33) are important in the next subsection.

3.4 Polynomial simplification

At step J of the item-by-item examination, it follows from (12) and (16) that the recurrence

$$G_{a_J(k),b_J(k)}y(k) = c_J(k), \tag{34}$$

where $c_J(k) = \xi_J(k)p_J(k)$, $J \in \mathbb{N}$ is considered. By (22)

$$\begin{aligned}
\xi_{J+1}(k)p_{J+1}(k) &= \xi_{J+1}(k)\Bigg(p_J(A_0,\ldots,A_J,n,k)s_2(n+J+1,k)+ \\
&\qquad A_{J+1}\prod_{i=1}^{J+1} s_1(n+i,k)\Bigg).
\end{aligned}$$

Therefore,

$$c_{J+1}(k) = \frac{\xi_{J+1}(k)}{\xi_J(k)} s_2(n+J+1,k)c_J(k) + \xi_{J+1}(k)A_{J+1}\prod_{i=1}^{J+1} s_1(n+i,k). \quad (35)$$

If the right hand side of the J-th recurrence (34) is simplified by means of a polynomial $f_J(k)$, then the right-hand side becomes c'_J where

$$c'_J = c_J - G_{a_J,b_J} f_J. \quad (36)$$

It follows from (22), (33) and (36) that

$$\begin{aligned}\frac{\xi_{J+1}(k)}{\xi_J(k)} s_2(n+J+1,k)c'_J(k) &= \xi_{J+1}(k)s_2(n+J+1,k)p_J(k) - \\ &\quad G_{a_{J+1},b_{J+1}} \frac{s_2(n+J+1,k)\xi_{J+1}(k)}{\xi_J(k)} f_J(k).\end{aligned}$$

This means that the change of c_J by c'_J in the right-hand side of (35) is equal to the change of c_{J+1} by

$$\tilde{c}_{J+1} = c_{J+1} - G_{a_{J+1},b_{J+1}} \frac{s_2(n+J+1,k)\xi_{J+1}}{\xi_J} f_J.$$

Once we found a polynomial g_{J+1} such that for

$$c'_{J+1} = \tilde{c}_{J+1} - G_{a_{J+1},b_{J+1}} g_{J+1},$$

we have $\deg_k c'_{J+1} < \deg_k c_{J+1}$, then set

$$f_{J+1} = \frac{s_2(n+J+1,k)\xi_{J+1}}{\xi_J} f_J + g_{J+1}. \quad (37)$$

Suppose $\deg_k c_J - \deg_k c'_J = m > 0$. Then evidently

$$\deg_k \frac{s_2(n+J+1,k)\xi_{J+1}}{\xi_J} c_J - \deg_k \frac{s_2(n+J+1,k)\xi_{J+1}}{\xi_J} c'_J = m. \quad (38)$$

Therefore, if we use c'_J instead of c_J in (35) to construct the right-hand side of the $(J{+}1)$-th equation from (34), then we decrease by m the high degrees of the monomials that do not depend on A_{J+1}. Thereby we increase by m the $(J{+}1)$-increment of the system of linear algebraic equations generated by $\mathcal{Z}$ at step $(J{+}1)$, and Theorem 2.2 is proven.

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