A METHOD FOR PROVING POLYNOMIAL ENUMERATION FORMULAS

Ilse Fischer

Institut für Mathematik, Universität Klagenfurt,
Universitätsstrasse 65 – 67, A-9020 Klagenfurt, Austria.
E-mail: `Ilse.Fischer@uni-klu.ac.at`

Abstract. On the base of a refinement of the Bender-Knuth (ex-)Conjecture I present an elementary method for proving enumeration formulas which are polynomial in at least one parameter. The Bender-Knuth (ex-)Conjecture gives the generating function of column-strict plane partitions with parts in $\{1, 2, \ldots, n\}$ and at most c columns. In our refinement of this result the number of parts equal to n is fixed in addition.

Résumé. Sur la base d'un raffinement de l'ex-conjecture de Bender-Knuth, je présente une méthode élémentaire pour prouver des formules d'énumérations qui sont polynômiales en au moins un paramètre. L'ex-conjecture de Bender-Knuth donne la fonction génératrice des partitions planes à colonnes strictes en parts dans $\{1, 2, \ldots, n\}$ et ayant au plus c colonnes. Dans notre raffinement de ce résultat, le nombre de parts égales à n est fixé en plus.

1. Introduction

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ be a partition. A *strict plane partition* of shape λ is an array $\pi_{1 \le i \le r, 1 \le j \le \lambda_i}$ of non-negative integers such that the rows are weakly decreasing and the columns are strictly decreasing. The *norm* $n(\pi)$ of a strict plane partition is defined as the sum of its parts and π is said to be a strict plane partition of the non-negative integer $n(\pi)$. In [2, p.50] Bender and Knuth conjectured that the generating function of strict plane partitions with at most c columns whose parts lie in the set $\{1, 2, \ldots, n\}$ and with respect to this norm is equal to

$$\sum q^{n(\pi)} = \prod_{i=1}^{n} \frac{[c+i;q]_i}{[i;q]_i},$$

where $[n;q] = 1 + q + \cdots + q^{n-1}$ and $[a;q]_n = \prod_{i=0}^{n-1} [a+i;q]$. This conjecture was independently proved by Andrews [1], Gordon [9], Macdonald [12, Ex. 19, p.53] and Proctor [14, Prop. 7.2]. For related papers, which mostly include generalisations of the Bender-Knuth (ex-)Conjecture, see [4, 5, 10, 11, 15, 18]. In our refinement of this result the number of parts equal to n in the strict plane partition is fixed in addition.

Theorem 1. *The generating function of strict plane partitions with parts in* $\{1, 2, \ldots, n\}$, *at most* c *columns and* k *entries equal to* n *is*

$$\frac{q^{kn}[k+1;q]_{n-1}[1+c-k;q]_{n-1}}{[1;q]_{n-1}} \prod_{i=1}^{n-1} \frac{[c+i+1;q]_{i-1}}{[i;q]_i}.$$

If we sum this generating function over all k's, $0 \leq k \leq c$, we easily obtain the Bender-Knuth (ex)-Conjecture. Probably this detour over Theorem 1 is so far the easiest and most elementary possibility to prove the Bender-Knuth (ex-)Conjecture. For the full version of this paper see [6].

2. The method

We first explain how to prove the case $q = 1$ of Theorem 1, i.e. we compute the number of strict plane partitions with parts in $\{1, 2, \ldots, n\}$, at most c columns and k entries equal to n, and later see that the method can be extended in order to prove the more general result. Note that in the special case $q = 1$ the formula in Theorem 1 is a polynomial in k, which factorises into linear factors over $\mathbb{Z}$. In the following I give a recipe which might be suitable for proving polynomial enumeration formulas of that type in general. It is divided into three steps.

(1) **Extension of the combinatorial interpretation.** Suppose we are given combinatorial objects we want to enumerate which depend on an integer parameter k and we suspect that there exists an enumeration formula of these objects which is polynomial in k and factorises into distinct linear factors over $\mathbb{Z}$. Typically the admissible domain of k is a set S of non-negative integers. In the first step of our method we have to find (most likely new) combinatorial objects indexed by an *arbitrary* integer k which are in bijection with the original objects for $k \in S$. In our example: With the help of new combinatorial objects, namely the generalised $(n-1, n, c)$-Gelfand-Tsetlin-patterns with fixed part k in the center of the first row, we extend the interpretation of strict plane partitions with parts in $\{1, 2, \ldots, n\}$, at most c columns and k parts equal to n to arbitrary integers k.

(2) **The extension is enumerated by a polynomial.** The extension of the combinatorial interpretation in the previous step has to be chosen such that we are able to prove that the new objects are enumerated by a polynomial in k. Moreover the degree of this polynomial has to be computed. In our example we show that for fixed n and c these new objects are enumerated by a polynomial $P_{n,c}(k)$ in k. Moreover it can be shown that this polynomial is of degree $2n-2$. However, this is the part of the proof of Theorem 1 we omit in this extended abstract. It can be found in [6].

(3) **Exploring 'natural' linear factors.** Finally one has to find the k's for which there exist none of these objects, i.e. one has to compute the (integer) zeros of the enumeration polynomial. Typically these zeros will not lie in S, which made the extension in Step 1 necessary. Moreover one has to find a non-zero evaluation of the polynomial which is easy to compute, and together with the

zeros the polynomial is finally computable. In our example we observe that there is no such $(n-1, n, c)$-pattern with fixed part k in the center of the first row for $k = -1, -2, \ldots, -n+1$ and for $k = c+1, c+2, \ldots, c+n-1$ which implies that the polynomial $P_{n,c}(k)$ must have the factor $(k+1)_{n-1}(1+c-k)_{n-1}$. By the degree estimation of the previous step we have determined $P_{n,c}(k)$ up to a factor which does not depend on k. Since it is possible to compute $P_{n,c}(c)$ inductively, i.e. by the use of $P_{n-1,c}(k)$, we are able to compute this factor and with this $P_{n,c}(k)$.

I plan to apply this method to other enumeration problems. The most ambitious project in this direction is probably my current effort to give another proof of the refined alternating sign matrix (ex-)Conjecture. There is some hope for a proof which is in the vein of the proof of Theorem 2: Let $A(n,k)$ denote the number of alternating sign matrices of order n, where the unique 1 in the first row is in the k-th column. It was conjectured by Mills, Robbins and Rumsey [13] (well-known as the refined alternating sign matrix Conjecture; see also [3] for a nice introduction to alternating sign matrices) and proved by Zeilberger [19] that

$$A(n,k) = \frac{(k)_{n-1}(1+n-k)_{n-1}}{(1)_{n-1}} \prod_{i=1}^{n-1} \frac{(1)_{3i-2}}{(1)_{n+i-1}}.$$

For me it came by surprise that the number of strict plane partitions with parts in $\{1, 2, \ldots, n\}$, at most $n-1$ columns and $k-1$ parts equal to n divided by $A(n,k)$ is independent of k. (In fact this fraction is the number of $(n-1)\times(n-1)\times(n-1)$ totally symmetric plane partitions.) In other words: Up to a shift and up to a constant the enumeration polynomials (in k) coincide in these two cases. Moreover strict plane partitions and alternating sign matrices are also closely related because of the following: Below we see that strict plane partitions are in bijection with Gelfand-Tsetlin patterns. Alternating sign matrices are in bijection with monotone triangles (see [3]) and monotone triangles are simply Gelfand-Tsetlin patterns with strictly increasing rows. Thus an application of the method to alternating sign matrices could be similar to the application presented in this paper.

Throughout our considerations we use the extended definition of the summation symbol, namely,

$$\sum_{i=a}^{b} f(i) = \begin{cases} f(a) + f(a+1) + \ldots + f(b) & \text{if } a \le b \\ 0 & \text{if } b = a-1 \\ -f(b+1) - f(b+2) - \ldots - f(a-1) & \text{if } b+1 \le a-1 \end{cases}.$$

This assures that for any polynomial $p(X)$ over an arbitrary integral domain I there exists a unique polynomial $q(X)$ over I such that $\sum_{x=0}^{y} p(x) = q(y)$ for all integers y. Moreover $\deg q = \deg p + 1$. (Thus we usually use $\sum_{x=0}^{y} p(x)$ as a synonym for $q(y)$.)

The following three sections refer to the three steps of our method given above.

3. Step 1: From strict plane partitions to generalised $(n-1,n,c)$-Gelfand-Tsetlin-patterns

Let r, n, c be integers, r non-negative and n positive. In this paper a *generalised (r,n,c)-Gelfand-Tsetlin-pattern* (short: (r,n,c)-pattern) is an array $(a_{i,j})_{1 \le i \le r+1, i-1 \le j \le n+1}$ of integers with

- $a_{i,i-1} = 0$ and $a_{i,n+1} = c$,
- if $a_{i,j} \le a_{i,j+1}$ then $a_{i,j} \le a_{i-1,j} \le a_{i,j+1}$
- if $a_{i,j} > a_{i,j+1}$ then $a_{i,j} > a_{i-1,j} > a_{i,j+1}$.

Thus

$$\begin{array}{cccccccccccccccc}
 & & & 0 & & 3 & & -5 & & 10 & & 4 & & & \\
 & & 0 & & 2 & & -2 & & 3 & & 8 & & 4 & & \\
 & 0 & & 2 & & -1 & & 2 & & 4 & & 7 & & 4 & \\
0 & & 0 & & 0 & & 1 & & 2 & & 5 & & 6 & & 4
\end{array}$$

is an example of an $(3,6,4)$-pattern. Note that a generalised $(n-1,n,c)$-Gelfand-Tsetlin pattern with $0 \le a_{n,n} \le c$ is what is said to be a Gelfand-Tsetlin pattern, see [17, p. 313] or [8, (3)] for the original reference. It is crucial to our considerations that $(n-1,n,c)$-patterns with $0 \le a_{n,n} = k \le c$ are in bijection with strict plane partitions with parts in $\{1,2,\ldots,n\}$, at most c columns and k parts equal to n: Given an $(n-1,n,c)$-pattern, the corresponding strict plane partition is such that the shape filled by entries greater than i corresponds to the partition given by the $(n-i)$-th row of the $(n-1,n,c)$-pattern, the topmost row being the first row.

Therefore it suffices to enumerate the $(n-1,n,c)$-patterns with $0 \le a_{n,n} = k \le c$. A pair $(a_{i,j}, a_{i,j+1})$ with $a_{i,j} > a_{i,j+1}$ and $i \ne 1$ is called an *inversion* of the (r,n,c)-pattern and $(-1)^{\#\text{ of inversions}}$ is said to be the *sign* of the pattern, denoted by $\operatorname{sgn}(a)$. The $(3,6,4)$-pattern in the example above has altogether 6 inversions and thus its sign is 1. We define the following expression

$$F(r,n,c;k_1,k_2,\ldots,k_{n-r}) = \sum_a \operatorname{sgn}(a),$$

where the sum runs over all (r,n,c)-patterns $(a_{i,j})$ with $k_i = a_{r+1,r+i}$ for $i = 1,\ldots,n-r$. Note that the number of $(n-1,n,c)$-patterns with $0 \le a_{n,n} = k \le c$ is equal to $F(n-1,n,c;k)$. This is because an $(n-1,n,c)$-pattern with $0 \le a_{n,n} \le c$ has no inversions.

Clearly it makes no sense to ask for the number of strict plane partitions with parts in $\{1,2,\ldots,n\}$, at most c columns and k parts equal to n if $k \notin \{0,1,\ldots,c\}$, since by the columnstrictness parts equal to n can only occur in the first row of a strict plane partition. However, $F(r,n,c;k)$ is well-defined for all integers k and therefore what we have done so far is 'naturally' extended the combinatorial interpretation of the number of these strict plane partitions to arbitrary integers k. In the following we see that 'naturally' here stands for the fact that the extension was chosen such that the extended objects are enumerated by a polynomial in k.

4. Step 2: $F(n-1,n,c;k)$ is a polynomial in k of degree $2n-2$

The definition of an (r,n,c)-pattern immediately implies the following recursion

$$F(r,n,c;k_1,k_2,\ldots,k_{n-r}) = \sum_{l_1=k_0}^{k_1} \sum_{l_2=k_1}^{k_2} \cdots \sum_{l_{n-r+1}=k_{n-r}}^{k_{n-r+1}} F(r-1,n,c;l_1,l_2,\ldots,l_{n-r+1}),$$

for $r>0$ and with $k_0=0$, $k_{n-r+1}=c$. This recursion and the fact that

$$F(0,n,c;k_1,k_2,\ldots,k_n)=1$$

shows inductively (with respect to r) that $F(r,n,c;k_1,\ldots,k_{n-r})$ can be expressed by a polynomial in $k_1,k_2,\ldots,k_{n-r}$. In the following $F(r,n,c;k_1,k_2,\ldots,k_{n-r})$ will be identified with this polynomial. It can be shown that this polynomial is of degree $2r$ in every k_i. However, this is the most difficult part in the proof of Theorem 1 and therefore we omit it here. It can be found in [6]. Consequently $F(n-1,n,c;k)$ is a polynomial in k of degree $2n-2$.

5. Step 3: Exploring 'natural' linear factors

Now observe that there exists no (r,n,c)-pattern with fixed first row $0,k_1,\ldots,k_{n-r},c$ such that $k_1\in\{-1,-2,\ldots,-r\}$ by induction with respect to r: For $r=0$ there is nothing to prove, otherwise the southwest neighbour of k_1 in an (r,n,c)-pattern must be in $\{-1,-2,\ldots,-r+1\}$ and the assertion follows by induction. Analogously there exists no such (r,n,c)-pattern with $k_{n-r}\in\{c+1,c+2,\ldots,c+r\}$. Consequently $(k+1)_{n-1}(1+c-k)_{n-1}$ is a factor of $F(n-1,n,c;k)$.

Lemma 1. *$F(n-1,n,c;k)/((k+1)_{n-1}(1+c-k)_{n-1})$ is independent of k.*

Proof. By the argument above and by Section 4 $(k+1)_{n-1}(1+c-k)_{n-1}$ is a factor of $F(n-1,n,c;k)$. But $F(n-1,n,c;k)$ is a polynomial of degree $2n-2$ in k and the assertion follows. □

Theorem 2. *The number of strict plane partitions with parts in $\{1,2,\ldots,n\}$, at most c columns and k parts equal to n is given by*

$$F(n-1,n,c;k) = \frac{(k+1)_{n-1}(1+c-k)_{n-1}}{(1)_{n-1}} \prod_{i=1}^{n-1} \frac{(c+i+1)_{i-1}}{(i)_i}.$$

Proof. Induction with respect to n. The formula is true for $n=1$ since $F(0,1,c;k)=1$. In order to show the assertion for n, observe that

$$F(n-1,n,c;k) = (k+1)_{n-1}(1+c-k)_{n-1}\frac{F(n-1,n,c;c)}{(c+1)_{n-1}(1)_{n-1}}$$

by Lemma 1. The fact that for an (r,n,c)-pattern $(a_{i,j})_{1\le i\le r+1, i-1\le j\le n+1}$ with $a_{n,n}=c$ we have $a_{i,n}=c$ for all i, implies the recursion

$$F(n-1,n,c;c) = \sum_{k=0}^{c} F(n-2,n-1,c;k).$$

With the help of this recursion, the hypergeometric identity

$$\sum_{k=0}^{c}(k+1)_{m-1}(1+c-k)_{m-1} = \frac{(1)_{m-1}^2(c+1)_{2m-1}}{(1)_{2m-1}} \tag{5.1}$$

(this identity can easily be deduced from the Chu-Vandermonde identity; see [16, (1.7.7); Appendix (III. 4)]) and the induction hypothesis for $F(n-2,n-1,c;k)$ we compute $F(n-1,n,c;k)$. □

6. Extension of the method to q-polynomials

A natural question to ask is whether it is possible to obtain a generating function version of Theorem 2. Of course only this would refine the Bender-Knuth (ex)-Conjecture. Clearly this generating function is not a polynomial in k. However, we introduce the notion of a q-polynomial below and find that the generating function is roughly such a q-polynomial. Thus we adapt our method to q-polynomials in this section.

Let I be an integral domain. A q-polynomial over I in the variable X is an ordinary polynomial over $I(q)$, the quotient field of $I[q]$, in the variable $[X;q]=(1-q^X)/(1-q)$. With the help of the following identity

$$[X+Y;q]=[X;q]+q^X[Y;q]$$

it is possible to express every q-polynomial in the basis $[X;q]_n=\prod_{i=0}^{n-1}[X+i;q]$ over $I(q)$, which is helpful below.

If we review the proof of Theorem 2 we see that the following two basic properties of polynomials were crucial.

- If $p(X)$ is a polynomial over an integral domain, then there exists a (unique) polynomial $r(X)$ with $\deg r = \deg p + 1$ and

$$\sum_{x=1}^{y} p(x) = r(y)$$

  for every integer y.
- If $p(X)$ is a polynomial over an integral domain and $a_1, a_2, \ldots, a_r$ are distinct zeros of $p(X)$, then there exists a polynomial $r(X)$ with

$$p(X)=(X-a_1)(X-a_2)\ldots(X-a_r)r(X).$$

The following analogs hold for q-polynomials.

- If $p(X)$ is a q-polynomial, then there exists a (unique) q-polynomial $r(X)$ with $\deg r = \deg p + 1$ and

$$\sum_{x=1}^{y} p(x)\, q^x = r(y)$$

  for all integers y. In order to see this note that

$$[X;q]_{n+1}-[X-1;q]_{n+1}=q^{X-1}[n+1;q][X;q]_n,$$

which implies

$$\sum_{x=1}^{y}[x;q]_n q^x = \frac{q}{[n+1;q]}[y;q]_{n+1}$$

for all integers y.

- If $p(X)$ is a q-polynomial and $a_1, a_2, \ldots, a_r$ are distinct zeros of $p(X)$, then there exists a q-polynomial $r(X)$ with

$$p(X) = ([X;q]-[a_1;q])([X;q]-[a_2;q])\ldots([X;q]-[a_r;q])r(X) = q^{a_1+a_2+\ldots+a_r}[X-a_1;q][X-a_2;q]\cdots[X-a_r;q]r(X).$$

The proof is analog to the proof for ordinary polynomials, namely the fundamental identity is

$$[X;q]^n - [Y;q]^n = ([X;q]-[Y;q])\sum_{i=0}^{n-1}[Y;q]^i[X;q]^{n-1-i} = q^Y[X-Y;q]\sum_{i=0}^{n-1}[Y;q]^i[X;q]^{n-1-i}.$$

We introduce a q-analog of $F(r,n,c;k_1,k_2,\ldots,k_{n-r})$. The norm of an (r,n,c)-pattern is defined as the sum of its parts, where we omit the first and the last part in each row. Then we define

$$F_q(r,n,c;k_1,k_2,\ldots,k_{n-r}) = \sum_a \operatorname{sgn}(a) q^{\mathrm{norm(a)}},$$

where the sum runs over all (r,n,c)-patterns $(a_{i,j})$ with $k_i = a_{r+1,r+i}$ for $i=1,\ldots,n-r$. Note that $F_q(n-1,n,c;k)$ is the generating function of strict plane partitions with parts in $\{1,2,\ldots,n\}$, at most c columns and k parts equal to n, since the bijection between these strict plane partitions and $(n-1,n,c)$-patterns is norm-preserving. If we run through the q-analog of the proof of Theorem 2 we observe that

$$F_q(r,n,c;k_1,k_2,\ldots,k_{n-r})/\prod_{i=1}^{n-r} q^{k_i}$$

is a q-polynomial in $k_1,k_2,\ldots,k_{n-r}$ and obtain the following q-analog of Lemma 1.

Lemma 2. *$F_q(n-1,n,c;k)/[1+k;q]_{n-1}[k-c-n+1;q]_{n-1}q^k)$ is independent of k.*

(Note that $[1+c-k;q]_{n-1}$ is not a q-polynomial in k and therefore we work with $[k-c-n+1;q]_{n-1}$ instead.) Thus we are now able to prove Theorem 1.

Proof of Theorem 1. Induction with respect to n. The formula is true for $n=1$ since $F(0,1,c;k)=q^k$. The rest follows from the lemma, the recursion

$$F_q(n-1,n,c;c) = q^{cn}\sum_{k=0}^{c} F_q(n-2,n-1,c;k)$$

and the following identity

$$\sum_{k=0}^{c}[k+1;q]_{n-1}[k-c-n+1;q]_{n-1}\, q^k = (-1)^{n-1}q^{(-n+1)(2c+n)/2}\frac{[1;q]_{n-1}^2[c+1;q]_{2n-1}}{[1;q]_{2n-1}} \quad (6.1)$$

which can be deduced from the q-Chu-Vandermonde identity, see [7, (1.5.3);Appendix (II. 6)]. □

Finally we are able to prove the Bender-Knuth (ex)-Conjecture.

Corollary 1. *The generating function for the strict plane partitions with parts in* $\{1, 2, \ldots, n\}$ *and at most* c *columns equals*

$$\sum q^{n(\pi)} = \prod_{i=1}^{n} \frac{[c+i; q]_i}{[i; q]_i}.$$

Proof. By Theorem 1 the generating function is equal to

$$\sum_{k=0}^{c} \frac{q^{k\,n}[k+1; q]_{n-1}[1+c-k; q]_{n-1}}{[1; q]_{n-1}} \prod_{i=1}^{n-1} \frac{[c+i+1; q]_{i-1}}{[i; q]_i}.$$

The assertion follows from (6.1). □

References

[1] G. E. Andrews, Plane Partitions II: The equivalence of the Bender-Knuth and the MacMahon conjecture, *Pacific J. Math* **72** (1977), no. 2, 283 – 291.

[2] E. A. Bender and D. E. Knuth, Enumeration of Plane Partitions, *J. Combin. Theory Ser. A* **13** (1972), 40 – 54.

[3] D. M. Bressoud, *Proof and Confirmations, The Story of the Alternating Sign Matrix Conjecture*, Cambridge University Press, Cambridge, 1999.

[4] J. Désarménien, La démonstration des identitiés de Gordon et MacMahon et de deux identitiés nouvelles, in: Actes de 15^{e} Seminaire Lotharingien, I.R.M.A. Strassbourg, 1987, 39 – 49.

[5] J. Désarménien, Une généralisation des formules de Gordon et de MacMahon, *Comptes Rendus Acad. Sci. Paris, Série I* **309**, (1989), no. 6, 269 – 272.

[6] I. Fischer, A method for proving polynomial enumeration formulas, preprint, `math.CO/0301103`.

[7] G. Gasper and R. Rahman, *Basic hypergeomtric series*, Encyclopedia of Mathematics and its Applications **35**, Cambridge University Press, Cambridge, 1990.

[8] I. M. Gelfand and M. L. Tsetlin, Finite-dimensional representations of the group of unimodular matrices (in Russian), *Doklady Akad. Nauk. SSSR (N. S.)* **71** (1950), 825 – 828.

[9] B. Gordon, A proof of the Bender-Knuth Conjecture, *Pacific J. Math.* **108** (1983), no. 1, 99 – 113.

[10] K. W. J. Kadell, Schützenberger's jeu de taquin and plane partitions, *J. Combin. Theory Ser. A* **77** (1997), no. 1, 110 – 133.

[11] C. Krattenthaler, The major counting of nonintersecting lattice paths and generating functions for tableaux, *Mem. Amer. Math. Soc.* **115** (1995), no. 552, vi+109 pp.

[12] I. G. Macdonald, *Symmetric Functions and Hall Polynomials*, Oxford University Press, New York/London, 1979.

[13] W. H. Mills, D. P. Robbins and H. Rumsey, Alternating sign matrices and descending plane partitions, *J. Combin. Theory Ser. A* **34** (1983), no. 3, 340 – 359.

[14] R. A. Proctor, Bruhat lattices, plane partitions generating functions, and minuscule representations, *Europ. J. Combin.* **5**, (1984), no. 4, 331 – 350.

[15] R. A. Proctor, New symmetric plane partition identities from invariant theory work of DeConcini and Procesi, *Europ. J. Combin.* **11**, (1990), no. 3, 289 – 300.

[16] L.J. Slater, *Generalized hypergeometric functions*, Cambridge University Press, Cambridge 1966.

[17] R. P. Stanley, Enumerative combinatorics, vol. 2, Cambridge University Press, 1999.

[18] J. R. Stembridge, Hall-Littlewood functions, plane partitions and Rogers-Ramanujan identities, *Trans. Amer. Math. Soc.* **319**, (1990), no. 2, 469 – 498.

[19] D. Zeilberger, Proof of the refined alternating sign matrix conjecture, *New York Journal of Mathematics* **2** (1996), 59 – 68, electronic.