RANDOM WALKS ON TREES AND RAMANUJAN GRAPHS

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Abstract. Different combinatorial techniques can be used successfully in the investigation of random walks on infinite graphs and more specifically, on infinite trees. We shall report on some properties of random walks on trees with finitely many cone types, which are examples of infinite tress satisfying a natural asymptotic finiteness condition. We shall then explain how these results can be applied to the study of Ramanujan graphs. In particular, an algorithm will be presented which determines in finite time whether a given finite graph is Ramanujan.

On étudie les propriétés des marches aléatoires sur des graphes infinis et, plus particulièrement, des arbres infinis, a l'aide des techniques combinatoires. On s'intéresse en particulier aux marches aléatoires sur les arbres avec un nombre fini de types coniques, une condition naturelle de finitude. Il se trouve que les résultats que nous obtenons sont utiles dans l'étude des graphes finis ayant des bonnes propriétés d'expansion. On en déduit un algorithme qui permet d'établir, en temps fini, si un graphe fini donné possède la propriété de Ramanujan.

1. Introduction

The notion of a cone type in a graph was introduced by Cannon in [1]. Probabilities on infinite trees with finitely many cone types were first studied by Lyons in [5]. An investigation of random walks on such trees was undertaken in the author's PhD thesis [6] and was pursued in a joint paper with Woess [7]. Two main motivations for this investigation are given by two important classes of trees with finitely many cone types: trees of geodesics of hyperbolic groups and of Coxeter groups, and universal covers of finite graphs. We were able to apply our results on random walks and get interesting implications for both these classes of trees. This paper discusses the latter class of universal covering trees.

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2. Random walks on graphs

Let Γ be an infinite, locally finite, connected, simple graph with the vertex set $V(\Gamma)$. For $x, y \in V(\Gamma)$ a *path* of length n from x to y is a sequence $x = x_1, x_2, ..., x_{n+1} = y$ of vertices, such that, for each $1 \le i \le n$, the vertices x_i and x_{i+1} are neighbours, $x_i \sim x_{i+1}$. The *distance* between two vertices $d(x, y)$ is the length of a shortest path from x to y. A shortest path is called *geodesic*. With a base point $x_0 \in V(\Gamma)$ fixed, the *norm* of a vertex x is $|x| = d(x_0, x)$.

A random walk on Γ is described by a stochastic transition matrix

$$P \;=\; \big(p(x,y)\big)_{x,y\in V(\Gamma)} \;.$$

The *Green kernel* on Γ associated with the random walk P is defined by

$$G(x, y \mid z) \;=\; \sum_{n=0}^{\infty} p^{(n)}(x,y) z^n \;.$$

The radius of convergence of the series $G(x, y \mid z)$ is $R_G = 1/\rho \ge 1$ with

$$\rho \;=\; \limsup_{n\to\infty} \big(p^{(n)}(x,y)\big)^{1/n} \;.$$

It is independent of x and y if P is irreducible. The number ρ is called the *spectral radius* of the random walk P.

Geometry of the graph is important for the study of random walks on it. Random walks on *trees* are particularly nice to deal with because any two vertices of a tree are connected by a unique path. On the other hand many different physical and applied problems can be modeled by a random walk on a tree. The book [8] contains much information and extensive bibliography on random walks on trees.

3. Trees with finitely many cone types

In this paper we shall turn our attention to *trees with finitely many cone types* which are defined as follows.

Definition 1. *Let T be an infinite, locally finite tree. We say that a vertex x is a predecessor of a vertex y and y is a successor of x, if x and y are neighbours (i.e., connected by an edge), and $|x| < |y|$. We define the cone $C(x)$ of a vertex x of T as the induced subgraph of T rooted at x with $V\big(C(x)\big) = \{y \in V(T) \mid x$ belongs to a geodesic from x_0 to $y\}$. Two vertices x and y are said to be of a same cone type if their cones are isomorphic as rooted graphs. Consider a function $t : V(T) \to \mathbb{Z}_+$, such that $t(x) = 0$ if and only if $x = x_0$ and $t(x_1) = t(x_2)$ if and only if $x_1, x_2 \in V(T) \setminus \{x_0\}$ and are of a same cone type. We will say that a vertex x is of type $t(x)$.*

Suppose that there exists a type function t on T which takes values in a finite set $\{0, 1, ..., K\}$. We say then that T has finitely many cone types.

Since T is a tree, each vertex x of T except x_0 has exactly 1 predecessor $pr(x)$. The number $s_{i,j}$ of successors of type j of a vertex of type i is well defined for every $i \in \mathbb{Z}_+$ and $j \in \mathbb{N}$.

4. The Green function and the spectral radius

Consider an infinite, locally finite tree T with $K+1$ cone types. On T consider a nearest neighbour random walk $P = \left(p^{(n)}(x, y)\right)_{x,y \in V(T)}$ such that $p(x, pr(x))$ depends only on the cone type of x for any $x \in V(T) \setminus \{x_0\}$, and $p(x, y)$ depends only on the cone types of x and y for any $x \in V(T)$ and for any successor y of x. Consequently, the probability $p(x, pr(x))$ for a vertex of type $i, i \geq 1$ will be denoted by p_{-i}, and the probability $p(x, y)$ will be denoted by $p_{i,j}$ for x of type i and y a successor of x of type j.

Theorem 1. *Let T be an infinite, locally finite tree with $K+1$ cone types. For every $x, y \in V(T)$, the Green function $G(x, y \mid z)$ is an algebraic element over the field $\mathbb{Q}(z, \{p_{-i}z\}_{i=1}^{K}, \{p_{i,j}z\}_{i,j=1}^{K})$.*

It can be deduced from Theorem 1 that the spectral radius of a tree with finitely many cone types is an algebraic number. Moreover, close inspection of systems of polynomial equations which appear in the proof of Theorem 1 shows that one can use them to actually compute the spectral radius. Computations of this type had appeared in different places, in a similar context, e.g. in Lalley's study of finite range random walks on a free group (see [3], [8].) A similar method was used in [7] to compute the rate of escape of the random walk P on T.

5. What is a Ramanujan graph?

We shall now turn our attention to finite graphs, and infinite trees will appear in what follows in the role of universal coverings of finite graphs. It can be shown that if a tree covers a finite graph then it covers infinitely many finite graphs. Trees which cover finite graphs are called *uniform.*

A random walk and a spectral radius can be defined for a finite graph in the same way as they were defined above for an infinite graph. In the irreducible case, the spectral radius of a random walk on a finite graph is the largest eigenvalue of the (finite) transition matrix P. This number is closely connected with another constant associated with the graph, namely the *isoperimetric constant*. Recall that the isoperimetric constant is defined as

$$i \;=\; inf\{\frac{|\partial A|}{|A|} \mid A \subset V(\Gamma),\ \text{finite}\}\ ,$$

where ∂A denotes the boundary of the set A.

Definition 2. *An infinite sequence of finite regular graphs of a fixed degree k is said to be a family of expanders if the isoperimetric constant of every graph in the sequence is bounded away from* 0 *by a constant which depends only on k.*

There appear to be geometric obstructions on how big the isoperimetric constant of a k-regular finite graph can be. The graphs with biggest possible isoperimetric constant are called *Ramanujan*. More precisely,

Definition 3. *A finite connected k-regular graph is called Ramanujan if the second largest eigenvalue of its adjacency matrix is not bigger than* $2\sqrt{k-1}$.

Existence of infinite families of such graphs of fixed degree k is a difficult and important open question. It is not known whether an infinite sequence of k-regular Ramanujan graphs exists for every $k \geq 3$. For some values of k explicit constructions were given by Lubotzky, Phillips and Sarnak using deep results from representation theory and number theory, in particular, a partial solution of Ramanujan conjecture (which explains the terminology). The problem can be reformulated as follows: does every regular tree of degree greater than or equal to 3 covers infinitely many finite Ramanujan graphs? $k = 7$ is the smallest value for which the answer is unknown.

It was noticed by Lubotzky and Greenberg that the property of a graph to be Ramanujan is in fact well-defined in the class of all finite connected (not necessarily regular) graphs.

Definition 4. *A finite connected graph Γ is called Ramanujan if the second largest eigenvalue of its adjacency matrix is not bigger than the spectral radius of the adjacency matrix of the universal covering tree of* Γ.

Greenberg justified this definition by generalizing several results on Ramanujan graphs to the non-regular case [2].

The question of existence of infinite families of Ramanujan, not necessarily regular graphs was studied in a joint paper by Alex Lubotzky and the author [4]. In this paper it is shown that there exist infinitely many uniform trees which cover no Ramanujan graph; a sufficient condition is given for a finite graph to be covered by such tree; and explicit examples are constructed.

6. How to recognize a Ramanujan graph?

While Greenberg has shown that many results about regular Ramanujan graphs remain true in the non-regular setting, the following problem arose. It is fairly easy, given a finite connected k-regular graph, to determine whether it is Ramanujan or not. It amounts to compute the second largest eigenvalue of its adjacency matrix and to compare it with $2\sqrt{k-1}$. But what about an arbitrary finite connected graph?

Given a finite graph Γ, is there an algorithm which decides in finite time whether Γ is Ramanujan?

The second largest eigenvalue of the adjacency matrix of a finite connected graph can be easily computed. The real question is what to compare it to, in other words, can one produce an algorithm which computes the spectral radius of the adjacency matrix of the universal covering tree.

The first step on the way of constructing such algorithm consists in introducing a type function on the set of vertices of the universal covering tree. The number of different values this function takes is at most the double of the number of edges of Γ.

Computation of the spectral radius of the adjacency matrix, on a non-regular tree, is not equivalent to the computation of the spectral radius of the matrix of transition probabilities of a random walk. Nevertheless the method to which we have alluded in Section 4, of computing the spectral radius of the transition matrix of a random walk preserving the cone types structure on a tree with finitely many cone types, can be carried out without any change for computing the spectral radius of the adjacency matrix of a uniform tree.

References

[1] J. Cannon, *The combinatorial structure of cocompact discrete hyperbolic groups,* Geometriae Dedicata **16** (1986), 123-148.

[2] Y. Greenberg, *Ph.D. thesis*, Hebrew University of Jerusalem (1995).

[3] S. Lalley, *Finite range random walk on free groups and homogeneous trees*, Ann. Probab. **21** (1993), 2087-2130.

[4] A. Lubotzky, T. Nagnibeda, *Not every uniform tree covers Ramanujan graphs*, J. Comb. Th., Series B **74** (1998), 202-212.

[5] R. Lyons, *Random walks and percolation on trees*, Ann. Probab. **18** (1990), 931-958.

[6] T. Nagnibeda, *Ph.D. thesis*, University of Geneva (1997).

[7] T. Smirnova-Nagnibeda, W. Woess, *Random walks on trees with finitely many cone types*, J. Theor. Prob. **15** (2002), 383-422.
[8] W. Woess, *Random walks on infinite graphs and groups*, Cambridge University Press (2002)