q-NARAYANA NUMBERS AND THE FLAG h-VECTOR OF $J(\mathbf{2} \times \mathbf{n})$

PETTER BRÄNDÉN

Abstract. The Narayana numbers are $N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k+1}$. There are several natural statistics on Dyck paths with a distribution given by $N(n,k)$. We show the equidistribution of Narayana statistics by computing the flag h-vector of $J(\mathbf{2} \times \mathbf{n})$ in different ways. In the process we discover new Narayana statistics and provide co-statistics for the Narayana statistics so that the bi-statistics have a distribution given by Fürlinger and Hofbauer's q-Narayana numbers. We interpret the flag h-vector in terms of semi-standard Young tableaux, which enables us to express the q-Narayana numbers in terms of Schur functions. We also introduce what we call pre-shellings of simplicial complexes.

Résumé. Les nombres de Narayana sont les $N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k+1}$ Ces nombres décrivent les distributions de certaines statistiques naturelles sur les chemins de Dyck. Nous démontrons que les statistiques en question sont équidistribuées en calculant le h-vecteur drapeau de $J(\mathbf{2}\times\mathbf{n})$ de plusieurs façons différentes. Ce faisant, nous trouvons de nouvelles statistiques de Narayana et donnons des co-statistiques telles que les distributions des paires statistiques/co-statistiques soient décrites par les nombres q-Narayana de Fürlinger et Hofbauer. Nous interprétons le h-vecteur drapeau en termes de tableaux de Young semi-standards, ce qui nous permet d'exprimer les nombres q-Narayana en termes de fonctions de Schur. Nous introduisons également la notion de pré-shelling de complexes simpliciaux.

1. Introduction

One of the most common refinements of the famous *Catalan numbers*, $\frac{1}{n+1}\binom{2n}{n}$, [12, Exercise 6.19] is given by the *Narayana numbers*,

$$N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k+1}.$$

They appear in many combinatorial problems. Some examples are the number of noncrossing partitions of $\{1,2,\ldots,n\}$ of rank k [5], the

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number of stack sortable permutations with k descents [10], and also several problems involving Dyck paths.

A *Dyck path* of length $2n$ is a path in $\mathbb{N}\times\mathbb{N}$ from $(0,0)$ to (n,n) using steps $v=(0,1)$ and $h=(1,0)$, which never goes below the line $x=y$. The set of all Dyck paths of length $2n$ is denoted $\mathscr{D}_n$. A statistic on $\mathscr{D}_n$ having a distribution given by the Narayana numbers will be referred to as a *Narayana statistic*. The first Narayana statistics to be discovered were

- $\operatorname{des}(w)$: the number of *descents (valleys)* (sequences hv) in w, Narayana [9],
- $\operatorname{ea}(w)$: the number of *even ascents*, i.e., the number of letters v in an even position in w, Kreweras [6],
- $\operatorname{lnfs}(w)$: the number of *long non-final sequences*, more precisely the number of sequences vvh and hhv in w, Kreweras and Moszkowski [7].

Recently, [3], Deutsch discovered a new Narayana statistic, hp, and it counts the number of *high peaks*, i.e., peaks that have vertices strictly above the line $y=x+1$. Also, in [13, 14] Sulanke found numerous new Narayana statistics with the help of a computer. See [15] for more information on Narayana numbers. For terminology on posets in what follows, we refer the reader to [11].

Our main objective is to show that the statistics $\operatorname{des},\operatorname{hp}$ and lnfs arise naturally when studying different shellings of $\Delta(J(\mathbf{2}\times\mathbf{n}))$, the order complex of the lattice of order ideals of the poset $\mathbf{2}\times\mathbf{n}$. More precisely, we show that they can be computed as invariants of certain shellings. From this follows not only that the statistics all have the same distribution, but that the results can be extended to set-valued statistics. In the process we will also find a new family of Narayana statistics. Since our methods are not restricted to Dyck paths, we will consider a more general setting.

In Section 2 we review some theory on shellings of simplicial complexes and order complexes and define what we call a pre-shelling of a complex. In Section 3 we give different pre-shellings of order complexes of certain plain distributive lattices, and finally in Section 4 we apply our results to Dyck paths. There is a q-analog of the Narayana numbers,

$$N_q(n,k)=\frac{1}{[n]}\begin{bmatrix}n\\k\end{bmatrix}\begin{bmatrix}n\\k+1\end{bmatrix}q^{k^2+k}, \tag{1}$$

introduced by Fürlinger and Hofbauer in [4]. Here $[n]$ and $\left[\begin{smallmatrix}n\\k\end{smallmatrix}\right]$ are the usual q-analogs. To each statistic we treat we associate a co-statistic together with which the Narayana statistic has a joint distribution given by the q-Narayana numbers.

2. Pre-shellings of Simplicial complexes

An (abstract) *simplicial complex* Δ on a vertex set V is a collection of subsets F of V satisfying:

(i) if $x \in V$ then $\{x\} \in \Delta$,
(ii) if $F \in \Delta$ and $G \subseteq F$, then $G \in \Delta$.

The elements of Δ are called *faces* and a maximal face (with respect to inclusion) is called a *facet*. A simplicial complex is said to be *pure* if all its facets have the same cardinality. A total order Ω on the set of facets of a pure simplicial complex Δ is a *shelling* if whenever $F <^\Omega G$ there is an $x \in G$ and $E <^\Omega G$ such that

$$F \cap G \subseteq E \cap G = G \setminus \{x\}.$$

A simplicial complex which allows a shelling is said to be *shellable*. Instead of finding a particular shellings we will find partial orders on the set of facets with the property that every linear extension is a shelling. In our attempts to prove that our partial orders had this property we found ourselves proving Theorem 1 and Corollary 2. We therefore take the opportunity to take a general approach and define what we call a *pre-shelling*. Though we have found examples of pre-shellings implicit in the literature we have not found explicit references, so we provide proofs.

Let Ω be a partial order on the set of facets of a pure simplicial complex Δ. The *restriction*, $r_\Omega(F)$, of a facet F is the set

$$r_\Omega(F) = \{x \in F : \exists E \text{ s.t } E <^\Omega F \text{ and } E \cap F = F \setminus \{x\} \ \}.$$

We say that Ω is a *pre-shelling* if any of the equivalent conditions in Theorem 1 are satisfied.

Theorem 1. *Let Ω be a partial order on the set of facets of a pure simplicial complex Δ. Then the following conditions on Ω are equivalent:*

(i) *For all facets F, G we have*

$$r_\Omega(F) \subseteq G \text{ and } r_\Omega(G) \subseteq F \quad \Longrightarrow \quad F = G.$$

(ii) *Δ is the disjoint union*

$$\Delta = \dot{\bigcup_F} [r_\Omega(F), F].$$

(iii) *For all facets F, G*

$$r_\Omega(F) \subseteq G \quad \Rightarrow \quad F \leq^\Omega G.$$

(iv) *For all facets F and G: if $F \not\geq^\Omega G$ then there is an $x \in G$ and $E <^\Omega G$ such that*

$$F \cap G \subseteq E \cap G = G \setminus \{x\}.$$

Proof. (i) $\Rightarrow$ (ii): Let F and G be facets of Δ. If there is an $H \in [r_\Omega(F), F] \cap [r_\Omega(G), G]$ then $r_\Omega(F) \subseteq G$ and $r_\Omega(G) \subseteq F$, so by (i) we have $F = G$. Hence the union is disjoint. Suppose that $H \in \Delta$, and let F_0 be a minimal element, with respect to Ω, of the set

$$\{F : F \text{ is a facet and } H \subseteq F\}.$$

If $r_\Omega(F_0) \nsubseteq H$ then let $x \in r_\Omega(F_0) \setminus H$ and let $E <^\Omega F_0$ be such that $F_0 \cap E = F_0 \setminus \{x\}$. Then $H \subseteq E$, contradicting the minimality of F_0. Therefore $H \in [r(F_0), F_0]$.

(ii) $\Rightarrow$ (i): If $r_\Omega(F) \subseteq G$ and $r_\Omega(G) \subseteq F$ we have that $F \cap G \in [r_\Omega(F), F] \cap [r_\Omega(G), G]$, which by (ii) gives us $F = G$.

(i) $\Rightarrow$ (iii): If $r_\Omega(F) \subseteq G$ then by (i) we have either $F = G$ or $r_\Omega(G) \nsubseteq F$. If $F = G$ we have nothing to prove, so we may assume that there is an $x \in r_\Omega(G) \setminus F$. Then, by assumption, there is a facet $E_1 <^\Omega G$ such that

$$r_\Omega(F) \subseteq G \cap E_1 = G \setminus \{x\} \subset E_1.$$

If $E_1 = F$ we are done. Otherwise we continue until we get

$$F = E_k <^\Omega E_{k-1} <^\Omega \cdots <^\Omega E_1 <^\Omega G,$$

and we are done.

(iii) $\Leftrightarrow$ (iv): It is easy to see that (iv) is just the contrapositive of (iii)

(iii) $\Rightarrow$ (i): Immediate. $\square$

The set of all partial orders on the same set is partially ordered by inclusion, i.e $\Omega \subseteq \Lambda$ if $x <^\Omega y$ implies $x <^\Lambda y$.

Corollary 2. *Let Δ be a pure simplicial complex. Then*

(i) *every shelling of Δ is a pre-shelling,*
(ii) *if Ω is a pre-shelling of Δ and Λ is a partial order such that $\Omega \subseteq \Lambda$, then Λ is a pre-shelling of Δ with $r_\Lambda(F) = r_\Omega(F)$ for all facets F. In fact, the set of pre-shellings of Δ is a principal upper ideal of the poset of all partial orders on the set of facets of Δ,*
(iii) *every linear extension of a pre-shelling is a shelling, with the same restriction function.*

Proof. (i): Follows immediately from Theorem 1(iv).

(ii): That Λ is a pre-shelling follows from Theorem 1(iv). If F is a facet then by definition $r_\Omega(F) \subseteq r_\Lambda(F)$, and if $r_\Omega(F) \subset r_\Lambda(F)$ for some facet F we would have a contradiction by Theorem 1(ii). It remains to show that there is a unique minimal order with r_Ω as a pre-shelling. Define a partial order Υ as the transitive closure of the relation $\mathcal{R}$ defined by: $F\mathcal{R}G$ if

$$F <^\Omega G \quad \text{and} \quad |F \cap G| = |F| - 1. \tag{2}$$

It follows that Υ is a partial order with r_Ω as restriction function, so Υ is a pre-shelling by Theorem 1(i) . Since (2) only depends on r_Ω, and $\Upsilon \subseteq \Omega$ we are done.

(iii): Is implied by (ii). $\square$

There are interesting examples of the unique minimal pre-shelling afforded by Corollary 2:

Example 3. Let Δ be the barycentric subdivision of a simplex of dimension $n-1$. Then there is a standard way of identifying the facets of Δ with the permutations in the symmetric group $\mathcal{S}_n$. The lexicographic order $<_L$ on $\mathcal{S}_n$ is then a shelling of Δ, and it follows that the unique minimal pre-shelling with the same restriction function as $<_L$ is the weak Bruhat order on $\mathcal{S}_n$.

See also Example 6 for another example of a minimal pre-shelling.

In Section 4 we will need some facts about flag h-vectors of order complexes which we state here for reference. Let P be any finite graded poset with a smallest element $\hat{0}$ and a greatest element $\hat{1}$ and let ρ be the rank function of P with $\rho(\hat{1}) = n$. For $S \subseteq [n-1]$ let

$$\alpha_P(S) := |\{c \text{ is a chain of } P : \rho(c) = S\}|,$$

and

$$\beta_P(S) := \sum_{T \subseteq S} (-1)^{|S-T|} \alpha_P(T).$$

The functions $\alpha_P, \beta_P : 2^{[n-1]} \to \mathbb{Z}$ are called the *flag f-vector* and the *flag h-vector* of P respectively. The *order complex*, $\Delta(P)$, of P is the simplicial complex of all chains of P. A simplicial complex Δ is *partitionable* if it can be written as

$$\Delta = [r(F_1), F_1] \dot\cup [r(F_2), F_2] \dot\cup \cdots \dot\cup [r(F_n), F_n], \tag{3}$$

where each F_i is a facet of Δ and r is any function on the set of facets such that $r(F) \subseteq F$ for all facets F. The right hand side of (3) is a *partitioning* of Δ. By Theorem 1(iii) we see that shellable complexes are partitionable. We need the following well known fact about partitionable order complexes. Let $\mathscr{M}(P)$ be the set of maximal chains of P.

Lemma 4. *Let $\Delta(P)$ be partitionable and let*

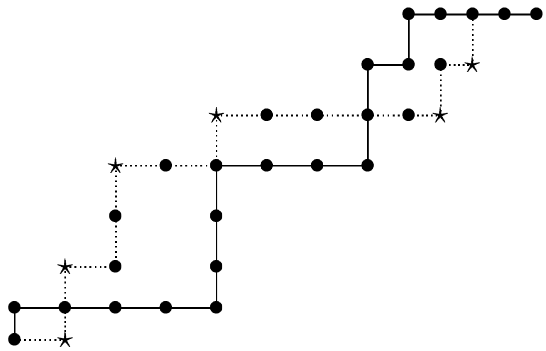
$$\Delta(P) = \dot{\bigcup_c} [r(c), c] \tag{4}$$

be a partitioning of $\Delta(P)$. Then the flag h-vector is given by

$$\beta_P(S) = |\{c \in \mathscr{M}(P) : \rho(r(c)) = S\}|.$$

Proof. See [11, Exercise 3.59 c]. $\square$

FIGURE 1. The vertices of the dotted path facing the other path are displayed as stars



3. SOME PRE-SHELLINGS OF PLANE DISTRIBUTIVE LATTICES

We will here study different pre-shellings of certain distributive lattices. When the distributive lattice is $J(\mathbf{2}\times\mathbf{n})$ the statistics des, hp and lnfs will be the equal to the number of elements in the corresponding restriction functions of the pre-shellings. Let V, W be lattice paths in $\mathbb{Z}^2$ using steps $v = (0,1)$ and $h = (1,0)$ with the same starting- and end-point, $\hat{0} = (0,0)$ and $\hat{1}$ respectively. We don't require V and W to be non-intersecting. Let $R = R(V,W)$ be the closed region in $\mathbb{R}^2$ bounded by V and W, and let $L = L(V,W) = R(V,W) \cap \mathbb{Z}^2$, ordered by the product ordering. It is not hard to see that L is a distributive lattice. The maximal chains in L are the lattice paths from $\hat{0}$ to $\hat{1}$ which stay inside R, and we denote them by $\mathscr{M} = \mathscr{M}(V,W)$. The set of Dyck paths, $\mathscr{D}_n$, is thus $\mathscr{M}(V,W)$ where $V = vhvh\cdots vh$ and $W = vv\cdots vhh\cdots h$ are of length $2n$. Fix a path $W_0 \in \mathscr{M}$. We say that a point $x = u_1 + u_2 + \cdots + u_i$ in a lattice path $u = u_1u_2\cdots u_n$ is *facing* W_0 if (see Figure 1)

- $u_iu_{i+1} = vh$ and x is strictly north-west of W_0 or
- $u_iu_{i+1} = hv$ and x is strictly south-east of W_0.

Define a function $r_{W_0} : \mathscr{M} \to L$ by

$$r_{W_0}(u) = \{x \in u : x \text{ is facing } W_0\}.$$

For any $W_0 \in \mathscr{M}$ we may now define a partial order on $\mathscr{M}$ (the facets of $\Delta(L(V,W))$) by

$$u \leq_{W_0} w \quad \text{if} \quad R(u, W_0) \subseteq R(w, W_0),$$

so that r_{W_0} is the restriction function of $\leq_{W_0}$.

Theorem 5. *For all $W_0 \in L(V,W)$, the partial order $\leq_{W_0}$ is a pre-shelling of $\Delta(L(V,W))$.*

Proof. If $x \in L$ let $R(x)$ be the region enclosed by W_0 and the horizontal and vertical lines emanating from x. If $u \in \mathscr{M}$ contains $r_{W_0}(w)$ then $R(u, W_0)$ contains the union of all $R(x)$, $x \in r_{W_0}(w)$. But this union is $R(w, W_0)$. This verifies condition (iii) of Theorem 1. □

Note that $\leq_{W_0}$ is the smallest pre-shelling with r_{W_0} as restriction function.

Example 6. Let $V = W_0 = vhvh \cdots vh$ and $W = vv \cdots vhh \cdots h$ be of length $2n$ in Theorem 5. Then $\leq_{W_0}$ is a distributive lattice with rank function, $\rho(w)$, given by the area of $R(w, W_0)$. Moreover, it is the unique smallest pre-shelling with r_{W_0} as restriction function. The rank generating function of this lattice is thus the well known Carlitz-Riordan q-analog of the Catalan numbers, $C_n(q)$, satisfying

$$C_{n+1}(q) = \sum_{k=0}^{n} q^k C_k(q) C_{n-k}(q).$$

See [2, 4].

We will now study a pre-shelling linked with long non-final sequences. For $j \geq 1$ let t_j be the mapping on lattice paths which transposes the letters in positions j and $j+1$. An element $x = u_1 + u_2 + \cdots + u_i \in L$ is a *long non-final vertex* of $u = u_1 u_2 \cdots u_n$ if $u_{i-1}u_i u_{i+1} = vvh$ or $u_{i-1}u_i u_{i+1} = hhv$. We say that x is an *inner long non-final vertex*, ILNFV, of u if x is a long non-final vertex and $t_i(u) \in \mathscr{M}$.

Let $S = \{s_1, s_2, \ldots, s_{n-2}\}$ denote the set of mappings

$$s_i(u) = \begin{cases} t_i(u) & \text{if } x = u_1 + \cdots + u_i \text{ is an ILNFV} \\ u & \text{otherwise.} \end{cases}$$

Thus the elements in S flip valleys into peaks, and vice versa, in long non-final sequences provided that the resulting path is still in $\mathscr{M}$. Define a relation Ω, by $u <^{\Omega} w$ whenever $u \neq w$ and $u = \sigma_1\sigma_2 \cdots \sigma_k(w)$ for some mappings $\sigma_i \in S$ (see Figure 2).

Lemma 7. *The relation Ω on $\mathscr{M}(V, W)$ is a partial order.*

Proof. We need to prove that Ω is anti-symmetric. To do this we define a mapping $\sigma : \mathscr{M} \to \mathbb{N} \times \mathbb{N}$, where $\mathbb{N} \times \mathbb{N}$ is ordered lexicographically, with the property

$$u <^{\Omega} w \Rightarrow \sigma(u) < \sigma(w).$$

Define $\sigma(w) = (\mathrm{da}(w), \mathrm{MAJ}(w))$, where $\mathrm{da}(w)$ is the number of double ascents (sequences vv) in w, and MAJ is defined as in (5). Now, suppose that $s_i \in S$ and $s_i(w) \neq w = a_1 a_2 \cdots a_{2n}$. Then $\mathrm{da}(s_i(w)) \leq \mathrm{da}(w)$, and if we have equality we must have $a_{i-1}a_i a_{i+1} a_{i+2} = vvhv$ or $a_{i-1}a_i a_{i+1} a_{i+2} = hhvh$ which implies $\mathrm{MAJ}(s_i(w)) < \mathrm{MAJ}(w)$, so σ has the desired properties. □

FIGURE 2. The partial order Ω on $\mathscr{D}_4$, with long non-final sequences marked with bars.

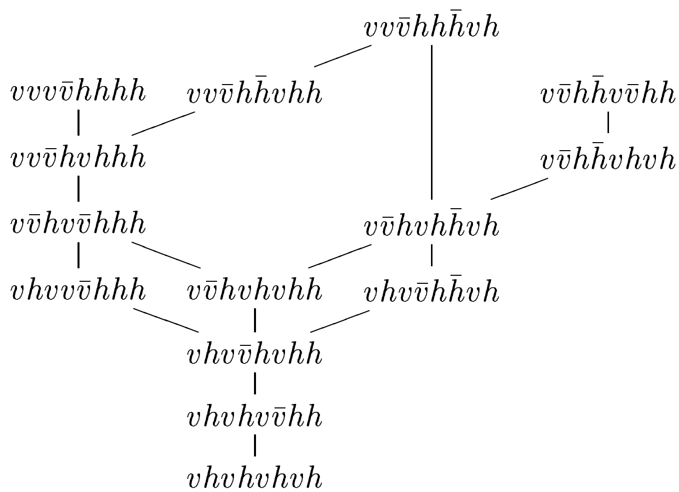
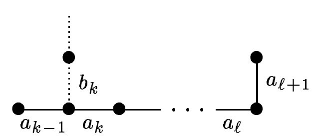


FIGURE 3. Illustration of the proof of Theorem 8.



If $u, w \in \mathscr{M}$ intersect maximally i.e., $|u \cap w| = |u| - 1$ we have either $s(u) = w$ or $s(w) = u$ for some $s \in S$, so the restriction $r_\Omega(u)$ is the set of inner long non-final vertices of u.

Theorem 8. *Suppose that V and W have the same initial step. Then Ω is a pre-shelling of $\Delta(L(V, W))$.*

Proof. We prove that Ω satisfies the contrapositive of condition (i) of Theorem 1. Suppose that $u = a_1 a_2 \cdots a_n \neq w = b_1 b_2 \cdots b_n$ and let k be the coordinate such that $a_i = b_i$ for $i < k$ and $a_k \neq b_k$. By symmetry we may assume that $a_k = h$. Now, if $a_{k-1} = h$ then the valley of u which is determined by the first v (at, say, coordinate $\ell + 1$) after k will correspond to an element

$$x = a_1 + \cdots + a_\ell \in r_\Omega(u) \setminus w$$

(see Figure 3).

If $a_{k-1} = v = b_{k-1}$, then if $\ell + 1$ is the coordinate for the first h after k we have that

$$x = b_1 + \cdots + b_\ell \in r_\Omega(w) \setminus u,$$

so Ω is a pre-shelling. □

4. The restriction to Dyck paths

When $V = vhvh \cdots vh$ and $W = vv \cdots vhh \cdots h$ are of length $2n$ we have that $L(V, W) = J(\mathbf{2} \times \mathbf{n})$, the set of order ideals of $\mathbf{2} \times \mathbf{n}$. See Example 3.5.5 in [11]. Moreover, $\mathscr{M}(V, W) = \mathscr{D}_n$, the set of Dyck paths of length $2n$. The *descent set* $D(w)$, the *set of high peaks* $HP(w)$ and the *set of long non-final sequences* $LS(w)$ are defined as

$$\begin{aligned} D(w) &= \{i \in [2n-1] : w_i w_{i+1} \text{ is a descent}\}, \\ HP(w) &= \{i \in [2n-1] : w_i w_{i+1} \text{ is a high peak}\}, \\ LS(w) &= \{i \in [2n-1] : w_{i-1} w_i w_{i+1} \text{ is a long non-final sequence}\}, \end{aligned}$$

where $w = w_1 w_2 \cdots w_{2n}$. For each $W_0 \in \mathscr{D}_n$ we also define

$$D_{W_0}(w) = \{i \in [2n-1] : w_0 + w_1 + \cdots + w_i \text{ is facing } W_0\}.$$

Theorem 9. *The set functions D, HP, LS and D_{W_0} have the same distribution and it is given by*

$$|\{w \in \mathscr{D}_n : D(w) = S\}| = \beta_{J(\mathbf{2} \times \mathbf{n})}(S).$$

Proof. We have that $D = D_{W_0}$ where $W_0 = vv \cdots vhh \cdots h$ and $HP = D_{W_0}$ where $W_0 = vhvh \cdots vh$. Since all long non-final vertices are inner when $L = J(\mathbf{2} \times \mathbf{n})$ the theorem follows from Lemma 4. □

In particular, for every choice of $W_0 \in \mathscr{D}_n$, we get a Narayana statistic, namely the number of vertices facing W_0.

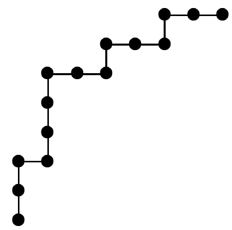
Remark 10. Some of the results in Theorem 9 can be given bijective proofs. The fact that D and LS have the same distribution can be deduced from the bijection given in [1]. The bijection in [3] does not give that the descent set and the set of high peaks have the same distribution. However, there is another bijection ω_1, given by Sulanke in [14], which is seen to take the set of high peaks to the descent set.

We will now take a closer look at the flag h-vector of $J(\mathbf{2} \times \mathbf{n})$ which we hereafter will denote by β_n. The function β_n can be described nicely in terms of partitions. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ be a partition of a nonnegative integer. The index ℓ is called the *length*, $\ell(\lambda)$, of λ. A *semistandard Young tableau* (SSYT) of *shape* λ is an array $T = (T_{ij})$ of positive integers, where $1 \le i \le \ell(\lambda)$ and $1 \le j \le \lambda_i$, that is weakly increasing in every row and strictly increasing in every column. For any SSYT of shape λ let

$$x^T := x_1^{\alpha_1(T)} x_2^{\alpha_2(T)} \cdots ,$$

FIGURE 4. An illustration of Theorem 11 for $n = 7$.

$$\begin{matrix} 1 & 2 \\ 3 & 5 \\ 5 & 6 \end{matrix} \quad \rightarrow$$



where $\alpha_i(T)$ denotes the number of entries of T that are equal to i. The *Schur function* $s_\lambda(x)$ of *shape* λ is the formal power series

$$s_\lambda(x) = \sum_T x^T,$$

where the sum is over all SSYTs T of shape λ. If T is any SSYT we let $\mathrm{row}(T) = (\gamma_1(T), \gamma_2(T), \ldots, \gamma_\ell(T))$ where $\gamma_i(T) = \sum_j T_{ij}$. Let $\langle 2^k \rangle$ be the partition $(2, 2, \ldots, 2)$ with k 2's. Note that if T is a SSYT of shape $\langle 2^k \rangle$ then the coordinates of $\mathrm{row}(T)$ are distinct. We therefore identify $\mathrm{row}(T)$ with its set of coordinates.

Theorem 11. *For any $n > 0$ and $S \subseteq [2n-1]$, $|S| = k$, we have that $\beta_n(S)$ counts the number of* SSYT*s T of shape $\langle 2^k \rangle$ with* $\mathrm{row}(T) = S$ *and with parts less than n.*

Proof. Let T be a SSYT as in the statement of the theorem. We want to construct a Dyck path $w(T)$ with descent set S.

Let $w(T) = w_1 w_1' w_2 w_2' \cdots w_{k+1} w_{k+1}'$ where

- w_1 is the word consisting of T_{12} vertical steps and w_1' is the word consisting of T_{11} horizontal steps,
- w_i is the word consisting of $T_{i2} - T_{(i-1)2}$ vertical steps and w_i' is the word consisting of $T_{i1} - T_{(i-1)1}$ horizontal steps, when $2 \leq i \leq k$,
- w_{k+1} is the word consisting of $n - T_{k2}$ vertical steps and w_{k+1}' is the word consisting of $n - T_{k1}$ horizontal steps.

It is clear that $w(T)$ is indeed a Dyck path with descent set S, and each such Dyck path is given by $w(T)$ for a unique SSYT T. □

The statistic MAJ on Dyck paths is defined as

$$\mathrm{MAJ}(w) = \sum_{i \in D(w)} i. \tag{5}$$

In [4] Fürlinger and Hofbauer defined the *q-Narayana numbers*, $N_q(n,k)$, by

$$N_q(n,k) := \sum_{w \in \mathscr{D}_n, \mathrm{des}(w) = k} q^{\mathrm{MAJ}(w)}.$$

We will see that $N_q(n,k)$ can be written in the explicit form of (1). For each set-valued statistic in Theorem 9 we get a bi-statistic with a distribution given by the q-Narayana numbers .

Theorem 12. *For all $n, k \geq 0$ we have*

$$N_q(n,k) = s_{\langle 2^k \rangle}(q, q^2, \ldots, q^{n-1}).$$

Proof. By Theorem 11 we have that

$$\begin{aligned} \sum_{w \in \mathscr{D}_n, \mathrm{des}(w)=k} q^{\mathrm{MAJ}(w)} &= \sum_{|S|=k} \beta_n(S) q^{\sum_{s \in S} s} \\ &= \sum_T q^{\sum T_{ij}}, \end{aligned}$$

where the last sum is over all $SSYT$s T of shape $\langle 2^k \rangle$ with parts less than n. By the combinatorial definition of the Schur function this is equal to $s_{\langle 2^k \rangle}(q, q^2, \ldots, q^{n-1})$, and the theorem follows. □

To re-derive the formula (1) of the q-Narayana numbers we apply the *hook-content formula*, Theorem 13, to the expression in Theorem 12. If we identify a partition λ with its diagram $\{(i,j) : 1 \leq j \leq \lambda_i\}$ then the *hook length*, $h(u)$, at $u = (x,y) \in \lambda$ is defined by

$$h(u) = |\{(x,j) \in \lambda : j \geq y\}| + |\{(i,y) \in \lambda : i \geq x\}| - 1,$$

and the *content*, $c(u)$, is defined by $c(u) = y - x$. Let $[n] := 1 + q + \cdots + q^{n-1}$, $[n]! := [n][n-1]\cdots[1]$ and

$$\begin{bmatrix} n \\ k \end{bmatrix} := \frac{[n]!}{[n-k]![k]!}.$$

Theorem 13 (Hook-content formula)**.** *For any partition λ and $n > 0$,*

$$s_\lambda(q, q^2, \ldots, q^n) = q^{\sum i\lambda_i} \prod_{u \in \lambda} \frac{[n + c(u)]}{[h(u)]}.$$

For a proof see Theorem 7.21.2 of [12]. We now have an alternative proof of the following result which was proved in [4], and is a special case of a result of MacMahon, stated without proof in [8, p. 1429].

Corollary 14 (Fürlinger, Hofbauer, MacMahon)**.** *The q-Narayana numbers are given by:*

$$N_q(n,k) = \frac{1}{[n]} \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} n \\ k+1 \end{bmatrix} q^{k^2+k}$$

Proof. The Corollary follows from Theorem 12 after an elementary application of the hook-content formula, which is left to the reader. □

Remark 15. The Narayana statistic ea (even ascents) cannot in a natural way be associated to a shelling of $\Delta(J(\mathbf{2} \times \mathbf{n}))$. However, it would be interesting to find a co-statistic s for ea such that the bi-statistic (ea, s) has the q-Narayana distribution.

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Matematik, Chalmers tekniska högskola och Göteborgs universitet, S-412 96 Göteborg, Sweden

E-mail address: `branden@math.chalmers.se`