ON A NATURAL CORRESPONDENCE BETWEEN BASES AND REORIENTATIONS, RELATED TO THE TUTTE POLYNOMIAL AND LINEAR PROGRAMMING, IN GRAPHS, HYPERPLANE ARRANGEMENTS, AND ORIENTED MATROIDS

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ABSTRACT. A comparison of two expressions of the Tutte polynomial of an ordered oriented matroid yields remarkable numerical relations between the numbers of bases and reorientations with given activities. We address here the bijection problem for these relations, by constructing a natural activity preserving correspondence with suitable multiplicities between bases and reorientations, called the *canonical active (basis-reorientation) correspondence*.

A decomposition of activities is used, reducing the problem to situations with one activity equal to 1 and the other equal to 0. This decomposition is closely related to a new expression of the Tutte polynomial in terms of beta invariants of minors.

This canonical active correspondence has strong duality properties, and can be constructed inductively using minors with respect to the greatest element. Furthermore, it can be refined into an *active bijection* between all subsets of elements, inducing an active bijection between faces of the NBC complex of the matroid and regions of the oriented matroid.

In the graphical case, we get *active bijections* between spanning trees and activity classes of orientations, resp. acyclic orientations with a unique sink at a given vertex, resp. acyclic orientations with adjacent unique source and unique sink at given vertices.

For the regions of an hyperplane arrangement, we get an *active bijection* between certain simplices and activity classes of regions. Its restriction to simplices with (1,0)-activities and bounded regions is a bijection. If the hyperplanes are in general position, the bijection can be obtained by maximizing or minimizing a same linear function over all bounded regions.

In general, we get extensions of linear and oriented matroid programming: each reorientation is decomposed into bounded regions, and for each bounded region, instead of optimizing a face with respect to one objective function, we optimize a sequence of nested faces with respect to a sequence of objective functions.

RÉSUMÉ. Une comparaison de deux expressions du polynôme de Tutte d'un matroïde orienté ordonné fournit des relations numériques remarquables entre les nombres de bases et de réorientations d'activités données. On résoud ici le problème d'une bijection pour ces relations, en construisant une correspondance naturelle, préservant les activités, avec des multiplicités convenables, entre les bases et les réorientations, appelée *correspondance (bases-réorientations) active canonique*.

On utilise une décomposition des activités pour réduire le problème à des situations où une activité est égale à 1 et l'autre à 0. Cette décomposition est étroitement liée à une nouvelle expression du polynôme de Tutte en termes d'invariants béta des mineurs.

La correspondance active canonique a de fortes propriétés de dualité, et peut être construite inductivement en utilisant les mineurs relativement au plus grand élément. De plus, on peut la raffiner en une *bijection active* entre tous les sous-ensembles d'éléments, induisant une bijection active entre les faces du complexe NBC du matroïde et les régions du matroïde orienté.

Dans le cas graphique, on obtient des *bijections actives* entre les arbres couvrants et les classes d'activités d'orientations resp. les orientations acycliques avec un unique puits fixé, ou les orientations acycliques avec une unique source et un unique puits adjacents fixés.

Pour les régions d'un arrangement d'hyperplans, on obtient une *bijection active* entre certains simplexes et des classes d'activités de régions. Sa restriction aux simplexes d'activités (1,0) et aux régions bornées est une bijection. Si les hyperplans sont en position générale, cette bijection s'obtient en maximisant ou minimisant une même forme linéaire pour toutes les régions bornées.

En général, on obtient des extensions de la programmation linéaire et de la programmation dans les matroïdes orientés: chaque réorientation est décomposée en régions bornées, et pour chaque région bornée, au lieu d'optimiser une face pour une fonction objective on optimise une suite de faces emboîtées relativement à une suite de fonctions objectives.

KEYWORDS: matroid, oriented matroid, Tutte polynomial, basis, reorientation, activity, orientation, graph, directed graph, spanning tree, source, sink, acyclic, bijective proof, pseudoline arrangement, hyperplane arrangement, bounded region, linear programming, flag programming.

1. Introduction

The *Tutte polynomial* of a matroid is a 2-variable polynomial invariant, introduced for graphs by W.T. Tutte [Tu54], and generalized to matroids by H.H. Crapo [Cr69]. Up to simple algebraic transformations, the Tutte polynomial of a matroid is equivalent to its *rank-generating function*, i.e. to the generating function of cardinality and rank of subsets of elements. The Tutte polynomial is a fundamental tool in the theory of numerical invariants of matroids, and has useful enumerative properties and numerous applications. We refer the reader to Section 2 for relevant definitions, and to [BrOx92] for an extensive survey on the subject.

Let M be a matroid on a linearly ordered set of elements E. By a classical theorem proved by W.T. Tutte for graphs [Tu54], and extended to matroids by H.H. Crapo [Cr69], we have

$$t(M;x,y)=\sum_{i,j} b_{i,j}x^iy^j$$

where $b_{i,j}$ is the number of bases of M such that i basis elements are smallest in their fundamental cocircuit and j non-basis elements smallest in their fundamental circuit.

On the other hand, if M is an oriented matroid, M. Las Vergnas has shown in [LV84] that

$$t(M;x,y)=\sum_{i,j} o_{i,j}2^{-i-j}x^iy^j$$

where $o_{i,j}$ is the number of reorientations of M with exactly i elements smallest in some positive cocircuit and j elements smallest in some positive circuit.

This formula contains several results of the literature (see below, Section 2). Comparing the above two expressions for $t(M;x,y)$, we get the relation

$$o_{i,j}=2^{i+j}b_{i,j}$$

for all i,j. A natural question arises of a bijective proof for these formulas. The problem is to define a correspondence between bases and reorientations, preserving parameters (i,j), called *activities*, and compatible with the above formulas. More precisely, the desired correspondence should associate with a (i,j)-*active* basis of M, a set of 2^{i+j} (i,j)-active reorientations, in such a way that each reorientation of M is in the image of a unique basis.

The construction of a natural correspondence with these properties in general oriented matroids, called the *canonical active basis-reorientation correspondence*, is described into details in [Gi02], and will be the object of a forthcoming series of papers [GiLV]. Two other papers deal with special cases: graphs in [GiLV02], uniform and rank 3 oriented matroids in [GiLV03].

In the present survey, we sketch the construction of the canonical active correspondence. We give the two converse algorithms defining it, its fundamental properties and some significant illustrations.

2. Preliminaries

Let M be a matroid on a set of elements E, and $B\subseteq E$ be a basis of M. For $e\in E\setminus B$, we denote by $C(B;e)$ the *fundamental circuit* of e with respect to B, i.e. the unique circuit contained in $B\cup\{e\}$. Dually, for $e\in B$, we denote by $C^*(B;e)$

the *fundamental cocircuit* of e with respect to B, i.e. the unique cocircuit contained in $(E \setminus B) \cup \{e\}$. For $e \in E \setminus B$ and $e' \in B$, we have clearly $e' \in C(B;e)$ if and only if $e \in C^*(B;e')$, and then $C(B;e) \cap C^*(B;e') = \{e, e'\}$.

We say that a matroid M is *ordered* if its set of elements E is linearly ordered. The notion of *activities* of a basis B in an ordered matroid M is essentially due to W.T. Tutte in the case of graphs [Tu54]. The *internal activity* $\iota(B)$ is the number of elements $e \in B$ smallest in their fundamental cocircuit $C^*(B;e)$, and the *external activity* $\epsilon(B)$ is the number of elements $e \in E \setminus B$ smallest in their fundamental circuit $C(B;e)$. We say that a basis B with $\iota(B) = i$ and $\epsilon(B) = j$ is an (i,j)-*basis*. We denote by $b_{i,j}(M)$ the number of (i,j)-bases of M.

Spanning tree activities have been introduced by Tutte to generalize, in a self-dual way, classical properties of the chromatic polynomial of a graph [Tu54]. The theorem for graphs extends to matroids [Cr69], we have

$$t(M;x,y) = \sum_{i,j} b_{i,j} x^i y^j$$

This expression readily implies that the coefficients $b_{i,j}$ are independent from the ordering of E. In fact, originally, the Tutte polynomial of a matroid is defined by the closed formula

$$t(M;x,y) = \sum_{A \subseteq E} (x-1)^{r(M)-r_M(A)} (y-1)^{|A|-r_M(A}$$

algebraically equivalent to the *rank generating function* of the matroid, and the above formula is proved by deletion/contraction of the greatest element (see [BrOx92]). A classical inductive definition of the Tutte polynomial is given by the following deletion/contraction relations:

if $e \in E$ is not a loop nor an isthmus then $t(M;x,y) = t(M/e;x,y) + t(M \setminus e);x,y)$
if $e \in E$ is an isthmus then $t(M;x,y) = x\ t(M/e;x,y) = x\ t(M \setminus e);x,y)$
if $e \in E$ is a loop then $t(M;x,y) = y\ t(M/e;x,y) = y\ t(M \setminus e);x,y)$
if $E = \emptyset$ then $t(\emptyset;x,y) = 1$

For usual definitions on oriented matroids, the reader is referred to [OM]. If the matroid M is oriented for $e \in E \setminus B$, we denote by $C(B;e)$ the unique signed circuit C contained in $B \cup \{e\}$ such that $e \in C^+$, and dually for $e \in B$, we denote by $C^*(B;e)$ the unique signed cocircuit D contained in $(E \setminus B) \cup \{e\}$ such that $e \in D^+$. We recall that two signed subsets X, Y are said *conformal* if their signs agree on their intersection.

An oriented matroid is *acyclic* if it contains no positive circuit, or equivalently, if every element is contained in a positive cocircuit. Dually, an oriented matroid is *totally cyclic* if it contains no positive cocircuit, or equivalently, if every element is contained in a positive circuit. An oriented matroid is acyclic if and only if the dual oriented matroid is totally cyclic.

A basic result in the domain of the present paper, is a theorem due to R. Stanley (1973): the number of acyclic orientations of a graph G is equal to $t(C(G);2,0)$, where $C(G)$ is the cycle matroid of G [St73]. This theorem has been generalized independently in 1975 by T. Zaslavsky to real spaces in terms of hyperplane arrangements [Za75] (see also [BrLu76]), and by M. Las Vergnas to oriented matroids [LV75] (see also [LV80]).

The paper [LV84] introduces a generalization of these results in terms of an *orientation generating function.* The *(primal) orientation activity* of an ordered oriented matroid M, or *O-activity*, denoted by $o(M)$, is the number of elements smallest in some directed circuit. The *dual orientation activity* of M, or O^**-activity*, denoted by $o^*(M)$, is the number of elements smallest in some directed cocircuit. We denote by $o_{i,j}(M)$ the number of subsets $A \subseteq E$ such that $o^*(-_AM) = i$ and $o(-_AM) = j$, where $-_AM$ denotes the *reorientation* of M obtained by reversing signs on A (note that this notation differs slightly from the notation $_{\overline{A}}M$ used in [3]). We say that a reorientation A such that $o^*(-_AM) = i$ and $o(-_AM) = j$ is a (i,j)*-reorientation.* The definitions of O- and O^*-activities have been introduced in [LV84] in relation with the formula

$$t(M; x, y) = \sum_{i,j} o_{i,j} 2^{-i-j} x^i y^j$$

This formula implies that $o_{i,j}$ does not depend on the ordering. The proof in [LV84] is by deletion/contraction of the greatest element. Note that $\sum_i o_{i,0}$ is the number of acyclic reorientations of M, hence the above formula generalizes results of [BrLu76][LV75][St73][Za75].

It follows from the comparison of the above two state models for the Tutte polynomial that

$$o_{i,j} = 2^{i+j} b_{i,j}$$

In particular we get the equality $o_{1,0} = 2b_{1,0}$. This special case is originally due to C. Greene and T. Zaslavsky [GrZa83] for acyclic orientations of graphs with adjacent unique source and sink (see also [GeSa00]), or bounded regions in real spaces, a result generalized in [LV77] to oriented matroids.

Parts of the present paper use the topological representation of oriented matroids. We refer the reader to [OM] Chap. 5 for the needed prerequisites. Some notions on linear programming in oriented matroids are also necessary in subsection 3.3. We refer the reader to [OM] Chap. 10.

3. Decomposition of activities

Our purpose in this section is to reduce the general case of activities (i,j) to the case when $(i,j) = (1,0)$ or $(i,j) = (0,1)$. Given a basis B of an ordered matroid, we define minors decomposing its set of elements, such that the bases of these minors induced by B, which partition B, have activities $(1,0)$ or $(0,1)$. Similarly, we define minors of an ordered oriented matroid decomposing its set of elements with $(1,0$- or $(0,1)$-orientation activities. The similarity of these decompositions reduces our main problem - defining the active correspondence - to the particular case of $(1,0)$ activities. This section develops and deepens ideas from [LV83] and [EtLV98].

More precisely, we introduce the notion of *decomposing sequences* of an ordered oriented matroid, from either a basis or a reorientation. A *decomposing sequence* is an increasing sequence of subsets of elements of the matroid. Minors are defined from a decomposing sequence on the differences of two consecutive sets of the sequence: they define an *active partition* of the matroid. The notion of active partition is a refinement of the notion of activities.

In the first part, we define an *active decomposition of a basis* in an ordered matroid. We obtain as corollary an expression of the Tutte polynomial in terms of beta invariants of minors of the matroid.

In the second part, similarly, we define an *active decomposition of a reorientation* of an ordered oriented matroid. The set of 2^{i+j} reorientations obtained by reorienting in all possible ways the $i+j$ parts of the active partition of a reorientation with activities (i,j) is called an *activity class of reorientations*. All reorientations in an activity class have the same active partition. The activity classes constitute a natural partition of the set of reorientations.

The similarity of these two constructions, will be used in a third part to define a general active correspondence by extending active correspondences obtained in the $(1,0)$ case.

3.1. **Activities of bases.** Let M be an ordered matroid on E, and B be a basis of M. Let $Ext(B)=\{a_1<a_2<\ldots<a_\ell\}$ be the set of externally active elements of B ($\ell=\varepsilon_M(B)$). We denote by $C_<(B;e)$ the set of elements $b\in C(B;e)$ with $b<e$,

For $X\subseteq E$ set

$$f^1(X)=f(X)=\ X\ \cup \bigcup_{e\in(E\setminus B)\cap X} C(B;e)\ \cup\ \{\ e\in E\mid\ \emptyset\subset C_<(B;e)\subseteq X\ \}$$

$$f^{i+1}(X)=f(f^i(X))$$

$$\hat f(X)=\bigcup_{i\geq 1} f^i(X)$$

Let $F=\hat f(Ext(B))$, and similarly F^* calculated for $E\setminus B$ in M^*. Set $B'=B\cap F$ and $B''=B\setminus F$, and $M'=M(F)$ and $M''=M/F$.

Proposition 1. $E=F+F^*$
$\iota_{M'}(B')=0$, $\epsilon_{M'}(B')=\epsilon_M(B)$
$\iota_{M''}(B'')=\iota_M(B)$, $\epsilon_{M''}(B'')=0$

For $0\leq i\leq \ell-1$, let $F_i=\hat f(\{a_{i+1},...,a_\ell\}$; we have $\emptyset\subset F_{\varepsilon-1}\subset ...\subset F_1\subset F_0=F$.

For $1\leq i\leq\ell$, set $A_i=F_{i-1}\setminus F_i$, we have $F=A_1+A_2+\ldots+A_\ell$, and $B_i=B\cap A_i$ is a basis of the matroid $M_i=M(F_{i-1})/F_i=M/\sum_{j<i}A_j\setminus\sum_{j>i}A_j$ on A_i, and $Min(A_i)=a_i$.

Proposition 2. *For* $1\leq i\leq \ell$, $\iota_{M_i}(B_i)=0$, $\epsilon_{M_i}(B_i)=1$

For a basis B of the matroid M, with $F_c=F$, $F'_i=F_i$ for $1\leq i\leq\varepsilon_M(B)$, and F''_i the complement of F_i calculated for $E\setminus B$ in M^* for $1\leq i\leq\iota_M(B)$, we have defined the *decomposing sequence associated with B in M* (or with $E\setminus B$ in M^* up to complementarity in E). The partition of E induced by the partitions of F and F^* is called the *active partition* of M with respect to B, and F, resp. F^*, is called the external, resp. internal, part of B.

The active partition of a basis depends only on its fundamental circuits (or cocircuits) but not on the whole matroid. From a constructive point of view, there is an algorithm to compute this sequence of subsets associated with B in a single pass of E.

Proposition 3. *Let B ba a basis of M on $E = e_1 < ... < e_n$, and α be an application from E in E which maps $e \in E$ on the minimal element of its part in the active partition of B. This application is defined by the following algorithm.*

For k from 1 to n do :
If $e_k \notin B$ then :
  if e_k is externally active, then e_k is external and $\alpha(e_k) := e_k$;
  else
    if there exists $c < e_k$ internal in $C(B; e_k)$ then
      e_k is internal
      $\alpha(e_k) := \alpha(c)$ the greatest possible with $c < e_k$ internal in $C(B; e_k)$;
    else
      e_k is external
      $\alpha(e_k) := \alpha(c)$ the smallest possible with c in $C(B; e_k)$.
If $e_k \in B$ then :
  if e_k is internally active, then e_k internal and $\alpha(e_k) := e_k$;
  else
    if there exists $c < e_k$ external in $C^*(B; e_k)$ then
      e_k is external
      $\alpha(e_k) := \alpha(c)$ the greatest possible with c external in $C^*(B; e_k)$;
    else
      e_k is internal
      $\alpha(e_k) := \alpha(c)$ the smallest possible with c in $C^*(B; e_k)$.

3.2. **Activities of reorientations.** Let M be an ordered oriented matroid on E, and $a_1 < a_2 < \ldots < a_\ell$ be the orientation-active elements of M ($\ell = o(M)$).

Let F the union of all positive circuits of M and F^* union of all positive cocircuits of M.

Proposition 'Farkás Lemma for oriented matroids' (see [OM] Cor. 3.4.6). We have $E = F + F^*$

For $0 \leq i \leq \ell - 1$, let F_i' be the union of all positive circuits with smallest element $a \geq a_{i+1}$. We have $\emptyset \subset F_{\varepsilon-1} \subset ... \subset F_1 \subset F_0 = F$.

For $1 \leq i \leq \ell$, set $A_i = F_{i-1} \setminus F_i$. We have $E = A_1 + A_2 + \ldots + A_\ell$. The minor $M_i = M(F_i)/F_{i+1} = M/\sum_{j<i} A_j \setminus \sum_{j>i} A_j$ is an oriented matroid on A_i with $Min(A_i) = a_i$.

Proposition 4. *For $1 \leq i \leq \ell$, we have $o^*(M_i) = 0$, $o(M_i) = 1$*

For an ordered oriented matroid M, with $F_c = F$, $F_i' = F_i$ for $1 \leq i \leq o(M)$, and F_i'' the complementary of the F_i calculated for M^* for $1 \leq i \leq o^*(M)$, we have defined the *decomposing sequence associated with M* (or with M^* up to complementarity in E).

The partition of E induced by the partitions of F and F^* is called the *(orientation) active partition* of M.

Example. Figure 1 shows some regions in an arrangement of rank 4. The active partition associated with a region is written above the region. The sequences of positive cocircuits corresponding to the decomposing sequences of flats used for these regions are drawn in bold. The linear ordering is $1 < 2 < 3 < 4 < a < ... < g$, the minimal basis is 1234.

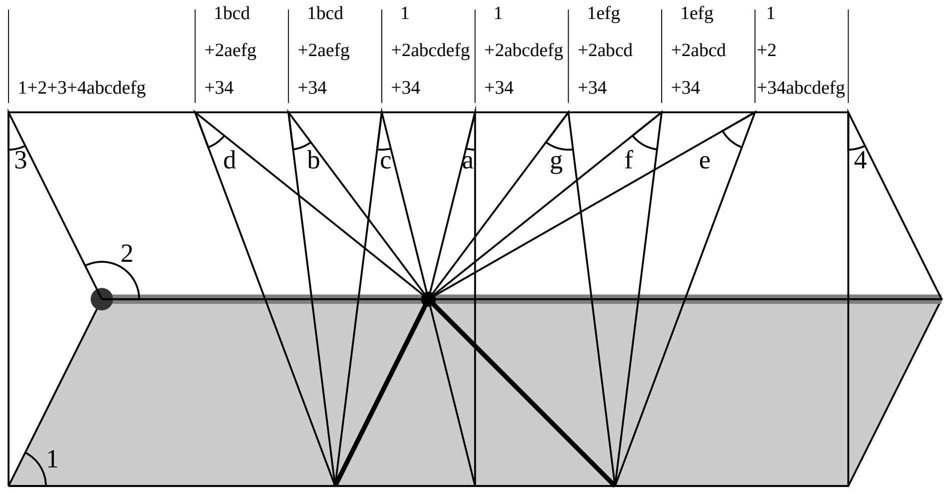

Figure 1

3.3. **Extension from the $(0,1)$ or $(1,0)$ case to the general case.** The sequences of subsets of E defined in the two previous subsections lead to the following definition. The results of this subsection are given quickly, see [GiLV] (or [Gi02] chapter 2) for precise formulations.

Definition. A *decomposing sequence* of an ordered matroid M on E

$$\emptyset = F'_\varepsilon \subset ... \subset F'_0 = F_c = F''_0 \subset ... \subset F''_\iota = E$$

is an increasing sequence of subsets of E satisfying:
- F_c is a cyclic flat of M (i. e. F_c is a flat of M and $E \setminus F_c$ is a flat of M^*) ;
- for all k, $0 \leq k \leq \iota$, F''_k is a flat of M, and $F_c = F''_0 \subset ... \subset F''_\iota = E$;
- for all k, $0 \leq k \leq \epsilon$, $E \setminus F'_k$ is a flat of M^*, and $\emptyset = F'_\varepsilon \subset ... \subset F'_0 = F_c$;
- the sequence $min(F''_k \setminus F''_{k-1})$, $1 \leq k \leq \iota$, is increasing with k;
- the sequence $min(F'_{k-1} \setminus F'_k)$, $1 \leq k \leq \epsilon$, is increasing with k;
- the matroids $M(F''_k)/F''_{k-1}$, $1 \leq k \leq \iota$, and $M(F'_{k-1})/F'_k$, $1 \leq k \leq \epsilon$ are such that $b_{1,0} = b_{0,1} \neq 0$ (or, equivalently, are connected).

A decomposing sequence induces by successive differences a partition of E, which is called *active partition* of E acording to this decomposing sequence:

$$E = F'_1 \setminus F'_0 + ... + F'_\varepsilon \setminus F'_{\varepsilon-1} + F''_1 \setminus F''_0 + ... + F''_\iota \setminus F''_{\iota-1}$$

For an ordered matroid M, when B runs through the set of all bases of M, all the $(0,1)$-active or $(1,0)$-active bases for all minors of M or M^* induced by all decomposing sequences of the matroid M are taken into account. As a corollary, we get two expressions for the Tutte polynomial. The first one, which uses only the cyclic flats F_c, implicit in [EtLV98]. is called the 'Convolution formula for the Tutte polynomial' in [KoReSt99]. The second one is new.

Corollary 5.

$$t(M;x,y) = \sum_{F \textit{ cyclic flat of } M} t(M/F;x,0)\ t(M(F);0,y)$$

$$t(M;x,y) = \sum_{\substack{\emptyset=F'_\varepsilon\subset\dots\subset F'_0=F_c \\ F_c=F''_0\subset\dots\subset F''_\iota=E \\ \textit{decomposing sequence}}} \Big(\prod_{1\leq k\leq\iota} \beta(M(F'_k)/F'_{k-1})\Big)\ \Big(\prod_{1\leq k\leq\varepsilon} \beta(M(F''_{k-1})/F''_k)\Big)\ x^\iota y^\varepsilon$$

For an ordered oriented matroid M, the *activity class* of M is the set of 2^{i+j} reorientations obtained by reorienting independently parts of the active partition of M.

Proposition 6. *All reorientations in an activity class have the same active partition.*

The activity classes form a remarkable partition of the set of reorientations of M (see for instance the graphic case Section 6 Part 2).

When $-_AM$, $A \subseteq E$, runs through the set of all reorientations of M, all the $(0,1)$-active and $(1,0)$-active reorientations of minors of M or M^* induced by the decomposing sequence of the matroid M are taken into account.

The two above decompositions use similarly the set of all decomposing sequences of the matroid.

Hence, as opposite reorientations define the same oriented matroid and so must naturally be associated with the same basis, one can extend a $(1-2)$ correspondence between $(1,0)$-bases and $(1,0)$-reorientations to a $1-2^{i+j}$ correspondence between bases and reorientations preserving activities (i,j). More precisely, we get a bijection between bases and activity classes of reorientations preserving active partitions. Of course, by duality of activities and of the previous decompositions of activities, the same property holds for $(0,1)$ activities.

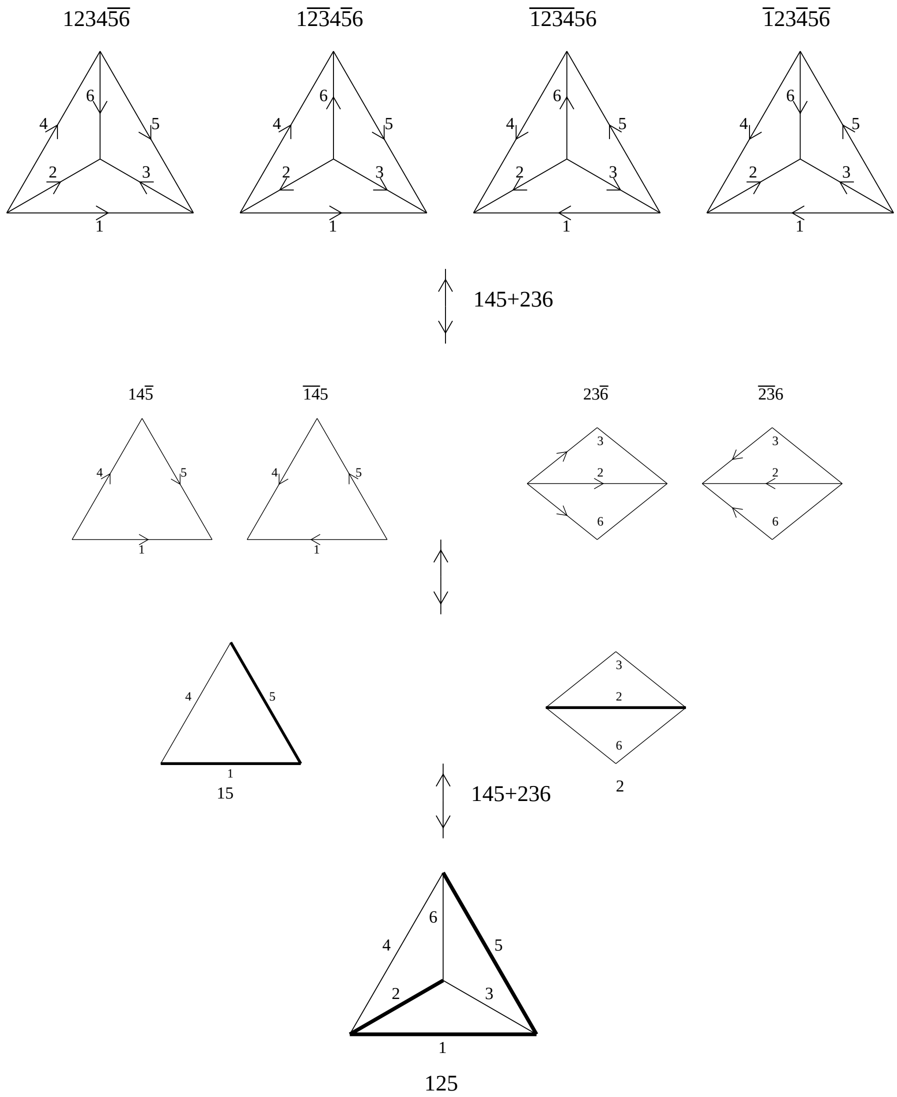

Figure 2

Example. Figure 2 shows the construction for the basis 125 of K_4 with internally active elements 1 2 and no externally active elements. In this simple example, the two minors $M(145)$ and $M/145$ have each one basis and a pair of opposite (re)orientations with activities $(1,0)$. The common active partition is $145+236$.

4. Fundamental bijection for $(1,0)$ activities.

Let M be an ordered oriented matroid. Let f_1 be the smallest non loop element of M, and f_2 be the smallest element independent from f_1.

Lemma 1. *It is easy to check that a basis* $B = b_1 < \dots < b_r$, $E\setminus = b'_1 < \dots < b'_{n-r}$ *has activities* $(1,0)$ *if and only if* $b_1 = f_1$, $b'_1 = f_2$, *for all* $1 < i \leq r$, $C^*_<(B;b_i) \subseteq \cup_{j<i} C^*(B;b_j)$ *and for all* $1 < i \leq n-r$, $C_<(B;b'_i) \subseteq \cup_{j<i} C(B;b'_j)$.

On the other hand, reorientations with activities $(1, 0)$ are in canonical bijection with regions (acyclic reorientations, i.e. with orientation-activity 0) which do not touch f_1, the smallest non loop element of M (dual-activity 1). These regions can be called *bounded regions* if f_1 is considered as the *plane at infinity*.

4.1. **From bases to reorientations: two dual algorithms.** Let M be an ordered oriented matroid on E, and $B = \{b_1, b_2, \dots, b_r\}_<$ a (1,0)-active basis of M with $E \setminus B = \{b'_1, b'_2, \dots, b'_{n-r}\}_<$.

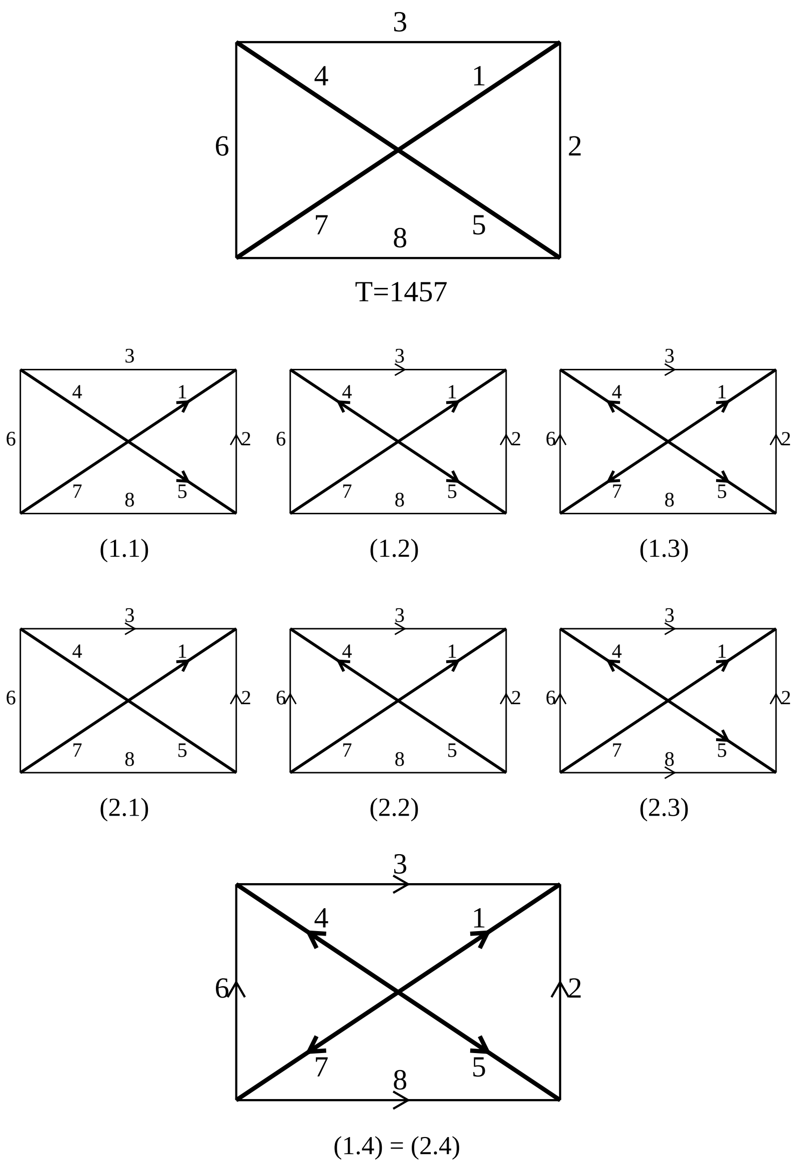

Figure 3

Algorithm 1

(1) reorient $C^*(B; b_1)$ to get all signs positive

(2) for $i = 2, \dots, r$ reorient $C^*(B; b_i) \setminus \bigcup_{j<i} C^*(B; b_j)$ to get all signs opposite to the sign of $MinC^*(B; b_i)$

Algorithm 2
(1) reorient $C(B; b'_1)$ to get $b'_1 = e_2$ negative and all other signs positive
(2) for $i = 2, \ldots, r$ reorient $C(B; b'_i) \setminus \bigcup_{j<i} C(B; b'_j)$ to get all signs opposite to the sign of $\mathrm{Min}\ C(B; b'_i)$

Proposition 7. *Algorithms (1) and (2) produce the same pair of opposite reorientations A and $E \setminus A$, such that $-_A M = -_{E\setminus A} M$ has (1,0) orientation activity.*

Note that we used here an algorithmic presentation, but in fact Algorithm 1 and 2 just describe two dual adjacency properties which characterize intrinsically the reorientation associated with a given (1,0)-basis (see Proposition 10).

Theorem 8. *The application defined by Algorithms (1) and (2) maps (1,0)-active bases of M to subsets $A \subseteq E \setminus \{e_1\}$ such that $-_A M$ has (1,0) orientation activity is a bijection.*

We denote $Oribas^{(1,0)}$ the reverse application which maps a $(1,0)$-orientation active oriented matroid onto its associated basis (since obviously a basis depends only on its associated image, as an oriented matroid, and not as a reorientation).

Examples.
Figure 3 shows these two dual equivalent algorithms for a (1,0)-base of W_4.
Figure 4 shows the algorithm 1 for a rank 3 arrangement. Geometrically, for a given basis B, here $B = 135$, the fundamental cocircuit of $b \in B$ corresponds to the two opposite vertices intersection of the $r-1$ elements $B \setminus b$ of the basis. The algorithm 1 comes to restrict step by step the set of possible associated regions, by choosing step by step, with respect to the linear ordering, which one of these two opposite vertices in on the same side as the region of the element of the basis. This is done geomerically by choosing the orientation of the element of the basis, and of the elements of its fundamental cocircuits that have not yet been oriented, using the orientation of the minimal element of the cocircuit, which has already been reoriented or not.
Figure 5 shows the whole bijection for the example of Figure 4.

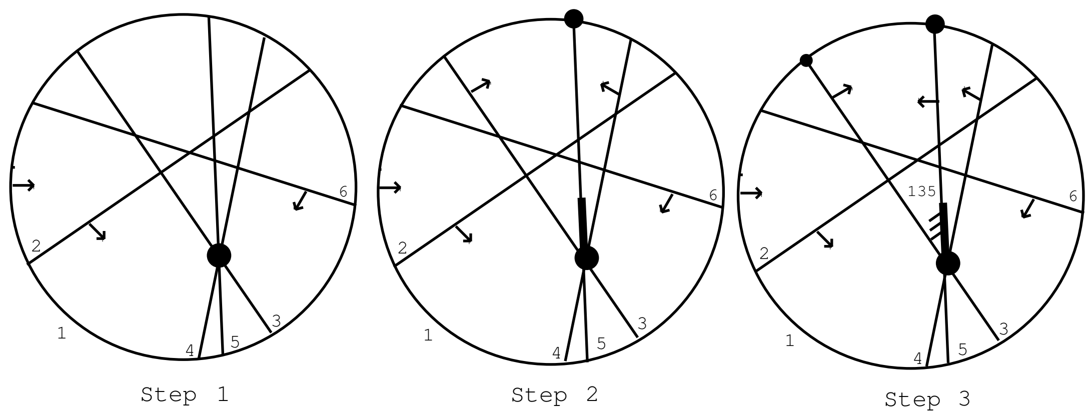

Figure 4

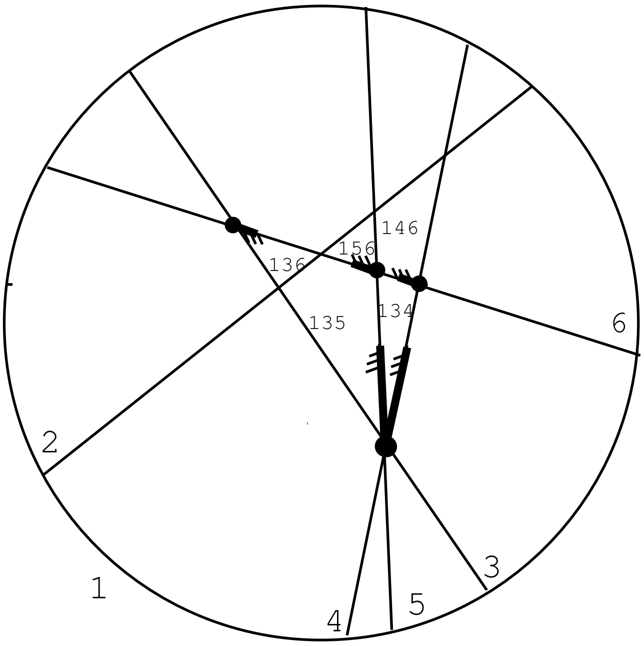

Figure 5

Figure 6 shows various possible situations in rank 4, corresponding to various shapes that can take the set of circuits and cocircuits of the bases (all situations in rank 3 are described in [GiLV 03]). The sequence of covectors of Algorithm 1 is represented on the drawing (i) by a sequence of nested faces: a black circle, a black segment, a grey triangle and a light grey tetraedron which defines the associated region. The elements of the minimal basis are $f_1 < f_2 < f_3 < f_4$. The smallest element of any cocircuit belongs to this minimal basis (easy lemma from greedy algorithm). With algorithm 1, the vertex corresponding to the fundamental cocircuit of $b_i > f_1$ in B for which b_i is positive must be opposite to the associated region, with respect to the minimal element f of this cocircuit. And this minimal element has already been reoriented at a previous step $j < i$, where j is minimal such that $f \in C^*(B; b_j)$. Hence the geometrical interpretation of Algorithm 1 is that, finally, f must cut the segment $[C^*(B; b_i), C^*(B; b_j)]$.

Drawing (ii) represents the simplest case: f_2 belongs to every fundamental cocircuit, and f_2 cuts every segment $[C * (B; f_1), C^*(B; b)]$ for $b \in B - f_1$. It is the only possible situation in the uniform case.

On drawings (i), (iii), (iv) and (v), we have $min(C^*(B; b_2)) = f_2$, and f_2 cuts the segment $[C^*(B; f_1), C^*(B; b_2)]$.

On drawing (i), we have $min(C^*(B; b_3)) = min(C^*(B; b_4)) = f_3$, and f_3 cuts the segments $[C^*(B; f_1), C^*(B; b_3)]$, and $[C^*(B; f_1), C^*(B; b_3)]$.

On drawing (iii), we have $min(C^*(B; b_3)) = min(C^*(B; b_4)) = f_3$, but this time $f_3 \notin C^*(B; f_1)$, and f_3 cuts the segments $[C^*(B; b_2), C^*(B; b_3)]$, and $[C^*(B; b_2), C^*(B; b_4)]$.

On drawing (iv), we have $min(C^*(B;b_3)) = f_4$, and f_4 cuts the segment $[C^*(B;f_1), C^*(B;b_3)]$, $min(C^*(B;b_4)) = f_3$, and f_3 cuts the segment $[C^*(B;f_1), C^*(B;b_4)]$.
On drawing (v), we have $min(C^*(B;b_3)) = f_4$, and f_4 cuts the segment $[C^*(B;f_1), C^*(B;b_3)]$, $min(C^*(B;b_4)) = f_3$, but $f_3 \notin C^*(B;f_1)$, and f_3 cuts the segment $[C^*(B;b_2), C^*(B;b_4)]$.

Eventually, this interpretation can be done exactly the same way in the dual, simply by reorienting f_1 and using Algorithm 2.

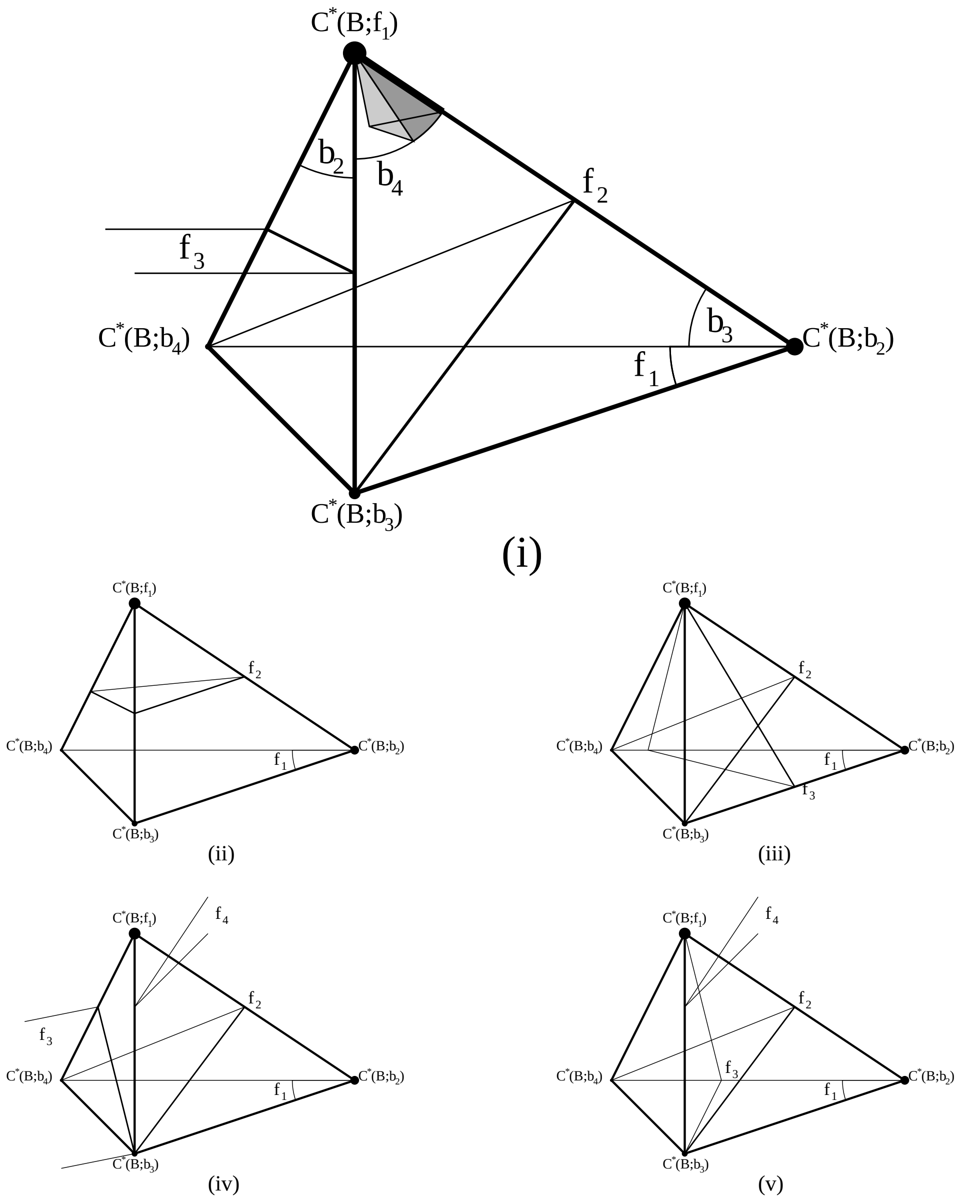

Figure 6

4.2. From reorientations to bases: inductive algorithm.

Theorem 9. *The application $Oribas^{(1,0)}$ satisfies the following inductive definition.*

Let M be an ordered oriented matroid M on a set E with greatest element ω, having orientation activities $(1,0)$. A function γ is defined so that $\gamma(M) = 1$ if and only if $\omega \in Oribas^{(1,0)}(M)$.
If $M = U_{1,1}$, the 1-element oriented matroid of rank 1, then $Oribas^{(1,0)} = \omega$.
If $M \neq U_{1,1}$ then :
  if $o^*(-_\omega M) = 0$ then set $\gamma(M) = 0$
  if $o^*(-_\omega M) > 1$ then set $\gamma(M) = 1$
  if $o^*(-_\omega M) = 1$ then :
    let $B' = Oribas^{(1,0)}(M \setminus \omega)$, $C = C_M(B'; \omega)$ and $e = Min(C)$
    if $\sigma_C(e) \neq \sigma_C(\omega)$ then $\gamma(M) := 0$;
    if $\sigma_C(e) = \sigma_C(\omega)$ then $\gamma(M) := 1$;
or equivalently :
    let $B'' = Oribas^{(1,0)}(M/\omega)$, $D = C^*_M(B'' \cup \omega; \omega)$ and $e = min(D)$
    if $\sigma_D(e) \neq \sigma_D(\omega)$ then $\gamma(M) := 1$;
    if $\sigma_D(e) = \sigma_D(\omega)$ then $\gamma(M) := 0$.
If $\gamma(M) = 0$ then

$$Oribas^{(1,0)}(M) := Oribas^{(1,0)}(M \setminus \omega)$$

if $\gamma(M) = 1$ then

$$Oribas^{(1,0)}(M) := Oribas^{(1,0)}(M/\omega) \cup \omega$$

Note that, as in the previous subsection, there are two dual, and equivalent, points of view in this definition. This equivalence is quite not obvious, its proof uses the fundamental Theorem 8.

4.3. From reorientations to bases: extensions of linear programming in oriented matroids.

Oriented matroid programming is a combinatorial extension of linear programming to oriented matroids (see [OM] Chap. 10). In this subsection, we define an extension of oriented matroid programming. Geometrical illustrations are given in the uniform case, and in the rank 3 case (see Section 6).

Let M be an acyclic oriented matroid on a linearly ordered set $E = \{e_1, e_2, \dots, e_n\}_<$ with dual-orientation activity 1. The plane at infinity in the topological representation is $f_1 = e_1$. The region R corresponding to M is bounded, i.e. does not touch f_1. Let $B = Oribas^{(1,0)}(M)$ be the (1,0)-active basis associated with M by the active correspondance.

In oriented matroid programming, a cocircuit C is optimal for the program (M, g, f) if and only if there exists a basis B such that the fundamental cocircuit $C^*(B; g)$ is positive except maybe on f and the fundamental circuit $C(B; f)$ is positive except maybe on g. Another formulation is that an element of a fundamental circuit belonging to $C^*(B; g)$, except f, has to be positive, and an element of a fundamental cocircuit belonging to $C(B; f)$, except g, has to be positive. In the extension we introduce, f_1 resp. f_2 plays the part of g resp. f, but the signs in all fundamental circuits and cocircuits are taken into account and not only two - the first circuit and cocircuit. Precisely the signs of all minimal elements of fundamental circuits and cocircuits, except for the first cocircuit, must be negative

A basis B with the properties of Proposition 10 can be considered, by analogy, as the *optimal basis* of M with respect to the ordering of E for an extended oriented matroid program.

Proposition 10. *The optimal basis B of the $(1,0)$-active oriented matroid M is characterized uniquely by the following properties, with $B = \{b_1 = f_1, b_2, \ldots, b_r\}_<$ and $E \setminus B = \{b'_1 = f_2, b'_2, \ldots, b'_{n-r}\}_<$:*

(i) The r covectors $C^*(B; b_1) \circ C^*(B; b_2) \circ \ldots \circ C^*(B; b_i)$ $i = 1, 2, \ldots, r$ are positive.

(ii) The $n - r$ vectors $C(B; b'_1) \circ C(B; b'_2) \circ \ldots \circ C(B; b'_i)$ $i = 1, 2, \ldots, n - r$ have all e_1 as unique negative element.

We give now an intermediate extension involving only the first fundamental cocircuit.

Proposition 11. *In the uniform case, the vertex v_1 of R given by the fondamental cocircuit $C^*(B; f_1)$ is the maximum resp. minimum of the matroid program with infinity plane f_1 and objective function f_2 if R is on the positive resp. negative side of f_2.*

In general, v_1 is the solution of simultaneous matroid programs with objective functions given by the elements of the lexicographically minimal basis of M. For a given objective function, the requirement - maximum or minimum - depends on the position of R with respect to the corresponding pseudohyperplane.

More precisely, instead of an optimal face when the kernel of the objective function is parallel to a face of the region, the active correspondence always determines a precise optimal vertex $C^*(B; f_1)$. In fact, the optimization is made according to f_2, then according to f_3 if the optimal face is not a vertex, then according to f_4, and so on, where $f_1 < \ldots < f_r$ is the minimal basis for the lexicographic ordering.

We define the *active cocircuit graph* as the directed graph whose vertices all are cocircuits of M and an edge suported by the coline F (flat of corank 2) is directed from f_q to f_p, where $f_p < f_q$ is the minimal basis of M/F.

Proposition 12. *$C^*(B; f_1)$ is the only vertex with no outgoing edge in the restriction of the active cocircuit graph to the positive cocircuits of M.*

This first extension is due to the fact that signs of the minimal element of every fundamental cocircuit except the first must be negative (not only the ones contained in $C(B; f_2)$). The cocircuit $C^*(B; f_1)$ is the unique optimal vertex of the *oriented matroid multiprogram* defined by M and its minimal basis for the ordering on E.

If we add the constraint that the sign of the minimal element of every fundamental circuit is negative, we get the general extension of Proposition 10. The second extension corresponds to the fact that, instead of an optimal vertex as in usual programming, the active correspondence determines an optimal basis, i.e. an optimal sequence of increasing faces, or *flag*, with respect to the ordering on E.

By analogy with usual linear programming, we say that B is the unique optimal basis solution of the *flag matroid program* defined by M and the ordering on E.

From an algorithmic point of view, the optimal basis of a bounded region is calculated with the algorithm of Theorem 9 (see also section 5.2).

Examples. Back to Figure 6 and its interpretation, when f cuts a usefal segment $[C^*(B;b_i), C^*(B;b_j)]$ with $j < i$, and thus is used for the reorientation or not of b_i in algorithm 1, one could say by analogy and langage abuse that f is the usefal objective function for the face $C^*(B;b_1) \circ ... \circ C^*(B;b_i)$. The active cocircuit graph and Proposition 12 are illustrated on next Figure 9. The notion of flag programming is ilustrated on next Figure 11.

4.4. The $(0,1)$ case and a strong duality property.

Proposition 13. *Let M be an ordered matroid with minimal basis $\{f_1 < f_2 < \dots\}$.*
(i) A basis B of M is (1,0) active if and only if $B \setminus \{f_1\} \cup \{f_2\}$ is (0,1) active
(ii) Suppose M is an oriented matroid. Then M is (1,0) orientation active if and only if $-_{f_1}M$ is (0,1) orientation active

Using this lemma, the previous bijection between $(1,0)-$bases and $(1,0)-$reorientations of M extends readily to bases and reorientations with $(0,1)$ activities.

Proposition 14. *'Strong duality property'*
If M has activities $(1,0)$, associated with B, then $-_{f_1}M^*$ (which has activities $(1,0)$) is associated with $(E \setminus B) \setminus f_2 \cup f_1$.

This property is an extension of the property of duality of linear programming ((M,g,f) and (M^*,f,g) are dual programs). In other words, f_1 and f_2 play dual parts in the extended program also.

5. The canonical active correspondence

The bijection mapping $(1,0)$-active reorientations to bases defined in Section 4 is denoted $Oribas^{(1,0)}$.

The *canonical active correspondence of an ordered oriented matroid M* is constructed by extending the fundamental bijection - or, more precisely (1-2) correspondence - for $(1,0)$ activities of Section 4 to all activities by means of the reduction of Section 3. The application, whose restriction to $(1,0)$ active oriented matroids is $Oribas^{(1,0)}$, that maps an oriented matroid M on its associated base, is denoted by $Oribas$.

The resulting correspondence not only preserves activities, but also the active elements, and in fact the active partitions.

The $2^{\iota(B)+\epsilon(B)}$ reorientations associated with a given basis form an activitiy class, they are obtained from any one of them by reorienting independently the $\iota(B)+\epsilon(B)$ parts of the active partition.

Moreover according to its definition and Proposition 14, the correspondence is invariant by duality:

$$Oribas(M^*) = E \setminus Oribas(M)$$

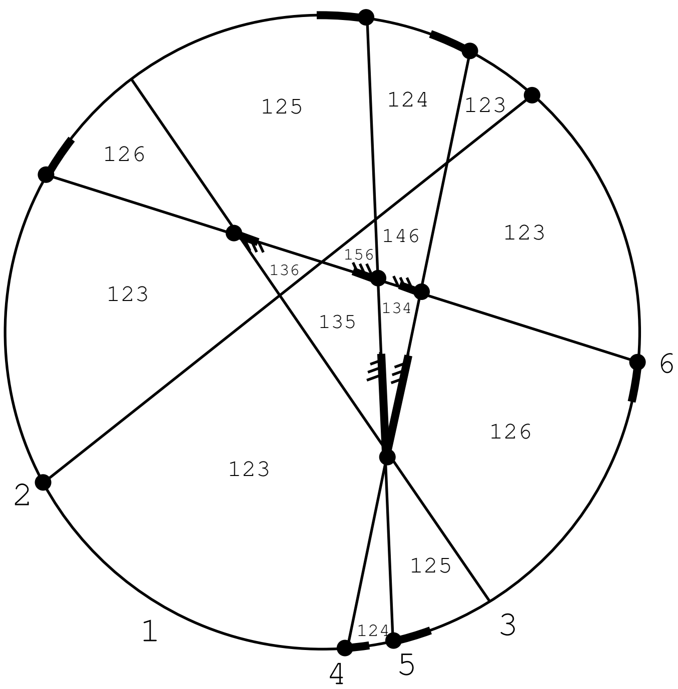

Figure 7

Example. Figure 7 shows the correspondence for all acyclic reorientations of Figure 5. In this example, since the oriented matroid obtained by contraction of 1 is uniform, then the bases associated to regions along 1 can be calculated immediately in this minor.

5.1. **From bases to reorientations.** The Algorithms 1 and 2 examine each element of E once, say they are *single pass*. Similarly the algorithm to construct the decomposition of activities of a basis is also single pass (Proposition 3). Hence the set of reorientations associated with a given basis can be computed in a single pass. Moreover, we need only to know the fundamental circuits and cocircuits.

Proposition 15. *Let B a base of M, ordered oriented matroid on $E = e_1 < ... < e_n$. The reorientations associated with B by the canonical active correspondence of M are calculated by the following algorithm. It constructs at the same time the active partition of B and a reorientation A associated with B. The $2^{\iota(B)+\varepsilon(B)}$ reorientations are obtained either by doing all possible choices $e \in A$ or $e \notin A$ during the algorithm or from a reorientation A by reorienting any part of the active partition of B.*

for k from 1 to n do
if $e_k \notin B$ then
    if e_k is externally active then

e_k external, $\alpha(e_k) := e_k$, and choose $e_k \in A$ or $e_k \notin A$
else
if there exists $c < e_k$ internal in $C(B; e_k)$ then
e_k is internal
let c in $C(B; e_k)$ with $c < e_k$, c internal and $\alpha(c)$ the greatest possible
let $\alpha(e_k) := \alpha(c)$
else
e_k is external
let c in $C(B; e_k)$ with $c < e_k$ and $\alpha(c)$ the smallest possible
let $\alpha(e_k) := \alpha(c)$
let e the smallest possible in $C(B; e_k)$ with $\alpha(e) = \alpha(e_k)$
if $\sigma_{C(B;e_k)}(e_k) \neq \sigma_{C(B;e_k)}(e)$ then
$e_k \notin A$ if an only if $e \notin A$
if $\sigma_{C(B;e_k)}(e_k) = \sigma_{C(B;e_k)}(e)$ then
$e_k \notin A$ if and only if $e \in A$
if $e_k \in B$ then
if e_k is internally active then
e_k internal, $\alpha(e_k) := e_k$, and choose $e_k \in A$ or $e_k \notin A$
else
if there exists $c < e_k$ external in $C^*(B; e_k)$ then
e_k is external
let c in $C^*(B; e_k)$ with $c < e_k$, c external and $\alpha(c)$ the greatest possible
let $\alpha(e_k) := \alpha(c)$
else
e_k is internal
let c in $C^*(B; e_k)$ with $c < e_k$ and $\alpha(c)$ the smallest possible
$\alpha(e_k) := \alpha(c)$
let e the smallest possible in $C^*(B; e_k)$ with $\alpha(e) = \alpha(e_k)$
if $\sigma_{C^*(B;e_k)}(e_k) \neq \sigma_{C^*(B;e_k)}(e)$ then
$e_k \notin A$ if and only if $e \notin A$
if $\sigma_{C^*(B;e_k)}(e_k) = \sigma_{C^*(B;e_k)}(e)$ then
$e_k \notin A$ if and only if $e \in A$

5.2. **From reorientations to bases.** Given a reorientation, the problem of finding the associated basis is far much harder in the sense of complexity. There is a natural exponential algorithm, which is a refinement of an set version of the definition of the Tutte polynomial by deletion/contraction. It is a generalization to all activities of the inductive definition for the $(1,0)$ case given in the previous section.

More precisely, there are essentially two ways to calculate $Oribas(M)$ for an ordered oriented matroid M. The first one is by decomposing the activities of M (section 3.2) and apply the inductive definition of $Oribas^{(1,0)}$ (section 4.2) to all obtained minors with activities $(1,0)$ and dually $(0,1)$. This comes to decompose a reorientation into bounded regions of minors of the matroid and its dual, and then calculate the optimal basis for each one of these bounded regions (section 4.3). The second is directly with the following theorem. The part 'choice' of the algorithm is due to fact that *Oribas* preserves active partitions, and the part 'equality case' is due to the adjacency property of Algorithm 1 or 2.

Theorem 16. *The application Oribas is determined by the following inductive definition, with* $\gamma(M) = 1$ *if and only if* $\omega \in Oribas(M)$*;* $O_\omega(M)$*, resp.* $O^*_\omega(M)$*, is the set of minimal elements of positive circuits, resp. cocircuits, of* M *containing* ω*;* $\sigma_C(e)$ *is the sign of* e *in the signed part* C*; and* $max(\emptyset) < e$ *for all* $e \in E$*.*

For all ordered oriented matroid on E with $max(E) = \omega$.
If ω is a loop of M then $\gamma(M) := 0$.
If ω is an isthmus of M then $\gamma(M) := 1$.
If ω is not an isthmus nor a loop of M then :
choice
if $max\ O^*_\omega(M) > max\ O^*_\omega(-_\omega M)$ or $max\ O_\omega(M) < max\ O_\omega(-_\omega M)$
then $\gamma(M) := 0$;
if $max\ O^*_\omega(M) < max\ O^*_\omega(-_\omega M)$ or $max\ O_\omega(M) > max\ O_\omega(-_\omega M)$
then $\gamma(M) := 1$;
if $max\ O^*_\omega(M) = max\ O^*(-_\omega M)$ and $max\ O_\omega(M) = max\ O(-_\omega M)$
then :
equality case
let $B' = Oribas(M \setminus \omega)$ and $C = C_M(B'; \omega)$
if ${O^*}_\omega(M) \neq \emptyset$ then let $e = min\Big(C \cap \bigcup_{\substack{D \text{ positive cocircuit of } M \\ min(D) \geq max(O^*_\omega(M))}} D\Big)$
if $O^*_\omega(M) = \emptyset$ then let $e = min(C)$
if $\sigma_C(e) \neq \sigma_C(\omega)$ then $\gamma(M) := 0$
if $\sigma_C(e) = \sigma_C(\omega)$ then $\gamma(M) := 1$
or equivalently :
let $B'' = Oribas(M/\omega)$ and $D = C^*_M(B'' \cup \omega; \omega)$
if $O_\omega(M) \neq \emptyset$ then let $e = min\Big(D \cap \bigcup_{\substack{C \text{ positive circuit of } M \\ min(C) \geq max(O_\omega(M))}} C\Big)$
if $O_\omega(M) = \emptyset$ then let $e = min(D)$
if $\sigma_D(e) \neq \sigma_D(\omega)$ then $\gamma(M) := 1$
if $\sigma_D(e) = \sigma_D(\omega)$ then $\gamma(M) := 0$
end
if $\gamma(M) = 0$ then

$$Oribas(M) := Oribas(M \setminus \omega)$$

if $\gamma(M) = 1$ then

$$Oribas(M) := Oribas(M/\omega) \cup \omega$$

The element $max\ O^*_\omega(M)$ is the greatest minimal element of a positive cocircuit containing ω in M, that is the minimal element of the part containing ω in the active partition of M.

Note that the two equivalent choices are dual each other, and that M and $-_\omega M$ play synmetric parts:

$$\gamma(M) = 1 - \gamma(-_\omega)M$$

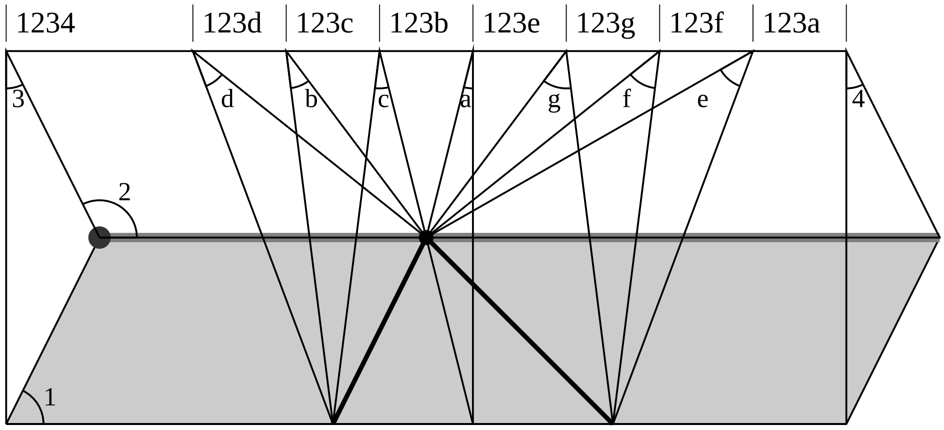

Figure 8

Example. Figure 8 is the sequel of the example of Figure 1. The order is $1 < 2 < 3 < 4 < a < ... < g$. It gives the basis associated with some regions of an arrangement of rank 4. One can read the active partitions on Figure 1. Admitting that in this particular picture all represented regions must be associated with bases containing 123 (because all these regions touch 1 and 2, that is have dual active elements 1, 2 and 3), one can easily apply the previous algorithm to all them. For instance, for the region between g and f, the face of the region embedded in 1 is embedded in g. Hence any positive cocircuit of the region containing $g = \omega$ will have smallest element 1. That is $max\ O^*_\omega(M) = 1$. On the contrary the opposite region with respect to g has a vertice on the face $1 \setminus 2$ which is not on g. That is $max\ O^*_\omega(-_\omega M) = 2$. Then by the inductive definition of $Oribas$, we have $g \in Oribas(M)$. Then on the one hand $Oribas(M) = 123g$, and on the other hand g is deleted, and so on...

5.3. **No Broken Circuit complex.** The canonical active correspondence does not depend on a particular reorientation. If we choose a particular reference reorientation M, and associate for a reorientation $-_A M$ associated with the base B, the subset $B \triangle (A \cap (Int(B) \cup Ext(B)))$ to $-_A M$ instead of B, we get an *activity preserving bijection between subsets and reorientations*, where the activity of a subset is the activity of the associated base by means of the classical partition ([Da81][Bj87][GoTr90][LV03]) of 2^E into intervals $[B \setminus Int(B), B \cup Ext(B)]$ for every base B [GiLV] (see also [Gi02] Part 4.3).

A broken circuit is a circuit whose smallest element is removed. A subset containing no broken circuit is associated with a basis whose external activity is 0 (the No Broken Circuit subsets form notably a basis of the Orlik-Solomon algebra)

Hence, the restriction of the above bijection to acyclic reorientations gives an *active bijection between the faces of the No Broken Circuit complex of the matroid and the regions of the oriented matroid.*

6. Particular cases

The uniform and rank 3 cases are studied into details in [GiLV03], and the graphic case in [GiLV02] (Proceedings of FPSAC02) (see also [Gi02] Parts 6.1, 6.2 and 6.4).

6.1. **Uniform case.** In the uniform case, a basis B is (1,0)-active if and only if $e_1 \in B$ and $e_2 \in E \backslash B$. Let B be a (1,0)-active basis. We have $C^*(B; e_1) \cap C(B; e_2) = \{e_1, e_2\}$.

Set $D = C^*(B; e_1)$ and $C = \pm C(B; e_2)$ such that e_2 has the same sign in C and D. Applying Algorithms 1 or 2, we get that *the reorientation associated with B by the active correspondence is*

$$A = (C^- \cup D^-) \setminus \{e_1\}$$

In $-_A M$ the fundamental cocircuit D is positive and the fundamental circuit C has $C^- = \{e_2\}$.

In the uniform case, a $(1,0)$-base B is determined by the fundamental cocircuit of e_1 according to B. The correspondence amounts to the usual oriented matroid programming: the application $Oribas^{(1,0)}$ maps a bounded region of M on its optimal vertex $C^*(Oribas(M); f_1)$ for the oriented matroid program (M, f_1, f_2) on the positive side of f_2 resp. $(M, f_1, -f_2)$ on the negative side of f_2.

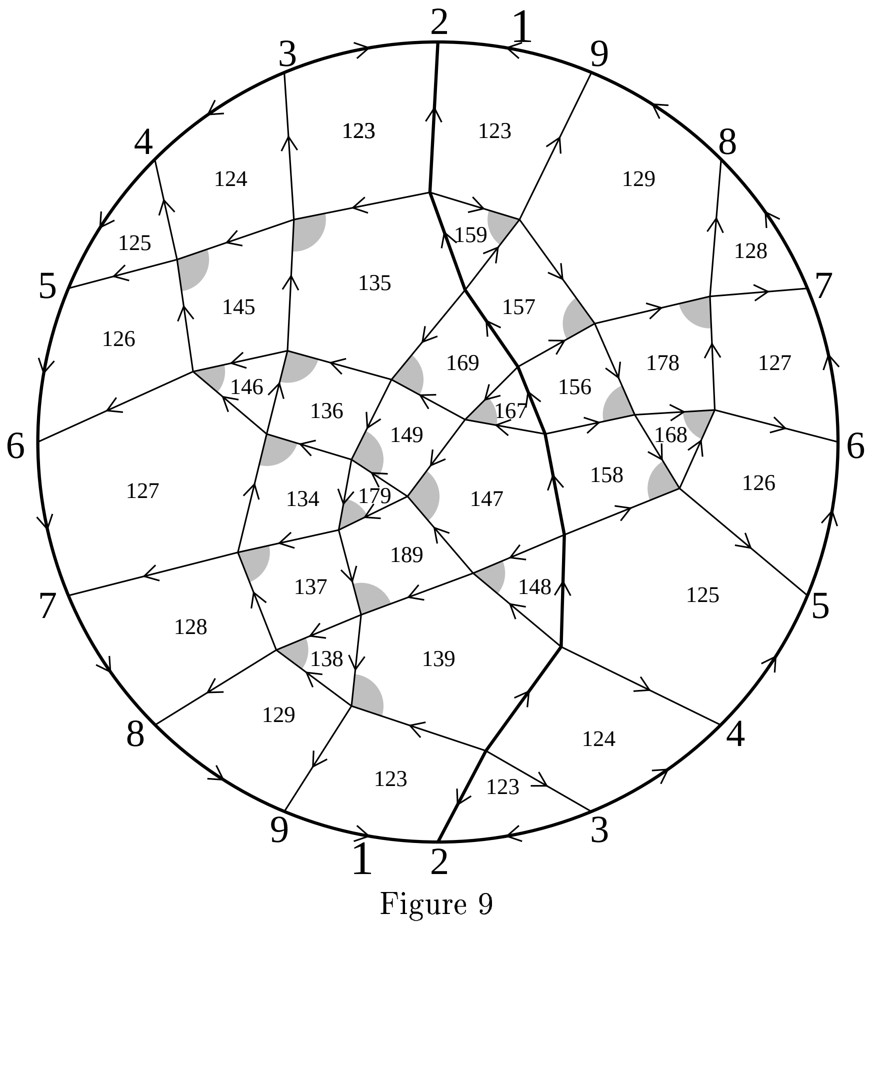

Figure 9

Example. Ringel arrangement $n = 9$ $r = 3$ (Figure 9). The vertex with no outgoing edge in each region is the fundamental cocircuit of e_1 according to the optimal basis of this region (Proposition 12).

6.2. **Graphic case.** In the graphic case, we get in general an active bijection between spanning trees and activity classes of orientations. In this case, orientations with $(1, 0)$ activities are acyclic orientations with unique source and unique sink, extremities of the smallest edge e_1.

We say that a spanning tree T in an ordered graph is *increasing with respect to a vertex s* if the edges increase for the ordering along any path of T beginning at s. As easily seen it is always possible to define a total order of the edges so that the lexicographically smallest spanning tree is increasing with respect to s.

Proposition 17. *Let G be an ordered graph such that the lexicographically smallest spanning tree is increasing with respect to a vertex s.*

Then there is exactly one acyclic orientation with a unique sink at s in each activity class of acyclic orientations of G.

Hence the correspondence gives an active bijection between spanning trees with external activity 0, and acyclic orientations with unique given sink.

Example. Figure 10 shows this bijection for W_4. The lexicographically smallest spanning tree 1236 is increasing with respect to the NE (North-East) vertex. For each acyclic orientation with unique sink at the NE vertex, we have indicated its image by *Oribas*: an internal spanning tree (its edges are drawn in heavy lines). We have also indicated the active partition. The internal activity is the number of parts of the active partitions, and the active edges are the first element of each part. By reversing all edge directions in arbitrarily chosen parts of the active partition, we get the activity class associated with the same tree. By Proposition 17, in each activity class exactly one acyclic orientation has a unique sink at the NE vertex: this orientation is shown on Figure 10.

The problem of counting orientations of graphs in connection with the Tutte polynomial has been addressed several times (for instance in [Vi86] for acyclic orientations and the chromatic polynomial of a graph with a totally ordered set of vertices, or in [GeSa00] for acyclic orientations with unique sink, or in [CoLB02] for the same orientations in connection with the sandpile model) but it is the first time that activities of orientations are taken into account (except in [LV83] where a different correspondence in the graphic case was considered).

Moreover, the oriented matroid point of view, for which only edges are considered and ordered, allows to take into account every orientation, not only acyclic ones, via duality properties.

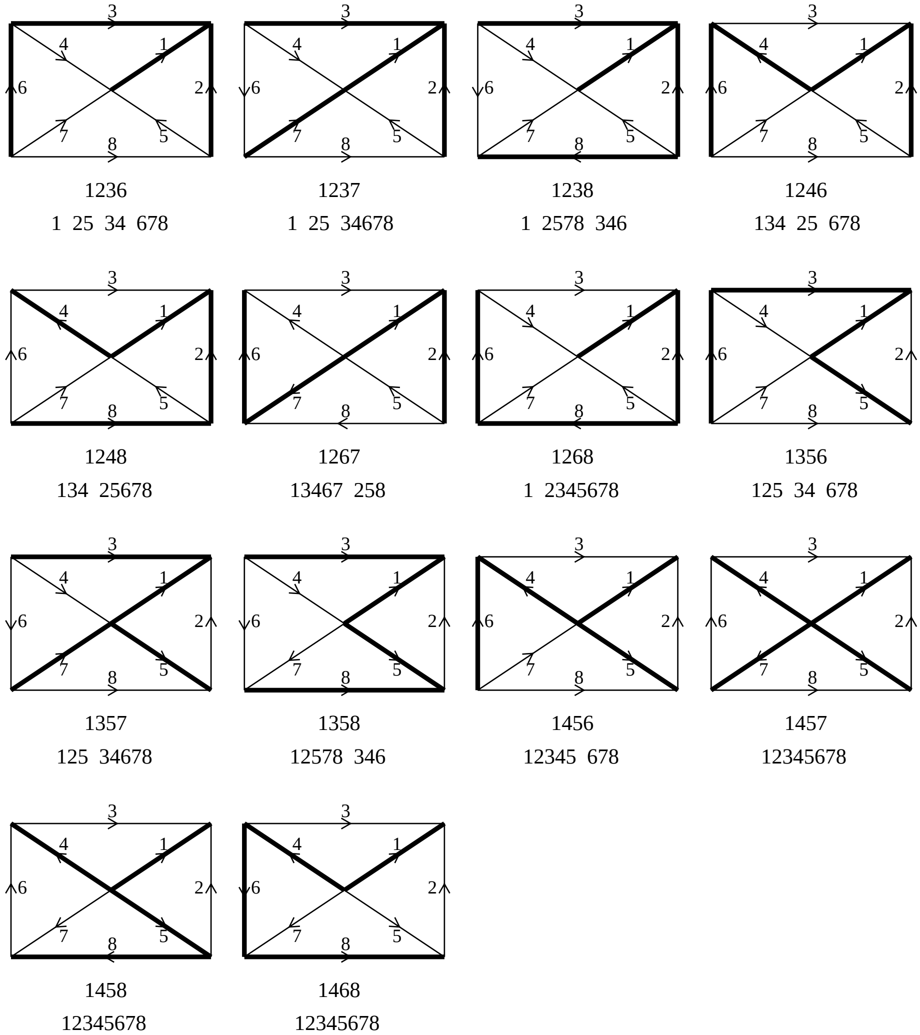

Figure 10

6.3. **Rank 3 case.**

Proposition 18. *In a rank 3 oriented matroid, the active canonical correspondence is the unique way to associate bijectively every* $(1,0)$*-basis* $B = f_1 < e_p < e_q$ *with a bounded region such that* e_q *is frontier of the region, and* e_p *contains an extremity of the segment contained in* e_q *in the region.*

Example. Figure 11 illustrates this property. The sequence of faces $e_p \cap e_q \subset e_q$ in each region illustrates the flag matroid programming of section 3 (see also the final picture in color).

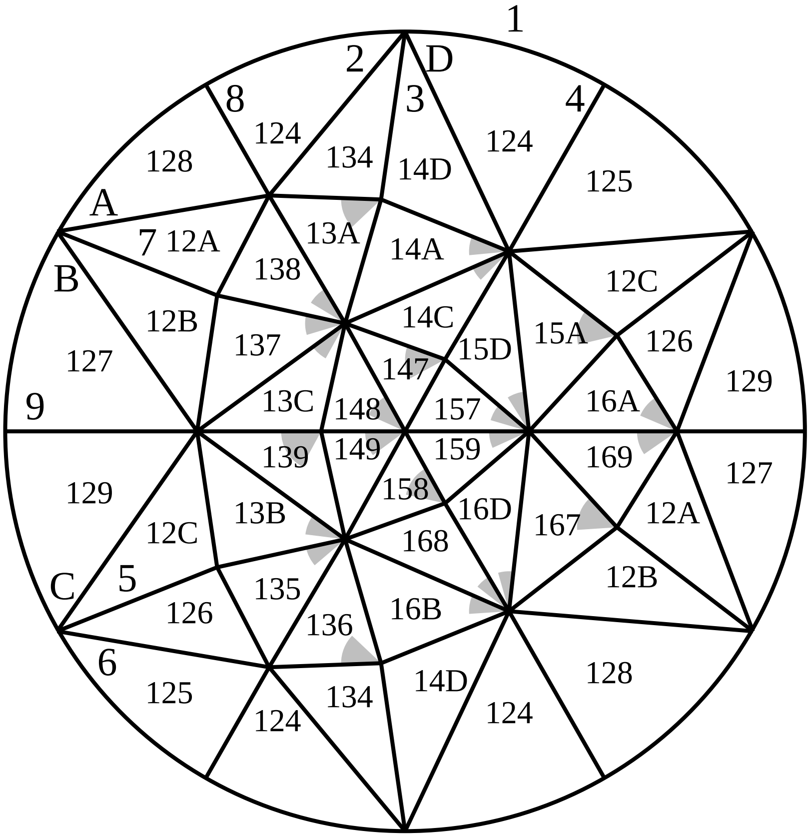

Figure 11

Intuitively, the application *Oribas* can be thought of as a phenomenon of attraction with respect to the linear ordering, related to activities, i. e. as an *(attr)active function of ordered oriented matroids* (see Figure in Annex).

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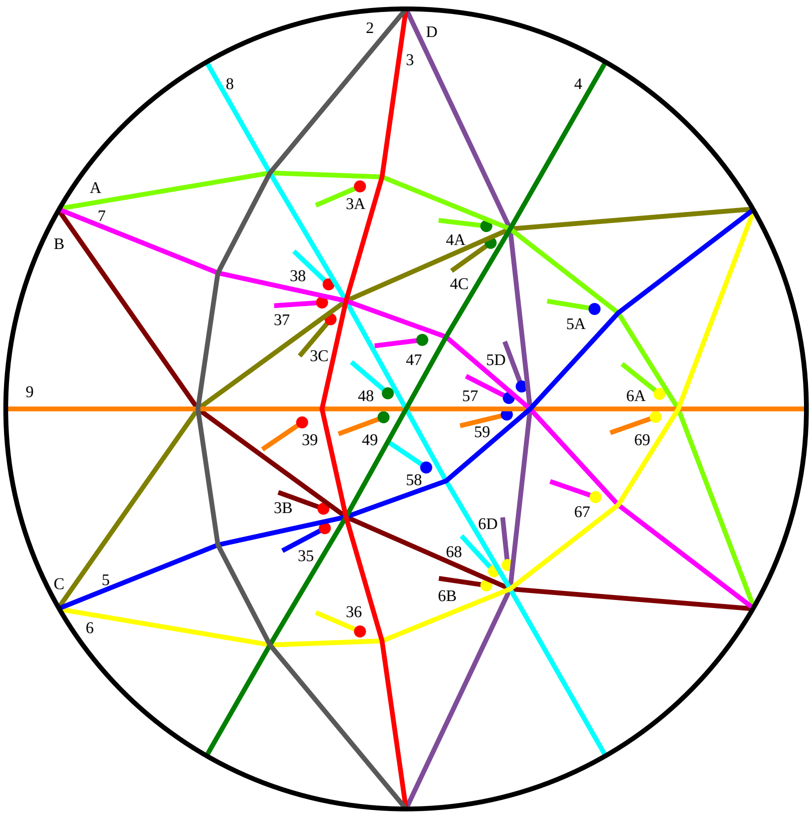
Annex. The following figure (it is Figure 11 with colors: only e_pe_q is written for each $(1,0)$-basis $e_1 < e_p < e_q$) illustrates the *attractivity phenomenon* described by the canonical active correspondence.



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