MULTICOMPLEXES AND POLYNOMIALS WITH REAL ZEROS.

JASON BELL AND MARK SKANDERA

ABSTRACT. We show that each polynomial $a(z) = 1 + a_1z + \cdots + a_dz^d$ in $\mathbb{N}[z]$ having only real zeros is the f-polynomial of a multicomplex. Evidently $a(z)$ is also the h-polynomial of a Cohen-Macaulay ring and is the g-polynomial of a simplicial polytope.

RÉSUMÉ. Nous montrons que chaque polynôme $a(z) = 1 + a_1z + \cdots + a_dz^d$ dans $\mathbb{N}[z]$ qui ne possède que des zéros réelles c'est le polynôme-f d'un multicomplexe. Évidemment $a(z)$ est aussi le polynôme-h d'un anneau de Cohen-Macaulay et est le polynôme-g d'un polytope simplicial.

1. Introduction

Several interesting results and open questions in algebraic combinatorics concern simplicial complexes and polynomials

$$a(z) = 1 + a_1z + \cdots + a_dz^d \in \mathbb{N}[z] \tag{1.1}$$

having only real zeros. A few examples are the following.

1. The f-polynomial of a $(\mathbf{3}+\mathbf{1})$-free poset has only real zeros [5], [14], [17].
2. The f-polynomial of a matching complex has only real zeros [8].
3. The f-polynomial of a distributive lattice is conjectured to have only real zeros [12].
4. If P is a series-parallel poset, then the f-polynomial of the distributive lattice $J(P)$ has only real zeros [19].
5. The question of whether the f-polynomial of a modular lattice has only real zeros is open [18].

Progress on the open questions above and on related open questions is obstructed somewhat by the lack of a known combinatorial interpretation for the coefficients of a polynomial (1.1) having only real zeros. A particularly nice combinatorial interpretation might involve faces in a simplicial complex.

Question 1.1. Let the polynomial $a(z) = 1+a_1z+\cdots+a_dz^d$ have nonnegative integer coefficients and only real zeros. Is $a(z)$ the f-polynomial of a simplicial complex?

The more general class of multicomplexes might also provide a combinatorial interpretation. In Section 2 we will define the f-polynomials of simplicial complexes and multicomplexes, and we will summarize the well-known characterizations of these

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polynomials. In Section 3 we will state inequalities satisfied by the coefficients of polynomials with real zeros. These inequalities lead to a proof in Section 4 that each polynomial (1.1) having only real zeros is the f-polynomial of a multicomplex.

2. Characterization of the f-vectors of multicomplexes and simplicial complexes

A *multicomplex* on a set $\{x_1, \ldots, x_n\}$ of variables is a collection Σ of monomials in $x_1, \ldots, x_n$ which satisfies

1. The monomial x_i belongs to Σ, for $i = 1, \ldots, n$.
2. If the monomial u belongs to Σ and w divides u, then w also belongs to Σ.

A multicomplex Σ is called a *simplicial complex* if each monomial in Σ is square-free.

Let Σ be a multicomplex on $x_1, \ldots, x_n$. We define the *f-vector* of Σ to be the sequence

$$f_\Sigma = (a_i)_{i \geq 0}, \tag{2.1}$$

where a_i is the number of monomials of degree i in Σ. Note that we necessarily have $a_0 = 1$, unless $n = 0$. Also note that the f-vector of a simplicial complex has only finitely many nonzero components.

Two well-known theorems characterize the f-vectors of multicomplexes and simplicial complexes in terms of functions based upon the following expression of a positive integer m as a sum of binomial coefficients. Given a positive integer i, we define the *ith Macaulay expansion* of m to be the unique expression

$$m = \binom{n_i}{i} + \binom{n_{i-1}}{i-1} + \cdots + \binom{n_j}{j}, \tag{2.2}$$

satisfying

$$n_i > n_{i-1} > \cdots > n_j \geq j \geq 1.$$

To obtain this expression we choose n_i to be the unique positive integer which satisfies

$$\binom{n_i}{i} \leq m < \binom{n_i + 1}{i},$$

and then we compute the $(i-1)$st Macaulay expansion of $m - \binom{n_i}{i}$.

We define the families $(\mu_i)_{i \geq 1}$, $(\kappa_i)_{i \geq 1}$ of functions on $\mathbb{N}$ by

$$\mu_i(m) = \begin{cases} \binom{n_i+1}{i+1} + \binom{n_{i-1}+1}{i} + \cdots + \binom{n_j+1}{j+1} & \text{if } m > 0, \\ 0 & \text{otherwise.} \end{cases}$$

$$\kappa_i(m) = \begin{cases} \binom{n_i}{i+1} + \binom{n_{i-1}}{i} + \cdots + \binom{n_j}{j+1} & \text{if } m > 0, \\ 0 & \text{otherwise.} \end{cases}$$

The characterization of f-vectors of multicomplexes is due to Macaualay [11].

Theorem 2.1. *An integer sequence* $(a_0, a_1, \dots)$ *is the* f*-vector of a nonempty multicomplex on* n *variables if and only if we have* $a_0 = 1$, $a_1 = n$ *and*

$$0 \leq a_{i+1} \leq \mu_i(a_i)$$

for $i \geq 1$.

The characterization of f-vectors of simplicial complexes is due (independently) to Kruskal [10], Katona [9], and Schützenberger [13].

Theorem 2.2. *An integer sequence* $(a_0, \dots, a_d)$ *is the* f*-vector of a nonempty simplicial complex on* n *variables if and only if we have* $a_0 = 1$, $a_1 = n$ *and*

$$0 < a_{i+1} \leq \kappa_i(a_i)$$

for $i = 1, \dots, d-1$.

(See [3], [6] for proofs of these theorems.)

The functions μ_i and κ_i may be expressed in terms of one another very easily.

Proposition 2.3. *For any positive integers* m, i, *we have*

$$\kappa_i(m) + m = \mu_i(m).$$

Proof. Omitted. □

It is customary to define the f-vector of a finite multicomplex Σ to be only the nonzero subsequence of the sequence (2.1),

$$f_\Sigma = (a_0, \dots, a_d).$$

We then define the *f-polynomial* of Σ to be

$$f_\Sigma(z) = a_0 + a_1 z + \cdots + a_d z^d.$$

We may also associate f-vectors and f-polynomials to posets. In particular, the set of chains of a poset P forms a simplicial complex $\mathcal{O}(P)$ called the *order complex* of P. (See [16, Ch. 3].) We then define the f-vector f_P and f-polynomial $f_P(z)$ of P to be $f_{\mathcal{O}(P)}$ and $f_{\mathcal{O}(P)}(z)$, respectively.

Multicomplexes have an important interpretation in commutative algebra: if R is a graded k-algebra generated by elements $x_1, \dots, x_n$, then R has a k-basis which is a multicomplex on $x_1, \dots, x_n$. Furthermore, $a(z)$ is the f-polynomial of a finite multicomplex if and only if for any nonnegative integer c there exists a c-dimensional Cohen-Macaulay ring whose Hilbert series is

$$\frac{a(z)}{(1-z)^c}.$$

(See [15, pp. 56-57].)

3. Inequalities pertaining to polynomials with real zeros

Let the polynomial $a(z) = 1 + a_1 z + \cdots + a_d z^d$ in $\mathbb{R}[z]$ have positive coefficients. Conditions on the sequence $(1, a_1, \ldots, a_d)$ which are both necessary and sufficient for $a(z)$ to have only real zeros are known but somewhat cumbersome. (See e.g., [1, Thm. 1], [4, p. 203].) On the other hand, several well-known conditions which are merely necessary are quite simple. In the event that $a(z)$ has only real zeros, the sequence $(1 = a_0, \ldots, a_d)$ is *unimodal*,

$$1 \leq \cdots \leq a_j \geq \cdots \geq a_d \quad \text{for some } j,$$

and *log-concave*,

$$a_i^2 \geq a_{i-1} a_{i+1} \quad \text{for } i = 1, \ldots, d-1.$$

It also has Newton's log-concavity property,

$$\frac{a_i^2}{\binom{d}{i}^2} \geq \frac{a_{i-1}}{\binom{d}{i-1}} \frac{a_{i+1}}{\binom{d}{i+1}} \quad \text{for } i = 1, \ldots, d-1, \tag{3.1}$$

from which one can derive Maclaurin's inequalities [7, p. 52],

$$\frac{a_1}{d} \geq \sqrt{\frac{a_2}{\binom{d}{2}}} \geq \sqrt[3]{\frac{a_3}{\binom{d}{3}}} \geq \cdots \geq \sqrt[d]{a_d}. \tag{3.2}$$

Note that we may interpret (3.2) as a generalization of the Artithmetic Mean - Geometric Mean Inequality by factoring $a(z)$ as $(1+\beta_1 z) \cdots (1+\beta_d z)$. From Maclaurin's inequalities we obtain the following upper bound for each coefficient a_i in terms of a_1.

Observation 3.1. *Let $a(z) = 1 + a_1 z + \cdots + a_d z^d \in \mathbb{R}[z]$ have positive coefficients and only real zeros. Then for $i = 2, \ldots, d$ we have*

$$a_i \leq \binom{d}{i} \left(\frac{a_1}{d}\right)^i.$$

Setting $i = d$ in Observation 3.1 and assuming that all coefficients are integers, we obtain an upper bound on the degree in terms of a_1.

Observation 3.2. *Let $a(z) = 1 + a_1 z + \cdots + a_d z^d \in \mathbb{N}[z]$ have only real zeros. Then d is no greater than a_1.*

The combination of these two facts yields a third.

Observation 3.3. *For any fixed c there are only finitely many polynomials of the form $1 + cz + a_2 z^2 \cdots + a_d z^d$ in $\mathbb{N}[z]$ which have only real zeros.*

Maclaurin's inequalities also give us a lower bound for each coefficient a_i in terms of a_d. In particular we have the following.

Observation 3.4. *Let* $a(z) = 1 + a_1 z + \cdots + a_d z^d \in \mathbb{N}[z]$ *have only real zeros. Then for* $i = 1, \ldots, d-1$ *we have*

$$a_i \geq \binom{d}{i}.$$

Thus it is easy to see that a polynomial such as $1 + 4z + 9z^2 + 10z^3 + 5z^4 + z^5$ has nonreal zeros.

A very different consequence of Maclaurin's inequalties relates polynomials with real zeros to the Upper Bound Conjecture for f-vectors of simplicial convex polytopes. (See [15, p. 59] for definitions.)

Proposition 3.5. *Let* $a(z) = 1 + a_1 z + \cdots + a_d z^d \in \mathbb{N}[z]$ *have only real zeros and let* $f = (f_{-1}, f_0, \ldots, f_{d-1})$ *be the* f*-vector of the cyclic polytope* $C(a_1, d)$*. Then for* $i = 1, \ldots, d$ *we have*

$$a_i \leq f_{i-1}.$$

Proof. Define the polynomial

$$b(z) = (1 + \tfrac{a_1}{d} z)^d = b_0 + b_1 z + \cdots + b_d z^d.$$

By (3.2) a_i is no greater than b_i for $i = 1, \ldots, d$. Therefore it suffices to show that b_i is no greater than f_{i-1} for $i = 1, \ldots, d$.

By a result of McMullen (see [15, p. 59]), the coefficients of $b(z)$ satisfy the conditions of the Upper Bound Conjecture if the coefficients of the polynomial

$$h(z) = (1 + \tfrac{a_1 - d}{d} z)^d = 1 + h_1 z + \cdots + h_d z^d$$

satisfy

$$h_i \leq \binom{a_1 - d + i - 1}{i}.$$

Computing an upper bound for h_i we have

$$h_i = \binom{d}{i}\left(\frac{a_1 - d}{d}\right)^i = \frac{d(d-1)\cdots(d-i+1)(a_1-d)^i}{i!\, d^i} \leq \frac{(a_1-d)^i}{i!},$$

which is clearly less than or equal to

$$\binom{a_1 - d + i - 1}{i} = \frac{1}{i!}\prod_{j=0}^{i-1}(a_1 - d + j).$$

□

Another consequence of Maclaurin's inequalities is a family of inequalities satisfied by the minors of totally nonnegative matrices. Denote by $\binom{[n]}{k}$ the collection of k-element subsets of $[n] = \{1, \ldots, n\}$. For any matrix A of size at least $n \times n$ and any elements S, T of $\binom{[n]}{k}$ define $\Delta_{S,T}$ to be the S, T *minor* of A, the determinant of the submatrix of A corresponding to rows S and columns T. A matrix is called *totally nonnegative* if all of its minors are nonnegative.

Proposition 3.6. *Let A be an $n \times n$ totally nonnegative matrix and let $k < \ell$ be two integers in $[n]$. Then we have*

$$\binom{n}{k}^{\ell} \Big(\sum_{S \in \binom{[n]}{\ell}} \Delta_{S,S} \Big)^{k} \leq \binom{n}{\ell}^{k} \Big(\sum_{S \in \binom{[n]}{k}} \Delta_{S,S} \Big)^{\ell}.$$

Proof. Suppose A is totally nonnegative. A well-known result states that A has only nonnegative real eigenvalues and therefore that the polynomial

$$\det(Az + I) = 1 + a_1 z + \cdots + a_n z^n$$

has only negative real zeros. Since these coefficients are given by

$$a_i = \sum_{S \in \binom{[n]}{i}} \Delta_{S,S},$$

we may apply (3.2) to obtain the desired result. □

4. MAIN RESULT

In comparing the functions obtained in Section 3 with those described in Section 2, it will be convenient to consider the expression

$$\binom{t}{i} = \frac{t(t-1)\cdots(t-i+1)}{i!}$$

to be a function of a real variable t for any nonnegative integer i. In particular we shall use $\binom{t}{i}$ to define the following function.

Lemma 4.1. *Let i be a positive integer. The real function*

$$\frac{\binom{t}{i+1}}{\binom{t}{i}^{\frac{i+1}{i}}}$$

increases with t on the interval $[i, \infty)$.

Proof. Omitted. □

Combining Lemma 4.1 with (3.2), we may now relate polynomials with real zeros to f-polynomials of multicomplexes.

Theorem 4.2. *Let $a(z) = 1 + a_1 z + \cdots + a_d z^d \in \mathbb{N}[z]$ have only real zeros. Then $a(z)$ is the f-polynomial of a multicomplex.*

Proof. Choose an integer i between 1 and $d-1$. By (3.2) we have

$$a_{i+1} \leq \binom{d}{i+1} \left(\frac{a_i}{\binom{d}{i}} \right)^{(i+1)/i}. \tag{4.1}$$

Now define n_i to be the unique nonnegative integer which satisfies

$$\binom{n_i}{i} \leq a_i < \binom{n_i+1}{i}.$$

Combining this inequality with Observation 3.4, we obtain

$$\binom{d}{i} \leq a_i < \binom{n_i+1}{i},$$

which implies that $n_i + 1$ is greater than d. We may therefore apply Lemma 4.1 to replace d by $n_i + 1$ in (4.1),

$$a_{i+1} < \binom{n_i+1}{i+1} \left(\frac{a_i}{\binom{n_i+1}{i}} \right)^{(i+1)/i}.$$

Since a_i is less than $\binom{n_i+1}{i}$, we also have

$$a_{i+1} < \binom{n_i+1}{i+1},$$

which clearly implies that a_{i+1} is less than $\mu_i(a_i)$. □

Corollary 4.3. *Let $a(z) = 1 + a_1 z + \cdots + a_d z^d \in \mathbb{N}[z]$ have only real zeros. Then for any $c \in \mathbb{N}$ there exists a Cohen-Macaulay ring whose Hilbert series is the rational function*

$$\frac{a(z)}{(1-z)^c}.$$

Equivalently, $a(z)$ is the h-vector of a Cohen-Macaulay complex.

A second consequence of Theorem 4.2 concerns simplicial polytopes. (See [2] for definitions.)

Corollary 4.4. *Let $a(z) = 1 + a_1 z + \cdots + a_d z^d \in \mathbb{N}[z]$ have only real zeros. Then for any $c \in \mathbb{N}$ greater than or equal to $2d$, there exists a simplicial c-polytope whose g-polynomial is $a(z)$.*

It would be interesting to strengthen Theorem 4.2 to provide an affirmative answer to Question 1.1. Such an answer seems plausible because if the polynomial

$$a(z) = 1 + a_1 z + \cdots + a_d z^d \in \mathbb{N}[z] \tag{4.2}$$

has only real zeros, then the function

$$\binom{d}{i+1} \left(\frac{a_i}{\binom{d}{i}} \right)^{(i+1)/i}$$

that bounds a_{i+1} is less than $\kappa_i(a_i)$ for a_i large enough. (For instance if a_i is at least $i^2 \binom{d}{i}$.) It follows that for fixed d, at most finitely many polynomials of the form (4.2) have only real zeros and are not f-polynomials of simplicial complexes. It is also possible to show that when $d \leq 4$ or $a_1 \leq 10$, all polynomials of the form (4.2) which have only real zeros are f-polynomials of simplicial complexes.

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Authors' address: University of Michigan, Dept. of Mathematics, 2074 East Hall, 525 East University, Ann Arbor, MI 48109-1109

E-mail address: belljp@umich.edu, mskan@umich.edu