Minimal matrices and minimal components in Kronecker products

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Abstract

We outline a combinatorial method for determining all minimal components, with respect to the dominance order of partitions, of the Kronecker product $\chi^\lambda \otimes \chi^\mu$ of two complex irreducible characters of the symmetric group. The method uses the notion of minimal matrix and a generalization of the dual RSK correspondence to 3-dimensional matrices. One of the main results is the description of the multiplicities of the minimal components of $\chi^\lambda \otimes \chi^\mu$ as the number of integral points in certain convex polytopes. A second result is the determination, in most cases, of all minimal components of $\chi^\lambda \otimes \chi^\mu$ and their multiplicities when λ has two parts and μ is arbitrary.

Résumé

Nous montrons une methode combinatoire por determiner toutes les composantes minimaux, par raport de l'ordre de domination des partages d'un entier, dans les produit de Kronecker $\chi^\lambda \otimes \chi^\mu$ de deux charactères irréductibles du groupe symétrique. Cet methode utilice la notion de matrice minimal et une generalization de la correspondence RSK dual au matrices des trois dimensions. Un de notres resultats plus importants est la description des multiplicités des composantes minimales de $\chi^\lambda \otimes \chi^\mu$ comme le nombre des pointes integrales dans certains politopes convexes. A second resultat d'importance est la determination, dans presque toutes les cases, du composantes minimaux de $\chi^\lambda \otimes \chi^\mu$ et de leur multiplicités, quand λ a deux parties et μ est arbitraire.

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1 Introduction

Let λ, μ, ν be partitions of n and denote by $c(\lambda,\mu,\nu)$ the multiplicity of the complex irreducible character χ^ν of the symmetric group S_n in the Kronecker product $\chi^\lambda \otimes \chi^\mu$ of other two irreducible characters of the same symmetric group S_n, that is,

$$\chi^\lambda \otimes \chi^\mu = \sum_{\nu \vdash n} c(\lambda,\mu,\nu)\chi^\nu.$$

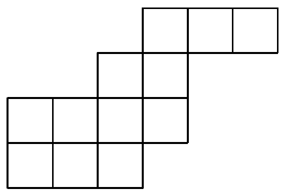
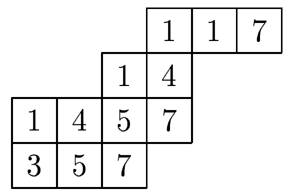
In this paper we will be concerned with minimal components of $\chi^\lambda \otimes \chi^\mu$ with respect to the dominance order $\trianglelefteq$ defined on the set of partitions of n. Our main results are Theorem 4.4, from which it follows a combinatorial description of the multiplicities of the minimal components of a Kronecker product $\chi^\lambda \otimes \chi^\mu$ as the number of integral points in certain convex polytopes; and Theorems 5.5 and 5.7 that give, in most cases, a complete description of all minimal components in $\chi^\lambda \otimes \chi^\mu$ and their multiplicities when λ has two parts and μ is arbitrary. More precisely. Let us write $\gamma \trianglelefteq \nu$ or $\nu \trianglerighteq \gamma$ to indicate that γ is dominated by ν; in case $\gamma \neq \nu$, we write $\gamma \triangleleft \nu$ or $\nu \triangleright \gamma$. We say that χ^ν is a *minimal component* of $\chi^\lambda \otimes \chi^\mu$ if $c(\lambda,\mu,\nu) > 0$ and $c(\lambda,\mu,\gamma) = 0$ for all $\gamma \triangleleft \nu$. The study of minimal components in Kronecker products was initiated in [12, 13], where it was shown that the notion of minimal matrix [11] is useful for understanding the minimal components in a Kronecker product. It was shown, for example, that any minimal plane partition with row sum vector λ and column sum vector μ yields a minimal component of $\chi^\lambda \otimes \chi^\mu$. Another observation from [13] that will be important in this paper is that the multiplicities of minimal components have combinatorial descriptions as the number of certain pairs of Littlewood-Richardson multitableaux. The derivation of this description is easy to obtain and follows from standard techniques. We believe, however, that this description, despite its simplicity, is important and deserves more attention. It is used, for example, in [1] to formulate a generalization of the dual RSK correspondence for 3-dimensional matrices of zeros and ones. This correspondence extends results of [13] and gives a combinatorial way to compute the contribution of each minimal matrix to some minimal component in some Kronecker product. Another application of the correspondence from [1] is a new description of the multiplicities of the minimal components in a Kronecker product as the number of integral points in certain convex polytopes. These methods are more efficient when, for given λ and μ, we know all minimal matrices with row sum vector λ and column sum vector μ. We illustrate this point of view by applying the classification of all minimal matrices of size $2 \times q$ given in [14]. In this way we determine, in most cases, all minimal components of $\chi^\lambda \otimes \chi^\mu$ and their multiplicities when λ has two parts and μ is arbitrary. These results are new except for small overlappings with [4, 5, 6, 7] when μ has two parts or is a hook.

The paper is organized as follows. In Section 2 we introduce Littlewood-Richardson multitableaux and give in Lemma 2.2 the first combinatorial description for the multilicities of minimal components. In Section 3 we introduce the notion of minimal matrix and review some results of [13] needed in this paper. In particular Proposition 3.8 was the starting point for the extension of the dual RSK correspondence to 3-dimensional matrices given in [1]. This correspondence is presented in Section 4. The main results of this section are Theorem 4.4 and its corollaries, which are new and contain a description for the multiplicities of minimal components as the number of integral points in certain convex polytopes. Finally, in Section 5 we apply the results from previous sections and the classification of minimal matrices of size $2 \times q$ [14] to determine, in most cases, all minimal components of $\chi^\lambda \otimes \chi^\mu$ when λ has two parts and μ is arbitrary.

2 Littlewood-Richardson multitableaux

Let $\alpha = (\alpha_1, \dots, \alpha_a)$ be a partition of a positive integer n, in symbols $\alpha \vdash n$. Its diagram is the set of pairs of positive integers $\{ (i,j) \mid 1 \leq i \leq a, 1 \leq j \leq \alpha_i \}$, which we also denote by α. The partition α' *conjugate* to α is obtained by transposing the diagram of α, that is, $\alpha' = \{ (i,j) \mid (j,i) \in \alpha \}$. If β is another partition and the diagram of β is a subset of the diagram of α, in symbols $\beta \subseteq \alpha$, we denote by α/β the *skew diagram* consisting of the points in α that are not in β, and by $|\alpha/\beta|$ its cardinality. It is customary to represent diagrams pictorially as a collection of boxes [2, 8, 10]. Any filling T of a skew diagram α/β with positive integers, formally a map $T : \alpha/\beta \longrightarrow \mathbb{N}$, will be called a *Young tableau* or just a *tableau* of *shape* α/β. We indicate the shape of T by the notation $\mathsf{sh}(T) = \alpha/\beta$, or $\mathsf{sh}(T) = \alpha$ if β is the empty partition. A tableau T is called *semistandard* if its rows are weakly increasing from left to right and its columns are strictly increasing from top to bottom. The *content* of T is the composition $\mathsf{cont}(T) = (\gamma_1, \dots, \gamma_c)$, where γ_i is the number of i's in T. The *word*, or *row word* of T, denoted by $w(T)$, is obtained from T by reading its entries from left to right, in successive rows, starting with the bottom row and moving up. Similarly, the *column word* of

T, denoted by $w_{\mathrm{col}}(T)$, is obtained from T by reading its entries from bottom to top, in successive columns, starting in the left column and moving to the right. For example, let

$D =$  and $T =$ 

then D is a diagram of shape $(6,4,4,3)/(3,2)$ and T is a semistandard tableaux of this shape and content $(4,0,1,2,2,0,3)$; its row and column words are, respectively, $w(T) = 357145714117$ and $w_{\mathrm{col}}(T) = 315475174117$. The *reverse word* w^* of $w = w_1 \cdots w_k$ is w read backwards: $w^* = w_k \cdots w_1$. Finally, a word $w = w_1 \cdots w_k$ in the alphabet $1, \ldots, n$ is called a *lattice permutation* if for all $1 \leq j \leq k$ and all $1 \leq i \leq n-1$ the number of occurrences of i in $w_1 \cdots w_j$ is not less than the number of occurrences of $i+1$ in $w_1 \cdots w_j$. A semistandard tableau T of skew shape is called a *Littlewood-Richardson* tableau if its *reverse word* $w(T)^*$ is a lattice permutation.

Let α and $\nu = (\nu_1, \ldots, \nu_r)$ be partitions of n, then a sequence $T = (T_1, \ldots, T_r)$ of tableaux is called a *Littlewood-Richardson multitableau* of *shape* α and *type* ν if there exists a sequence of partitions

$$\emptyset = \alpha(0) \subseteq \alpha(1) \subseteq \cdots \subseteq \alpha(r) = \alpha$$

such that T_i is a Littlewood-Richardson tableau of shape $\alpha(i)/\alpha(i-1)$ and $|\alpha(i)/\alpha(i-1)| = \nu_i$ for all $1 \leq i \leq r$. If each T_i has content $\rho(i)$, then we say that T has *content* $(\rho(1), \ldots, \rho(r))$. Note that, since T_i is a Littlewood-Richardson tableau, $\rho(i)$ is a partition of ν_i. Given partitions α, β and $\nu = (\nu_1, \ldots, \nu_r)$ of n we denote by $\mathsf{LR}^*(\alpha, \beta; \nu)$ the set of all pairs (T, S) of Littlewood-Richardson multitableaux of shape (α, β), type ν and *conjugate* content, that is, if T has content $(\rho(1), \ldots, \rho(r))$, then S has content $(\rho(1)', \ldots, \rho(r)')$. Let also

$$lr^*(\alpha, \beta; \nu) := \#\mathsf{LR}^*(\alpha, \beta; \nu).$$

denote the cardinality of $\mathsf{LR}^*(\alpha, \beta; \nu)$. Finally, let $\phi^\nu = \mathrm{Ind}_{S_\nu}^{S_n}(1_\nu)$ denote the character of S_n induced from the trivial character 1_ν of the Young subgroup S_ν associated to ν, and let (φ, ψ) denote the inner product of two complex characters φ and ψ of S_n. Throughout this paper we will make frequent use of Young's rule that expresses ϕ^ν as a linear combination of irreducible characters. Let $K_{\gamma\nu}$ denote the number of all semistandard Young tableaux of shape γ and content ν, then

$$\phi^\nu = \sum_{\gamma \trianglerighteq \nu} K_{\gamma\nu} \chi^\gamma. \tag{1}$$

The following lemma is a consequence of Frobenius reciprocity and the Littlewood-Richardson rule, compare with [3, 2.9.17].

2.1. Lemma. *Let λ, μ, ν be partitions of n. Then*

$$lr^*(\lambda, \mu; \nu) = \left(\chi^\lambda \otimes \chi^\mu, \phi^\nu \otimes \chi^{(1^n)}\right).$$

This lemma and Young's rule (1) imply the next lemma.

2.2. Lemma. *Let χ^ν be a component of $\chi^\lambda \otimes \chi^\mu$. Then χ^ν is a minimal component of $\chi^\lambda \otimes \chi^\mu$ if and only if $c(\lambda, \mu, \nu) = lr^*(\lambda, \mu; \nu')$.*

As we mentioned in the introduction this lemma is easy to obtain. We think, however, that the numbers $lr^*(\lambda, \mu; \nu)$ are important and deserve more attention. For example, we made use of them in Proposition 3.8, Theorem 4.4 and Corollary 4.6. Moreover, all computations in Section 5 depend on Corollary 4.6, and thus on these numbers.

3 Minimal matrices

Minimal matrices were introduced in [11] to characterize 3-dimensional matrices with zeros and ones that are uniquely determined by its plane sums. It was shown in [13] that minimal matrices yield information on minimal components in Kronecker products. We will review some results needed in this paper.

3.1. Notation. Let $\lambda = (\lambda_1, \ldots, \lambda_p)$, $\mu = (\mu_1, \ldots, \mu_q)$, $\nu = (\nu_1, \ldots, \nu_r)$ be partitions of n. Let $\mathsf{M}(\lambda, \mu)$ denote the set of all matrices $A = (a_{ij})$ of size $p \times q$ with non-negative integer entries, row sum vector λ and column sum vector μ, that is, $\sum_j a_{ij} = \lambda_i$ for $1 \le i \le p$, and $\sum_i a_{ij} = \mu_j$ for $1 \le j \le q$. For any matrix with non-negative integer entries, we denote by $\pi(A)$ the decreasing sequence of its entries and call it a *π-sequence*. Also let $\mathsf{M}_\nu(\lambda, \mu)$ denote the subset of $\mathsf{M}(\lambda, \mu)$ formed by all matrices A with $\pi(A) = \nu$, and let $m_\nu(\lambda, \mu) = \#\mathsf{M}_\nu(\lambda, \mu)$ be its cardinality. Similarly, let $\mathsf{M}^*(\lambda, \mu, \nu)$ denote the set of all 3-dimensional matrices $A = (a_{ijk})$ of size $p \times q \times r$ with entries equal to zero or one that have plane sums λ, μ, ν, that is, $\sum_{jk} a_{ijk} = \lambda_i$ for $1 \le i \le p$, $\sum_{ik} a_{ijk} = \mu_j$ for $1 \le j \le q$ and $\sum_{ij} a_{ijk} = \nu_k$, for $1 \le k \le r$, and let $m^*(\lambda, \mu, \nu) = \#\mathsf{M}^*(\lambda, \mu, \nu)$ be its cardinality.

3.2. Definition. [11] A matrix A in $\mathsf{M}(\lambda, \mu)$ is called *minimal* if there is no other matrix B in $\mathsf{M}(\lambda, \mu)$ such that $\pi(B) \lhd \pi(A)$.

3.3. Example. Let

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix},$$

then A, B and C have the same row and column sum vectors, and $\pi(A) = \pi(B) = (2, 2, 1, 1)$ and $\pi(C) = (3, 3)$. Thus A and B are minimal and C is not.

To each matrix $A = (a_{ij})$ in $\mathsf{M}_\nu(\lambda, \mu)$ we associate a 3-dimensional matrix $\overline{A} = (a_{ijk})$ by

$$a_{ijk} = \begin{cases} 1 & \text{if } a_{ij} \le k, \\ 0 & \text{otherwise.} \end{cases}$$

Since $\overline{A}$ is in $\mathsf{M}^*(\lambda, \mu, \nu')$, the correspondence $A \mapsto \overline{A}$ defines an injective map

$$F_{\lambda,\mu,\nu} : \mathsf{M}_\nu(\lambda, \mu) \longrightarrow \mathsf{M}^*(\lambda, \mu, \nu'). \tag{2}$$

We have the following characterization of minimality.

3.4. Proposition. [13, Prop. 3.1] *Let $A \in \mathsf{M}_\nu(\lambda, \mu)$. Then A is minimal if and only if $F_{\lambda,\mu,\nu}$ is surjective.*

The proof given in [13] is algebraic and uses character theory. It is also possible to give a direct combinatorial proof. The following two lemmas establish a connection between minimal matrices and multiplicities of minimal components in Kronecker products.

3.5. Lemma. *Let $A \in \mathsf{M}_\nu(\lambda, \mu)$ be minimal. Then*
(i) $c(\alpha, \beta, \gamma) = 0$ for all $\alpha \unrhd \lambda$, $\beta \unrhd \mu$, $\gamma \lhd \nu$.
(ii) $c(\alpha, \beta, \nu) = lr^*(\alpha, \beta, \nu')$ for all $\alpha \unrhd \lambda$, $\beta \unrhd \mu$.

Note that (i) is Proposition 3.2 in [13], and (ii) follows from Lemma 2.1, (1) and (i).

3.6. Lemma. *If χ^ν is a minimal component of $\chi^\lambda \otimes \chi^\mu$, then there is a minimal matrix A in $\mathsf{M}(\lambda, \mu)$ such that $\pi(A) \trianglelefteq \nu$.*

3.7. Remark. This lemma can be restated in the following way. Let $\Phi(\lambda, \mu)$ be the set of all partitions ν such that there is a minimal matrix A in $\mathsf{M}_\nu(\lambda, \mu)$, and let $\Xi(\lambda, \mu)$ be the set of all partitions ν such that χ^ν is a minimal component of $\chi^\lambda \otimes \chi^\mu$. Then every element in $\Xi(\lambda, \mu)$ is bounded below by some element of $\Phi(\lambda, \nu)$.

Recall that $m^*(\lambda, \mu, \nu) = \#\mathsf{M}^*(\lambda, \mu, \nu)$ and that $m_\nu(\lambda, \mu) = \#\mathsf{M}_\nu(\lambda, \mu)$. Lemmas 3.5 and 3.6 follows easily from a theorem due to Snapper [9] which expresses $m^*(\lambda, \mu, \nu)$ as an inner product of characters

$$m^*(\lambda, \mu, \nu) = \left(\phi^\lambda \otimes \phi^\mu \otimes \phi^\nu, \chi^{(1^n)}\right).$$

Then by Young's rule (1), this identity can be rewritten as

$$m^*(\lambda, \mu, \nu) = \sum_{\alpha,\beta} K_{\alpha\lambda} K_{\beta\mu}\, lr^*(\alpha, \beta; \nu). \tag{3}$$

Then from Proposition 3.4 we obtain

3.8. Proposition. [13, Cor. 3.3.2] *Let $A \in \mathsf{M}_\nu(\lambda, \mu)$ be a minimal matrix. Then*

$$m_\nu(\lambda, \mu) = \sum_{\alpha,\beta} K_{\alpha\lambda} K_{\beta\mu}\, lr^*(\alpha, \beta; \nu'). \tag{4}$$

Note that for each $\alpha \trianglerighteq \lambda$ and $\beta \trianglerighteq \mu$ Lemma 3.5 implies that $c(\alpha, \beta, \nu) = lr^*(\alpha, \beta; \nu')$. Thus the number $m_\nu(\lambda, \mu)$ of minimal matrices in $\mathsf{M}(\lambda, \mu)$ with π-sequence ν contributes to various multiplicities $c(\alpha, \beta, \nu)$. For those pairs (α, β) such that $c(\alpha, \beta, \nu)$ is positive, one has, by Lemma 2.2, that χ^ν is a minimal component of $\chi^\alpha \otimes \chi^\beta$. Moreover, by Theorem 3.4 in [13], all minimal plane partitions in $\mathsf{M}_\nu(\lambda, \mu)$ contribute to the summand $lr^*(\lambda, \mu; \nu')$. It is natural to ask which other matrices in $\mathsf{M}_\nu(\lambda, \mu)$ contribute to $lr^*(\lambda, \mu; \nu')$, and more general, to look for a bijection that realizes identity (4) and extends Theorem 3.4 in [13]. This will be expained in the next section.

4 An RSK correspondence for 3-dimensional matrices

The main result in [1] is the construction of a bijection that realizes identity (3) above. This bijection was motivated by the necessity to understand combinatorially the contribution of each minimal matrix to the right hand side of (4).

4.1. Theorem. [1, Thm. 3.2] *There is a one-to-one correspondence between* $\mathsf{M}^*(\lambda, \mu, \nu)$ *and the set of triples* $(Q, P, (T, S))$ *where* Q *and* P *are semistandard Young tableaux of content* λ *and* μ *respectively,* (T, S) *is in* $\mathsf{LR}^*(\alpha, \beta; \nu)$*, and* α *and* β *are the shapes of* Q *and* P *respectively.*

The correspondence is the composition of three bijections. The first one is tautological, the second one is the dual RSK correspondence applied simultaneously several times, and the third is an application of a convenient description of the Littlewood-Richardson rule. The three bijections are defined in the following way.

Split $A = (a_{ijk}) \in \mathsf{M}^*(\lambda, \mu, \nu)$ into its *level matrices* $A_k = \left(a_{ij}^{(k)}\right)$, $1 \leq k \leq r$, where $a_{ij}^{(k)} = a_{ijk}$; then $A \longmapsto (A_1, \ldots, A_r)$ is the first bijection.

Let (P_k, Q_k) be the pair of semistandard Young tableaux of conjugate shape that corresponds to A_k under the dual RSK correspondence, then

$$(A_1, \ldots, A_r) \longmapsto ((P_1, \ldots, P_r), (Q_1, \ldots, Q_r))$$

is the second bijection. Note that P_k and Q_k have each ν_k boxes, $1 \leq k \leq r$. Also $\sum_{k=1}^r \mathsf{cont}(P_k) = \mu$ and $\sum_{k=1}^r \mathsf{cont}(Q_k) = \lambda$, that is, the total number of j's in $P_1, \ldots, P_r$ is μ_j, $1 \leq j \leq q$ and the total number of i's in $Q_1, \ldots, Q_r$ is λ_i, $1 \leq i \leq p$. The third bijection is more elaborate. To each r-tuple $(P_1, \ldots, P_r)$ of semistandard tableaux obtained from the second bijection we associate a pair (P, S) such that

(1) P is a semistandard tableau such that $\mathsf{cont}(P) = \sum_{k=1}^r \mathsf{cont}(P_k) = \mu$.

(2) S is a Littlewood-Richardson multitableau of the same shape as P, type ν and content $(\mathsf{sh}(P_1), \ldots, \mathsf{sh}(P_r))$.

This is done in the following way. For each partition $\gamma = (\gamma_1, \ldots, \gamma_c)$ let $U(\gamma)$ denote the semistandard tableau of shape γ that has γ_i i's in row i. Let $\gamma(k) = \mathsf{sh}(P_k)$, $1 \leq k \leq r$. Then set $P^{(1)} = P_1$ and $S_1 = U(\gamma(1))$. Inductively define $P^{(k+1)}$ and S_{k+1} by considering the column words $w_{\mathrm{col}}(P_{k+1}) = v_m \cdots v_1$ and $w_{\mathrm{col}}(U(\gamma(k+1))) = u_m \cdots u_1$. Then $P^{(k+1)}$ is obtained by column inserting v_1 in $P^{(k)}$, then v_2 in the resulting tableau $v_1 \to P^{(k)}$ and son on, in other words,

$$P^{(k+1)} = v_m \to (\cdots (v_2 \to (v_1 \to P^{(k)})) \cdots),$$

and S_{k+1} is the tableau obtained by placing $u_1, \ldots, u_m$ successively in the new boxes. Let $P = P^{(r)}$ and $S = (S_1, \ldots, S_r)$. It is proved in [1] that (P, S)

satisfies the desired properties. Similarly from the r-tuple $(Q_1, \dots, Q_r)$ we construct a pair (Q, T) with properties analogue to (1) and (2). Then

$$((P_1, \dots, P_r), (Q_1, \dots, Q_r)) \longmapsto (Q, P, (T, S))$$

is the third bijection. Finally, the composition of the three bijections yields the correspondence in the statement of the Theorem.

4.2. Remark. A different bijection could be constructed from Proposition 1 in [2, p. 58]. However, the bijection presented in [1] is simpler to compute.

The bijection from Theorem 4.1 yields a combinatorial description of $\mathsf{LR}^*(\lambda, \mu; \nu)$ as the set of integral points in a convex polytope. To explain this we need some notation.

4.3. Notation. For $A \in \mathsf{M}^*(\lambda, \mu, \nu)$ of size $p \times q \times r$ with level matrices $A_1, \dots, A_r$ we define matrices

$$A^{\mathrm{row}} = \begin{bmatrix} A_r & A_{r-1} & \cdots & A_1 \end{bmatrix} \quad \text{and} \quad A^{\mathrm{col}} = \begin{bmatrix} A_1 \\ \vdots \\ A_r \end{bmatrix}$$

of sizes $p \times qr$ and $pr \times q$ respectively. Let $\mathsf{c}_1(A^{\mathrm{row}}), \dots, \mathsf{c}_{qr}(A^{\mathrm{row}})$ denote the columns of A^{row} and $\mathsf{r}_1(A^{\mathrm{col}}), \dots, \mathsf{r}_{pr}(A^{\mathrm{col}})$ denote the rows of A^{col}. We say that A satisfies the *row condition* if $\sum_{i=1}^{j} \mathsf{r}_i(A^{\mathrm{col}})$ is weakly decreasing for all $1 \le j \le pr$, and we say that A satisfies the *column condition* if $\sum_{i=j}^{qr} \mathsf{c}_i(A^{\mathrm{row}})$ is weakly decreasing for all $1 \le j \le qr$. Then the following is a consequence of Theorem 4.1.

4.4. Theorem. *Let λ, μ, ν be partitions of n, then $lr^*(\lambda, \mu; \nu)$ equals the number of matrices A in $\mathsf{M}^*(\lambda, \mu, \nu)$ that satisfy the row and column conditions. Thus $lr^*(\lambda, \mu; \nu)$ is the number of integral points in a convex polytope.*

4.5. Corollary. *Let χ^ν be a minimal component of $\chi^\lambda \otimes \chi^\mu$. Then $c(\lambda, \mu, \nu)$ equals the number of matrices $A \in \mathsf{M}^*(\lambda, \mu, \nu')$ that satisfy the row and column conditions. Thus $c(\lambda, \mu, \nu)$ is the number of integral points in a convex polytope.*

The Theorem is of particular interest when we have a minimal matrix $A \in \mathsf{M}_\nu(\lambda, \mu)$. Then, by Proposition 3.4, we can identify $\mathsf{M}_\nu(\lambda, \mu)$ with $\mathsf{M}^*(\lambda, \mu, \nu')$ and we obtain.

4.6. Corollary. *Let A be a minimal matrix in $\mathsf{M}_\nu(\lambda,\mu)$. Then $c(\lambda,\mu,\nu)$ equals the numbers of matrices in $\mathsf{M}_\nu(\lambda,\mu)$ that satisfies the row and column conditions. Moreover, if $c(\lambda,\mu,\nu)\neq 0$, then χ^ν is a minimal component of $\chi^\lambda\otimes\chi^\mu$.*

5 Minimal components of $\chi^\lambda\otimes\chi^\mu$ when λ has two parts

Let $\lambda=(\lambda_1,\lambda_2)$, $\mu=(\mu_1,\dots,\mu_q)$ be partitions of n. In this section we apply the results of Section 4 and the classification of minimal matrices of size $2\times q$ given in [14] to determine all minimal components of $\chi^\lambda\otimes\chi^\mu$ and their multiplicities. This is fully accomplished when $\lambda_1-\lambda_2\geq\#\{\,j\mid\mu_j\text{ is odd}\,\}$; all minimal components are obtained from minimal matrices of type I and all have multiplicity 1. In the complementary situation $\lambda_1-\lambda_2<\#\{\,j\mid\mu_j\text{ is odd}\,\}$, the behavior of minimal components is more complex and minimal matrices are not always sufficient to describe the minimal components of $\chi^\lambda\otimes\chi^\mu$. In some cases, however, there is a unique minimal component, it is determined by the minimal matrices of type II and it can have large multiplicity. We start by recalling the classification of minimal matrices of size $2\times q$ given in [14].

5.1. Definition. Let $A=(a_{ij})$ be a matrix in $\mathsf{M}(\lambda,\mu)$. Then A is called a matrix of *type I* if A has weakly decreasing columns, that is $a_{1j}\geq a_{2j}$, and any 2×2 submatrix

$$\begin{bmatrix}a_{1k} & a_{1l}\\ a_{2k} & a_{2l}\end{bmatrix},\quad k<l,$$

is either a plane partition or has one of the forms

$$\begin{bmatrix}c+1 & c\\ d & d+1\end{bmatrix}\quad\text{or}\quad\begin{bmatrix}c & c+1\\ d+1 & d\end{bmatrix};$$

and A is called a matrix of *type II* if A has at least one increasing column ($a_{1k}<a_{2k}$ for some $1\leq k\leq q$) and $|a_{1j}-a_{2j}|\leq 1$ for all $1\leq j\leq q$.

For example the matrix

$$\begin{bmatrix}10&10&9&9&9&7&7&6&6&4&4&3&3&2&2\\10&9&9&8&8&7&4&5&5&2&2&3&3&2&1\end{bmatrix}$$

is of type I, and the matrix

$$\begin{bmatrix}9&8&7&6&6&5&5&4&3&3&1&0&1\\10&7&8&6&6&4&4&5&3&2&0&1&0\end{bmatrix}.$$

is of type II. The main Theorem in [14] is

5.2. Theorem. *Let $\lambda = (\lambda_1, \lambda_2)$, $\mu = (\mu_1, \ldots, \mu_q)$ be partitions of n. Then a matrix $A \in \mathsf{M}(\lambda, \mu)$ is minimal if and only if A is either of type I or of type II.*

It follows [14] from the conditions on columns given in Definition 5.1 that

(1) If A is of type I, then $\lambda_1 - \lambda_2 \geq \#\{\, j \mid \mu_j \text{ is odd}\,\}$.

(2) If A is of type II, then $\lambda_1 - \lambda_2 < \#\{\, j \mid \mu_j \text{ is odd}\,\}$.

This implies that there cannot be simultaneously in a set $\mathsf{M}(\lambda, \mu)$ minimal matrices of both types. Recall the definitions of $\Phi(\lambda, \mu)$ and $\Xi(\lambda, \mu)$ from Remark 3.7. By looking at the structure of matrices of type I [14, Section 3] one can show

5.3. Lemma. *If $\lambda_1 - \lambda_2 \geq \#\{\, j \mid \mu_j \text{ is odd}\,\}$, then for any $\nu \in \Phi(\lambda, \mu)$ there is a minimal matrix $A \in \mathsf{M}_\nu(\lambda, \mu)$ that satisfies the row and column conditions.*

This, Remark 3.7 and Corollary 4.6 imply

5.4. Lemma. If $\lambda_1 - \lambda_2 \geq \#\{\, j \mid \mu_j \text{ is odd}\,\}$, then $\Phi(\lambda, \mu) = \Xi(\lambda, \mu)$.

This lemmas and Corollary 4.6 imply

5.5 Theorem. *If $\lambda_1 - \lambda_2 \geq \#\{\, j \mid \mu_j \text{ is odd}\,\}$, then every minimal matrix $A \in \mathsf{M}(\lambda, \mu)$ with π-sequence ν yields a minimal component χ^ν of $\chi^\lambda \otimes \chi^\mu$ with $c(\lambda, \mu, \nu) = 1$. Moreover, any minimal component of $\chi^\lambda \otimes \chi^\mu$ is obtained in this way.*

For type II the situation is more complex. First, Lemma 4.1 in [14] and Remark 3.7. imply

5.6. Lemma. *If $\lambda_1 - \lambda_2 < \#\{\, j \mid \mu_j \text{ is odd}\,\}$, then $\#\Phi(\lambda, \mu) = 1$, and the unique element ν in $\Phi(\lambda, \mu)$ has the property that for every minimal component χ^γ of $\chi^\lambda \otimes \chi^\mu$, $\gamma \trianglerighteq \nu$.*

The main result for matrices of type II requires the following notation. Let $A = (a_{ij})$ be a matrix of type II with q columns, let $d_j = a_{1j} - a_{2j} \in \{\, -1, 0, 1\,\}$, and let $\boldsymbol{d}_A = (d_q, \ldots, d_1)$ and $\mathbf{0} = (0, \ldots, 0)$ be the vector with q zeros. Then Corollary 4.6 implies

5.7. Theorem. *If $\lambda_1 - \lambda_2 < \#\{\, j \mid \mu_j \text{ is odd}\,\}$ and ν is the unique element in $\Phi(\lambda, \mu)$, then $c(\lambda, \mu, \nu)$ equals the number of matrices $A \in \mathsf{M}(\lambda, \mu)$*

of type II whose first row is weakly decreasing and such that $\mathbf{d}_A \trianglerighteq \mathbf{0}$. Thus if $c(\lambda,\mu,\nu)>0$, χ^ν is the only minimal component of $\chi^\lambda \otimes \chi^\mu$.

The following examples illustrate Theorem 5.7.

5.8. Example. Let $\lambda=(9,7)$, $\mu=(7,5,3,1)$. The four minimal matrices of type II are

$$A=\begin{bmatrix}3&3&2&1\\4&2&1&0\end{bmatrix},\quad B=\begin{bmatrix}4&2&2&1\\3&3&1&0\end{bmatrix},\quad C=\begin{bmatrix}4&3&1&1\\3&2&2&0\end{bmatrix},$$

$$D=\begin{bmatrix}4&3&2&0\\3&2&1&1\end{bmatrix}.$$

Then Theorem 5.7 implies that $\nu=\pi(A)=(4,3^2,2^2,1^2)$ yields the only minimal component of $\chi^\lambda\otimes\chi^\mu$ and $c(\lambda,\mu,\nu)=3$. For any integer $N>0$ similar examples can be constructed such that $c(\lambda,\mu,\nu)=N$.

5.9. Example. Let $\lambda=(6,6)$, $=\mu=(3^4)$. There are six minimal matrices, all obtained by permuting the columns of

$$A=\begin{bmatrix}2&2&1&1\\1&1&2&2\end{bmatrix}.$$

Let $\nu=\pi(A)=(2^4,1^4)$. In this case $c(\lambda,\mu,\nu)=0$ and we have to look for the minimal components of $\chi^\lambda\otimes\chi^\mu$ in the interval $\{\gamma\vdash n \mid \nu\trianglelefteq\gamma\}$. The partitions covering ν in the dominance order are $\gamma(1)=(3,2^2,1^5)$ and $\gamma(2)=(2^5,1^2)$. It turns out that $c(\lambda,\mu,\gamma(i))=lr^*(\lambda,\mu;\gamma(i)')=0$ for $i=1,2$. The partitions covering $\gamma(1)$ and $\gamma(2)$ in the dominance order are $\delta(1)=(3^2,1^6)$, $\delta(2)=(3,2^3,1^3)$ and $\delta(3)=(2^6)$. For them we have $c(\lambda,\mu,\delta(i))=lr^*(\lambda,\mu,\delta')=1$ for $i=1,2,3$. Thus $\delta(1)$, $\delta(2)$ and $\delta(3)$ are all the minimal components of $\chi^\lambda\otimes\chi^\mu$.

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