TYPE $D_n^{(1)}$ CRYSTALS AND RIGGED CONFIGURATIONS

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Abstract. Finite-dimensional affine crystals are not yet very well understood. Virtual crystals yield a presentation of crystals of nonsimply-laced type in terms of crystals of simply-laced type. Hence it is important to understand the crystal structure of simply-laced types. The combinatorial structure of crystals of simply-laced type $A_n^{(1)}$ is known. In this note we describe the combinatorial structure of crystals of type $D_n^{(1)}$. We also give a bijection between crystal paths and rigged configurations for the antisymmetric tensor and spinor representation of type $D_n^{(1)}$. This bijection reflects two different methods to solve lattice models in statistical mechanics: the corner-transfer-matrix method and the Bethe Ansatz.

Résumé. Les cristaux affines de dimension finie ne sont pas encore très bien compris. Les cristaux virtuels permettent une présentation de cristaux non simplement entrelacés, en termes de cristaux simplement entrelacés. D'où l'importance de comprendre la structure de ces derniers. La structure combinatoire de cristaux simplement entrelacés $A_n^{(1)}$ est connue. Cet article décrit la structure combinatoire de cristaux de type $D_n^{(1)}$. Nous produisons aussi une bijection entre les chemins cristallins et les configurations 'gréées', pour un produit tensoriel antisymmetrique et une représentation spineur de type $D_n^{(1)}$. Cette bijection reflète l'existence de deux méthodes pour trouver la solution de modèles sur réseau en mécanique statistique: la méthode 'corner-transfer-matrix' et l'Ansatz de Bethe.

1. INTRODUCTION

The quantized universal enveloping algebra $U_q(\mathfrak{g})$ associated with a symmetrizable Kac–Moody Lie algebra $\mathfrak{g}$ was introduced independently by Drinfeld [1] and Jimbo [4] in their study of two dimensional solvable lattice models in statistical mechanics. The parameter q corresponds to the temperature of the underlying model. Kashiwara [6] showed that at zero temperature or $q = 0$ the representations of $U_q(\mathfrak{g})$ have bases, which he coined crystal bases, with a beautiful combinatorial structure and favorable properties such as uniqueness and stability under tensor products.

The underlying algebras of affine crystals are affine Kac–Moody algebras $\mathfrak{g}$. There are two affine crystal categories: (i) crystal bases of infinite dimensional integrable $U_q(\mathfrak{g})$-modules and (ii) crystal bases of finite dimensional $U_q'(\mathfrak{g})$-modules, where $U_q'(\mathfrak{g})$ is a subalgebra of $U_q(\mathfrak{g})$ without the degree operator. The first category is well-understood. For instance, it is known [7] that an irreducible integrable $U_q(\mathfrak{g})$-module has a unique crystal basis. On the other hand, the latter are still far from well-understood. For example, it is not even known which finite dimensional $U_q'(\mathfrak{g})$-modules have a crystal basis.

Recently, the existence of a family of finite dimensional $U_q'(\mathfrak{g})$-modules $\{W_s^{(r)}\}$ with crystal bases $\{B^{r,s}\}$ $(1 \le r \le n, s \ge 1)$, where $n+1$ is the number of vertices in the Dynkin diagram of $\mathfrak{g}$, were conjectured [3, 2]. The existence of such crystals allows the definition of one dimensional configuration sums, which play an important role in the study of phase transitions of two dimensional exactly solvable lattice models. For $\mathfrak{g}$

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of type $A_n^{(1)}$, the existence of the crystal $B^{r,s}$ was settled in [9], and the one dimensional configuration sums contain the Kostka polynomials, which arise in the theory of symmetric functions, combinatorics, the study of subgroups of finite abelian groups, and Kazhdan–Lusztig theory. In certain limits they are branching functions of integrable highest weight modules.

In [16, 18] virtual crystals were introduced. Virtual crystals are a realization of crystals of type X as crystals of type Y, based on well-known natural embeddings $X \hookrightarrow Y$ of affine algebras [5]:

$$\begin{array}{ccc} C_n^{(1)}, A_{2n}^{(2)}, A_{2n}^{(2)\dagger}, D_{n+1}^{(2)} & \hookrightarrow & A_{2n-1}^{(1)} \\ A_{2n-1}^{(2)}, B_n^{(1)} & \hookrightarrow & D_{n+1}^{(1)} \\ E_6^{(2)}, F_4^{(1)} & \hookrightarrow & E_6^{(1)} \\ D_4^{(3)}, G_2^{(1)} & \hookrightarrow & D_4^{(1)}. \end{array}$$

Note that under these embeddings every affine Kac–Moody algebra is embedded into one of simply-laced type $A_n^{(1)}$, $D_n^{(1)}$ or $E_6^{(1)}$. Hence, by the virtual crystal method the combinatorial structure of any finite-dimensional affine crystal can be expressed in terms of the combinatorial crystal structure of the simply-laced types. Whereas the affine crystals $B^{r,s}$ of type $A_n^{(1)}$ are already well-understood [19], this is not the case for $B^{r,s}$ of types $D_n^{(1)}$ and $E_6^{(1)}$.

In this note we discuss the affine crystals $B^{k,1}$ for type $D_n^{(1)}$ and the corresponding rigged configurations. The crystal structure of $B^{k,1}$ of type $D_n^{(1)}$ is given by Koga [11]. However, here we need a different combinatorial description of these crystals compatible with rigged configurations. Rigged configurations first arose in the Bethe Ansatz studies of exactly solvable lattice models. They yield fermionic formulas for the one-dimensional configuration sums of the underlying lattice model. The term fermionic formula was coined by the Stony Brook group [12, 13], who interpreted fermionic-type formulas for characters and branching functions of conformal field theory models as partition functions of quasiparticle systems with "fractional" statistics obeying Pauli's exclusion principle.

We present a bijection between crystal paths and rigged configurations of type $D_n^{(1)}$ generalizing the bijection of type $A_n^{(1)}$ [14, 15] and the bijections for crystals $B^{1,1}$ for any nonexceptional type [17]. This bijection reflects two different methods to solve lattice models in statistical mechanics: the corner-transfer-matrix method and the Bethe Ansatz.

2. Crystal bases

2.1. **Axiomatic definition of crystals.** Let $\mathfrak{g}$ be a symmetrizable Kac-Moody algebra, P the weight lattice, I the index set for the vertices of the Dynkin diagram of $\mathfrak{g}$, $\{\alpha_i \in P \mid i \in I\}$ the simple roots, and $\{h_i \in P^* \mid i \in I\}$ the simple coroots. Let $U_q(\mathfrak{g})$ be the quantized universal enveloping algebra of $\mathfrak{g}$ [8]. A $U_q(\mathfrak{g})$-crystal is a nonempty set B equipped with maps $\mathrm{wt} : B \to P$ and $e_i, f_i : B \to B \cup \{\varnothing\}$ for all $i \in I$, satisfying

$$f_i(b) = b' \Leftrightarrow e_i(b') = b \text{ if } b, b' \in B \tag{2.1}$$

$$\mathrm{wt}(f_i(b)) = \mathrm{wt}(b) - \alpha_i \text{ if } f_i(b) \in B \tag{2.2}$$

$$\langle h_i\, , \, \mathrm{wt}(b)\rangle = \varphi_i(b) - \epsilon_i(b). \tag{2.3}$$

Here for $b \in B$

$$\epsilon_i(b) = \max\{n \geq 0 \mid e_i^n(b) \neq \varnothing\}$$
$$\varphi_i(b) = \max\{n \geq 0 \mid f_i^n(b) \neq \varnothing\}.$$

(It is assumed that $\varphi_i(b), \epsilon_i(b) < \infty$ for all $i \in I$ and $b \in B$.) A $U_q(\mathfrak{g})$-crystal B can be viewed as a directed edge-colored graph (the crystal graph) whose vertices are the elements of B, with a directed edge from b to b' labeled $i \in I$, if and only if $f_i(b) = b'$.

2.2. **Tensor products of crystals.** Let $B_1, B_2, \dots, B_L$ be $U_q(\mathfrak{g})$-crystals. The Cartesian product $B_L \times \cdots \times B_2 \times B_1$ has the structure of a $U_q(\mathfrak{g})$-crystal using the so-called signature rule. The resulting crystal is denoted $B = B_L \otimes \cdots \otimes B_2 \otimes B_1$ and its elements $(b_L, \dots, b_1)$ are written $b_L \otimes \cdots \otimes b_1$ where $b_j \in B_j$. The reader is warned that our convention is opposite to that of Kashiwara [8]. Fix $i \in I$ and $b = b_L \otimes \cdots \otimes b_1 \in B$. The i-signature of b is the word consisting of the symbols $+$ and $-$ given by

$$\underbrace{-\cdots-}_{\varphi_i(b_L)\text{ times}} \quad \underbrace{+\cdots+}_{\epsilon_i(b_L)\text{ times}} \cdots \underbrace{-\cdots-}_{\varphi_i(b_1)\text{ times}} \quad \underbrace{+\cdots+}_{\epsilon_i(b_1)\text{ times}}.$$

The reduced i-signature of b is the subword of the i-signature of b, given by the repeated removal of adjacent symbols $+-$ (in that order); it has the form

$$\underbrace{-\cdots-}_{\varphi\text{ times}} \quad \underbrace{+\cdots+}_{\epsilon\text{ times}}.$$

If $\varphi = 0$ then $f_i(b) = \varnothing$; otherwise

$$f_i(b_L \otimes \cdots \otimes b_1) = b_L \otimes \cdots \otimes b_{j+1} \otimes f_i(b_j) \otimes \cdots \otimes b_1$$

where the rightmost symbol $-$ in the reduced i-signature of b comes from b_j. Similarly, if $\epsilon = 0$ then $e_i(b) = \varnothing$; otherwise

$$e_i(b_L \otimes \cdots \otimes b_1) = b_L \otimes \cdots \otimes b_{j+1} \otimes e_i(b_j) \otimes \cdots \otimes b_1$$

where the leftmost symbol $+$ in the reduced i-signature of b comes from b_j. It is not hard to verify that this well-defines the structure of a $U_q(\mathfrak{g})$-crystal with $\varphi_i(b) = \varphi$ and $\epsilon_i(b) = \epsilon$ in the above notation, with weight function

$$\mathrm{wt}(b_L \otimes \cdots \otimes b_1) = \sum_{j=1}^{L} \mathrm{wt}(b_j). \tag{2.4}$$

This tensor construction is easily seen to be associative. The case of two tensor factors is given explicitly by

$$f_i(b_2 \otimes b_1) = \begin{cases} f_i(b_2) \otimes b_1 & \text{if } \epsilon_i(b_2) \geq \varphi_i(b_1) \\ b_2 \otimes f_i(b_1) & \text{if } \epsilon_i(b_2) < \varphi_i(b_1) \end{cases} \tag{2.5}$$

and

$$e_i(b_2 \otimes b_1) = \begin{cases} e_i(b_2) \otimes b_1 & \text{if } \epsilon_i(b_2) > \varphi_i(b_1) \\ b_2 \otimes e_i(b_1) & \text{if } \epsilon_i(b_2) \leq \varphi_i(b_1). \end{cases} \tag{2.6}$$

2.3. **Combinatorial R-matrix.** Let B_1 and B_2 be two $U_q(\mathfrak{g})$-crystals such that $B_1 \otimes B_2$ and $B_2 \otimes B_1$ are connected and such that there is a weight $\lambda \in P$ with unique elements $u \in B_1 \otimes B_2$ and $\tilde{u} \in B_2 \otimes B_1$ of weight $\mathrm{wt}(u) = \mathrm{wt}(\tilde{u}) = \lambda$. Then there is a unique crystal isomorphism

$$R : B_1 \otimes B_2 \to B_2 \otimes B_1,$$

called the combinatorial R-matrix. Explicitly, $R(u) = \tilde{u}$ and the action of R on any other element in $B_1 \otimes B_2$ is determined by using

$$R(e_i(b)) = e_i(R(b)) \qquad \text{and} \qquad R(f_i(b)) = f_i(R(b)).$$

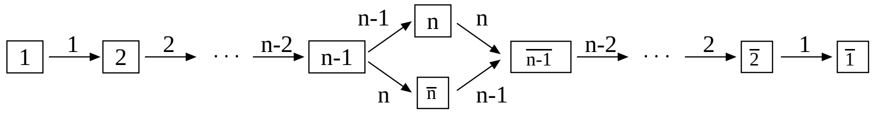


FIGURE 1. Crystal $B(\omega_1)$ of the vector representation

3. THE CRYSTAL $B^{k,1}$ OF TYPE $D_n^{(1)}$

From now on we restrict our attention to crystals of type D_n and $D_n^{(1)}$.

3.1. **Dynkin data.** For type D_n, the simple roots are

$$\begin{aligned} \alpha_i &= \varepsilon_i - \varepsilon_{i+1} \qquad \text{for } 1 \le i < n \\ \alpha_n &= \varepsilon_{n-1} + \varepsilon_n \end{aligned}$$

and the fundamental weights are

$$\begin{aligned} \Lambda_i &= \varepsilon_1 + \cdots + \varepsilon_i \qquad \text{for } 1 \le i \le n-2 \\ \Lambda_{n-1} &= (\varepsilon_1 + \cdots + \varepsilon_{n-1} - \varepsilon_n)/2 \\ \Lambda_n &= (\varepsilon_1 + \cdots + \varepsilon_{n-1} + \varepsilon_n)/2 \end{aligned}$$

where $\varepsilon_i \in \mathbb{Z}^n$ is the i-th unit standard vector. We also define

$$\begin{aligned} \omega_i &= \Lambda_i = \varepsilon_1 + \cdots + \varepsilon_i \qquad \text{for } 1 \le i \le n-2 \\ \omega_{n-1} &= \Lambda_{n-1} + \Lambda_n = \varepsilon_1 + \cdots + \varepsilon_{n-1} \\ \omega_n &= 2\Lambda_n = \varepsilon_1 + \cdots + \varepsilon_n \\ \overline{\omega}_n &= 2\Lambda_{n-1} = \varepsilon_1 + \cdots + \varepsilon_{n-1} - \varepsilon_n. \end{aligned}$$

3.2. **The crystals of antisymmetric tensor representations.** The crystal graphs of the antisymmetric tensor representations [10] are denoted by $B(\omega_\ell)$ (resp. $B(\overline{\omega}_n)$). The crystal graph $B(\omega_1)$ of the vector representation is given in Figure 1. The crystal $B(\omega_\ell)$ (resp. $B(\overline{\omega}_n)$) is the connected component of $B(\omega_1)^{\otimes \ell}$ (resp. $B(\omega_1)^{\otimes n}$) containing the element $\ell \otimes \ell - 1 \otimes \cdots \otimes 1$ (resp. $\overline{n} \otimes n - 1 \otimes \cdots \otimes 1$).

Explicitly, the elements of $B(\omega_\ell)$ for $1 \le \ell < n$ are single columns of height ℓ with entries from the alphabet $\{1 < 2 < \cdots < n-1 < {n \atop \overline{n}} < \overline{n-1} < \cdots < \overline{1}\}$. If $m_\ell \dots m_1$ is such a column it has to satisfy

(3.1)
1. $(m_j, m_{j+1}) = (n, \overline{n}), (\overline{n}, n)$ or $m_j < m_{j+1}$ for all j;
2. If $m_a = p$ and $m_b = \overline{p}$, then $\ell + 1 + a - b \le p$.

The elements of $B(\omega_n)$ (resp. $B(\overline{\omega}_n)$) are single columns of height n. An element $m_n \dots m_1 \in B(\omega_n)$ (resp. $B(\overline{\omega}_n)$) satisfies (3.1) and in addition

(3.2)
3. If $m_j = n$, then $n - j$ is even (resp. odd);
if $m_j = \overline{n}$, then $n - j$ is odd (resp. even).

The action of e_i, f_i for $1 \le i \le n$ is given by embedding $B(\omega_\ell) \hookrightarrow B(\omega_1)^{\otimes \ell}$ (resp. $B(\overline{\omega}_n) \hookrightarrow B(\omega_1)^{\otimes n}$) in the natural way by mapping $m_\ell \cdots m_1$ to $m_\ell \otimes \cdots \otimes m_1$ and using the tensor product rules.

3.3. **The crystals of spinor representations.** There are two crystals $B(\Lambda_n)$ and $B(\Lambda_{n-1})$ associated with the spinor representations of D_n [10]. The elements of $B(\Lambda_n)$ (resp. $B(\Lambda_{n-1})$) are given by single columns $m_n \dots m_1$ such that

(3.3)
1. $m_j < m_{j+1}$;
2. m and $\overline{m}$ cannot occur simultaneously;
3. If $m_j = n$, then $n-j$ is even (resp. odd);
   if $m_j = \overline{n}$, then $n-j$ is odd (resp. even).

If $b \in B(\Lambda_n)$ or $B(\Lambda_{n-1})$ and b contains i and $\overline{i+1}$ for $1 \le i < n$, then $f_i b$ is obtained by replacing i by $i+1$ and $\overline{i+1}$ by $\overline{i}$. Else $f_i b = 0$. If $b \in B(\Lambda_n)$ or $B(\Lambda_{n-1})$ and b contains $n-1$ and n, then $f_n b$ is obtained by replacing $n-1$ by $\overline{n}$ and n by $\overline{n-1}$. Otherwise $f_n b = 0$.

Note that for $k = n-1, n$ there is the following embedding of crystals

$$B(\Lambda_k) \hookrightarrow B(2\Lambda_k)$$

where e_i and f_i act as e_i^2 and f_i^2, respectively.

3.4. **Affine crystals.** As a classical crystal the affine crystal $B^{k,1}$ is isomorphic to

$$B^{k,1} \cong \begin{cases} B(\Lambda_k) \oplus B(\Lambda_{k-2}) \oplus \cdots \oplus B(0) & \text{if } k \text{ is even, } 1 \le k \le n-2 \\ B(\Lambda_k) \oplus B(\Lambda_{k-2}) \oplus \cdots \oplus B(\Lambda_1) & \text{if } k \text{ is odd, } 1 \le k \le n-2 \\ B(\Lambda_k) & \text{if } k = n-1, n. \end{cases} \tag{3.4}$$

The rules for f_0 on $B^{k,1}$ for $k = n, n-1$ is as follows. If $b = \overline{1}\overline{2} m_{n-2} \dots m_1$ then $f_0 b = m_{n-2} \dots m_1 21$. Otherwise $f_0 b = 0$. The operator f_0 on $B^{k,1}$ when $1 \le k < n-1$ was given by Koga [11] in terms of a tensor product of two spinor representations. For our purposes, we will need a combinatorial description of f_0 on elements $b \in B^{k,1}$ represented by a column of height k. This description of b will also be used in the description of the bijection from crystals to rigged configurations.

As explained in the section on the crystals of antisymmetric tensor representations, $b \in B(\Lambda_\ell)$ can be represented by a column of height ℓ. If $b \in B(\Lambda_\ell) \subset B^{k,1}$ then fill the column of height ℓ of b by pairs $(\overline{i}_j, i_j)$ for $1 \le j \le (k-\ell)/2$ in the following way to obtain a column of height k. Say that position p is connected to the pair $(\overline{q}, q) \in b$ if, for all $p < i < q$, b contains either i or $\overline{i}$, but not both $i, \overline{i}$. Set $i_0 = 0$. Let i_j be minimal such that $p := i_j > i_{j-1}$ and b contains neither p, $\overline{p}$. Furthermore, p is not connected to any pair $(\overline{q}, q) \in b$ with $q > p$. It is not hard to see that all added pairs $(\overline{i}_j, i_j)$ are disallowed according to (3.1) point 2. It can also be shown that f_i and the filling map commute, that is, the classical crystal graph with edges $f_1, \dots, f_n$ does not change. Denote the filling map to height k by F_k.

Let D_k, the dropping map, be the inverse of F_k. Explicitly, given a one-column tableau b of height k, set $i_0 = 0$. Let i_j be minimal such that $i_j > i_{j-1}$, $(\overline{i_j}, i_j) \in b$, $i_j - 1$ or $\overline{i_j - 1} \in b$, and if both $p, \overline{p} \notin b$ for some $i_{j-1} < p < i_j - 1$ then $(\overline{p+1}, p+1) \in b$. If no such i_j exists, set $a = j-1$. $D_k(b)$ is obtained by dropping all pairs $(\overline{i_j}, i_j)$ for $1 \le j \le a$ from b.

For the definition of f_0 we need slight variants of F_k and D_k, denoted by $\tilde{F}_k$ and $\tilde{D}_k$, respectively. $\tilde{F}_k$ and $\tilde{D}_k$ are defined in the same way as F_k and D_k, but with $i_0 = 2$.

Example 3.1. Let $b = \overline{3}\overline{5}532 \in B(\Lambda_5) \subset B^{9,1}$. Then $F_9(b) = \overline{3}\overline{5}\overline{6}\overline{7}76532$. Similarly if $b = \overline{2}\overline{3}\overline{4}\overline{5}65421 \in B^{9,1}$ then $D_9(b) = \overline{3}61$. Finally, $\tilde{F}_7(\overline{4}6421) = \overline{4}\overline{5}65421$.

Theorem 3.2. *Let $b \in B^{k,1}$. Then*

$$f_0 b = \begin{cases} F_k(\tilde{D}_{k-2}(x)) & \text{if } b = \overline{1}\overline{2}x \\ \tilde{F}_{k-1}(x)2 & \text{if } b = \overline{1}\overline{2}x2 \\ \tilde{F}_{k-1}(x)1 & \text{if } b = \overline{1}\overline{2}x1 \\ \tilde{F}_{k-2}(x)21 & \text{if } b = \overline{1}\overline{2}x21 \\ F_k(\tilde{D}_{k-1}(x)2) & \text{if } b = \overline{1}x \\ F_k(\tilde{D}_{k-1}(x)1) & \text{if } b = \overline{2}x \\ x21 & \text{if } b = \overline{1}x1 \text{ and } \tilde{D}_{k-2}(x) = x \\ 0 & \text{otherwise} \end{cases} \tag{3.5}$$

where x does not contain $1, 2, \overline{2}, \overline{1}$.

The proof will be given in a subsequent paper.

Alternatively, the action of f_0 is given by $f_0 = \sigma \circ f_1 \circ \sigma$ where σ is an involution on $B^{k,1}$ based on the type $D_n^{(1)}$ Dynkin diagram automorphism that interchanges nodes 0 and 1. If $b \in B^{k,1}$ contains either 1 or $\overline{1}$, $\sigma(b)$ is obtained from b by interchanging 1 and $\overline{1}$. If $b = \overline{1}x1$, then $\sigma(b) = \hat{F}_k(\hat{D}(x))$. If b contains neither 1 nor $\overline{1}$ then $\sigma(b) = F_k(\hat{D}(b))$. Here $\hat{D}$ and $\hat{F}$ are defined in the same way as D and F with $i_0 = 1$.

For later use, it will be convenient to define the sets $\hat{B}^{k,1}$ for $1 \le k \le n$ and $\hat{\hat{B}}^{n,1}$. For $1 \le k \le n-2$, $\hat{B}^{k,1} = B^{k,1}$. $\hat{B}^{n-1,1}$ is the set of all single columns of height $n-1$ satisfying condition (3.1) point 1. $\hat{B}^{n,1}$ (resp. $\hat{\hat{B}}^{n,1}$) is the set of all single columns of height n satisfying (3.1) point 1 and (3.2). An affine crystal structure can be defined on $\hat{B}^{n-1,1}$, $\hat{B}^{n,1}$ and $\hat{\hat{B}}^{n,1}$ by extending the rules of Theorem 3.2.

Theorem 3.3. *As affine crystals*

$$\begin{aligned} \hat{B}^{n-1,1} &\cong B^{n,1} \otimes B^{n-1,1} \\ \hat{B}^{n,1} &\cong B^{n,1} \otimes B^{n,1} \\ \hat{\hat{B}}^{n,1} &\cong B^{n-1,1} \otimes B^{n-1,1}. \end{aligned}$$

There are affine crystal embeddings

$$\mathrm{emb}_B : B^{k,1} \hookrightarrow B^{k,1} \otimes B^{k,1} \tag{3.6}$$

mapping $u(B^{k,1}) \mapsto u(B^{k,1}) \otimes u(B^{k,1})$ for $1 \le k \le n$ and sending e_i and f_i to e_i^2 and f_i^2, respectively, for $0 \le i \le n$. Here $u(B^{k,1}) = k(k-1)\cdots 1$ for $1 \le k \le n-2$, $u(B^{n-1,1}) = \overline{n}(n-1)\cdots 21$ and $u(B^{n,1}) = n(n-1)\cdots 21$. Note that the elements in the image of this embedding are aligned (see [16, Section 6.4]), meaning in this case that $\varphi_i(b)$ and $\epsilon_i(b)$ are even for all $0 \le i \le n$. Denote the image of $B^{k,1}$ under this embedding by $E^{k,1} = \mathrm{emb}_B(B^{k,1})$. For $k = n-1, n$ we view $E^{n-1,1} \subset \hat{\hat{B}}^{n,1}$ and $E^{n,1} \subset \hat{B}^{n,1}$.

As classical crystals

$$\begin{aligned} \hat{B}^{n-1,1} &\cong B(\omega_{n-1}) \oplus B(\omega_{n-3}) \oplus B(\omega_{n-5}) \oplus \cdots \\ \hat{B}^{n,1} &\cong B(\omega_n) \oplus B(\omega_{n-2}) \oplus B(\omega_{n-4}) \oplus \cdots \\ \hat{\hat{B}}^{n,1} &\cong B(\overline{\omega}_n) \oplus B(\omega_{n-2}) \oplus B(\omega_{n-4}) \oplus \cdots . \end{aligned}$$

3.5. **Paths.** Let $B = B^{k_L,1} \otimes B^{k_{L-1},1} \otimes \cdots \otimes B^{k_1,1}$ be a type $D_n^{(1)}$ crystal with $1 \le k_i \le n$. For a dominant integral weight λ define the set of paths as follows

$$\mathcal{P}(\lambda, B) = \{ b \in B \mid \mathrm{wt}(b) = \lambda,\, e_i(b) = \varnothing \text{ for all } i \in J \}$$

where $J = I \backslash \{0\} = \{1, 2, \ldots, n\}$.

4. Bijection between paths and rigged configurations

Let $B = B^{k_L,1} \otimes B^{k_{L-1},1} \otimes \cdots \otimes B^{k_1,1}$ be a type $D_n^{(1)}$ crystal and λ a dominant integral weight. In this section we define the set of rigged configurations $\mathrm{RC}(\lambda, B)$ and a bijection $\Phi : \mathrm{RC}(\lambda, B) \to \mathcal{P}(\lambda, B)$.

4.1. **Rigged configurations.** Denote by L^a the number of factors $B^{a,1}$ in B, so that $B = \bigotimes_{a=1}^n (B^{a,1})^{\otimes L^a}$. For later use, let us also define $\hat{L}^{n-1}$ (resp. $\hat{L}^n$, $\hat{\hat{L}}^n$) to be the number of tensor factors $\hat{B}^{n-1,1}$ (resp. $E^{n,1}$, $E^{n-1,1}$) in B. Let

$$\boldsymbol{L} = \sum_{a \in J} L^a \Lambda_a + \sum_{a \in \{n-1,n\}} \hat{L}^a \omega_a + \hat{\hat{L}}^n \overline{\omega}_n. \tag{4.1}$$

Then the sequence of partitions $\nu^\bullet = (\nu^{(1)}, \nu^{(2)}, \ldots, \nu^{(n)})$ is a (λ, B)-configuration if

$$\sum_{\substack{a \in J \\ i \ge 0}} i m_i^{(a)} \alpha_a = \boldsymbol{L} - \lambda, \tag{4.2}$$

where $m_i^{(a)}$ is the number of parts of length i in partition $\nu^{(a)}$. A (λ, B)-configuration is admissible if $p_i^{(a)} \ge 0$ for all $a \in J$ and $i \ge 0$, where $p_i^{(a)}$ is the vacancy number

$$p_i^{(a)} = \Big(\alpha_a \mid \boldsymbol{L} - \sum_{b \in J} \alpha_b \sum_{j \ge 1} \min(i,j) m_j^{(b)} \Big). \tag{4.3}$$

Denote the set of admissible (λ, B)-configurations by $\mathrm{C}(\lambda, B)$. A rigged configuration $(\nu^\bullet, J^\bullet)$ is a sequence of partitions $\nu^\bullet = \{\nu^{(a)}\}_{a \in J} \in \mathrm{C}(\lambda, B)$ together with a double sequence of partitions $J^\bullet = \{J^{(a,i)}\}_{\substack{a \in J \\ i \ge 0}}$ such that the partition $J^{(a,i)}$ is contained in a box of size $m_i^{(a)} \times p_i^{(a)}$. The set of rigged configurations is denoted by $\mathrm{RC}(\lambda, B)$.

4.2. **The bijection Φ.** Let B' be a tensor product of type $D_n^{(1)}$ crystals of the form $B^{a,1}$ or $\hat{B}^{n-1,1}$.

Theorem 4.1. *There exists a unique family of bijections $\Phi_B : \mathrm{RC}(\lambda, B) \to \mathcal{P}(\lambda, B)$ such that:*

(1) *Suppose $B = B^{1,1} \otimes B'$. Then the diagram*

$$\begin{array}{ccc}
\mathrm{RC}(\lambda, B) & \xrightarrow{\Phi_B} & \mathcal{P}(\lambda, B) \\
\delta \big\downarrow & & \big\downarrow \mathrm{lh} \\
\bigcup_{\lambda^-} \mathrm{RC}(\lambda^-, B') & \xrightarrow{\Phi_{B'}} & \bigcup_{\lambda^-} \mathcal{P}(\lambda^-, B')
\end{array}$$

commutes.

(2) *Suppose* $B = \hat{B}^{k,1} \otimes B'$ *with* $2 \le k \le n-1$ *and let* $\widetilde{B} = B^{1,1} \otimes \hat{B}^{k-1,1} \otimes B'$. *Then the diagram*

$$\begin{tikzcd} \mathrm{RC}(\lambda, B) \arrow[r, "\Phi_B"] \arrow[d, "\mathrm{tj}"'] & \mathcal{P}(\lambda, B) \arrow[d, "\mathrm{ts}"] \\ \mathrm{RC}(\lambda, \widetilde{B}) \arrow[r, "\Phi_{\widetilde{B}}"] & \mathcal{P}(\lambda, \widetilde{B}) \end{tikzcd}$$

commutes.

(3) *Suppose* $B = B^{k,1} \otimes B'$ *with* $k = n-1, n$. *Then the diagram*

$$\begin{tikzcd} \mathrm{RC}(\lambda, B) \arrow[r, "\Phi_B"] \arrow[d, "\delta_s"'] & \mathcal{P}(\lambda, B) \arrow[d, "\mathrm{lh}_s"] \\ \bigcup_{\lambda^{-s}} \mathrm{RC}(\lambda^{-s}, B') \arrow[r, "\Phi_{B'}"] & \bigcup_{\lambda^{-s}} \mathcal{P}(\lambda^{-s}, B') \end{tikzcd}$$

commutes.

A proof of this theorem will be given in a subsequent paper. In the following we define the various yet undefined maps.

4.3. **Various maps.** The map $\mathrm{lh} : \mathcal{P}(\lambda, B^{1,1} \otimes B') \to \bigcup_{\lambda^-} \mathcal{P}(\lambda^-, B')$ in point 1 of Theorem 4.1 is given by $b \otimes b' \mapsto b'$. The union is over all λ^- such that $\lambda - \lambda^-$ is the weight of an element in $B^{1,1}$.

Similarly, the map $\mathrm{lh}_s : \mathcal{P}(\lambda, B) \to \bigcup_{\lambda^{-s}} \mathcal{P}(\lambda^{-s}, B')$ of point 3 is given by $b \otimes b' \mapsto b'$. The union is over all λ^{-s} such that $\lambda - \lambda^{-s}$ is the weight of an element in $B^{k,1}$ for $k = n-1$ or n.

The map $\mathrm{ts} : \mathcal{P}(\lambda, B) \to \mathcal{P}(\lambda, \widetilde{B})$ in point 2 of the theorem is obtained by sending $m_k m_{k-1} \cdots m_1 \otimes b' \in \mathcal{P}(\lambda, \hat{B}^{k,1} \otimes B')$ to $m_k \otimes m_{k-1} \cdots m_1 \otimes b' \in \mathcal{P}(\lambda, B^{1,1} \otimes \hat{B}^{k-1,1} \otimes B')$ for $1 \le k \le n-1$. Similarly we may also define ts to map $m_n m_{n-1} \cdots m_1 \otimes b' \in \mathcal{P}(\lambda, E^{n-1,1} \otimes B')$ or $\mathcal{P}(\lambda, E^{n,1} \otimes B')$ to $m_n \otimes m_{n-1} \cdots m_1 \otimes b' \in \mathcal{P}(\lambda, B^{1,1} \otimes \hat{B}^{n-1,1} \otimes B')$.

Define $\mathrm{tj} : \mathrm{RC}(\lambda, B) \to \mathrm{RC}(\lambda, \widetilde{B})$ in the following way. Let $(\nu^\bullet, J^\bullet) \in \mathrm{RC}(\lambda, B)$. If $B = \hat{B}^{k,1} \otimes B'$ for $1 \le k \le n-1$, $\mathrm{tj}(\nu^\bullet, J^\bullet)$ is obtained from $(\nu^\bullet, J^\bullet)$ by adding a singular string of length 1 to each of the first $k-1$ rigged partitions of $\nu^\bullet$. For $B = E^{n,1} \otimes B'$, add a singular string of length 1 to $(\nu^\bullet, J^\bullet)^{(a)}$ for $1 \le a \le n-1$. For $B = E^{n-1,1} \otimes B'$, add a singular string of length 1 to $(\nu^\bullet, J^\bullet)^{(n)}$ and $(\nu^\bullet, J^\bullet)^{(a)}$ for $1 \le a \le n-2$.

Lemma 4.2. tj *is a well-defined injection that preserves the vacancy numbers.*

The maps δ and δ_s are defined in the next two subsections.

4.4. **Algorithm for δ.** Let $(\nu^\bullet, J^\bullet) \in \mathrm{RC}(\lambda, B^{1,1} \otimes B')$. The map $\delta : \mathrm{RC}(\lambda, B^{1,1} \otimes B') \to \bigcup_{\lambda^-} \mathrm{RC}(\lambda^-, B')$ is defined by the following algorithm [17]. The partition $J^{(a,i)}$ is called singular if it has a part of size $p_i^{(a)}$.

Set $\ell^{(0)} = 1$ and repeat the following process for $a = 1, 2, \ldots, n-2$ or until stopped. Find the smallest index $i \ge \ell^{(a-1)}$ such that $J^{(a,i)}$ is singular. If no such i exists, set $b = a$ and stop. Otherwise set $\ell^{(a)} = i$ and continue with $a+1$.

If the process has not stopped at $a = n-2$, find the minimal indices $i, j \ge \ell^{(n-2)}$ such that $J^{(n-1,i)}$ and $J^{(n,j)}$ are singular. If neither i nor j exist, set $b = n-1$ and stop. If i exists, but not j, set $\ell^{(n-1)} = i$, $b = n$ and stop. If j exists, but not i, set $\ell^{(n)} = j$, $b = \overline{n}$ and stop. If both i and j exist, set $\ell^{(n-1)} = i$, $\ell^{(n)} = j$ and continue with $a = n-2$.

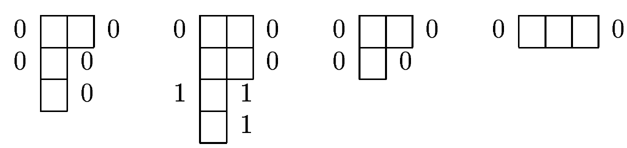
Now continue for $a = n-2, n-3, \ldots, 1$ or until stopped. Find the minimal index $i \geq \overline{\ell}^{(a+1)}$ where $\overline{\ell}^{(n-1)} = \max(\ell^{(n-1)}, \ell^{(n)})$ such that $J^{(a,i)}$ is singular (if $i = \ell^{(a)}$ then there need to be two parts of size $p_i^{(a)}$ in $J^{(a,i)}$). If no such i exists, set $b = \overline{a+1}$ and stop. If the process did not stop, set $b = \overline{1}$. Set all yet undefined $\ell^{(a)}$ and $\overline{\ell}^{(a)}$ to ∞.

The rigged configuration $(\tilde{\nu}^\bullet, \tilde{J}^\bullet) = \delta(\nu^\bullet, J^\bullet)$ is obtained by removing a box from the selected strings and making the new strings singular again. Explicitly

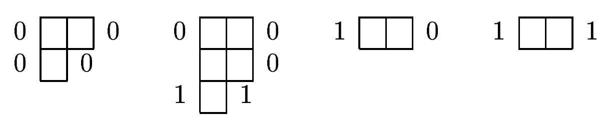
$$m_i^{(a)}(\tilde{\nu}^\bullet) = m_i^{(a)}(\nu^\bullet) + \begin{cases} 1 & \text{if } i = \ell^{(a)} - 1 \\ -1 & \text{if } i = \ell^{(a)} \\ 1 & \text{if } i = \overline{\ell}^{(a)} - 1 \text{ and } 1 \leq a \leq n-2 \\ -1 & \text{if } i = \overline{\ell}^{(a)} \text{ and } 1 \leq a \leq n-2 \\ 0 & \text{otherwise.} \end{cases}$$

The partition $\tilde{J}^{(a,i)}$ is obtained from $J^{(a,i)}$ by removing a part of size $p_i^{(a)}(\nu^\bullet)$ for $i = \ell^{(a)}$ and $i = \overline{\ell}^{(a)}$, adding a part of size $p_i^{(a)}(\tilde{\nu}^\bullet)$ for $i = \ell^{(a)} - 1$ and $i = \overline{\ell}^{(a)} - 1$, and leaving it unchanged otherwise.

Example 4.3. Let $(\nu^\bullet, J^\bullet) \in \mathrm{RC}(\lambda, B)$ with $\lambda = \Lambda_2$ and $B = (B^{1,1})^{\otimes 2} \otimes (B^{2,1})^{\otimes 3}$ of type $D_4^{(1)}$ given by



Here the vacancy number $p_i^{(a)}$ is written on the left of the parts of length i in $\nu^{(a)}$ and the partition $J^{(a,i)}$ is given by the labels on the right of the parts of length i in $\nu^{(a)}$. In this case $\ell^{(1)} = \ell^{(2)} = \ell^{(3)} = 1$, $\ell^{(4)} = 3$, $\overline{\ell}^{(2)} = \overline{\ell}^{(1)} = \infty$ and $b = \overline{3}$, so that $\delta(\nu^\bullet, J^\bullet)$ is



4.5. **Algorithm for δ_s.** The embedding (3.6) can be extended to paths

$$\mathrm{emb}_{\mathcal{P}} : \mathcal{P}(\lambda, B) \hookrightarrow \mathcal{P}(\hat{\lambda}, \hat{B}) \tag{4.4}$$

where $\hat{\lambda} = 2\lambda$, $B = \bigotimes_{a=1}^n (B^{a,1})^{\otimes L^a}$, and $\hat{B} = \bigotimes_{a=1}^n (B^{a,1})^{\otimes \hat{L}^a}$ where $\hat{L}^a = 2L^a$ for $1 \leq a \leq n$. A path $b = b_L \otimes \cdots \otimes b_1 \in \mathcal{P}(\lambda, B)$ is mapped to $\mathrm{emb}_{\mathcal{P}}(b) = \mathrm{emb}_B(b_L) \otimes \cdots \otimes \mathrm{emb}_B(b_1)$.

An analogous embedding can be defined on rigged configurations

$$\mathrm{emb}_{\mathrm{RC}} : \mathrm{RC}(\lambda, B) \hookrightarrow \mathrm{RC}(\hat{\lambda}, \hat{B}) \tag{4.5}$$

where $\mathrm{emb}_{\mathrm{RC}}(\nu^\bullet, J^\bullet) = 2(\nu^\bullet, J^\bullet)$.

Theorem 4.4. *For B a tensor product of crystals of the form $B^{a,1}$ with $1 \le a \le n-2$ the following diagram commutes:*

$$\begin{tikzcd} \mathrm{RC}(\lambda, B) \arrow[r, "\Phi_B"] \arrow[d, "\mathrm{emb}_{\mathrm{RC}}"'] & \mathcal{P}(\lambda, B) \arrow[d, "\mathrm{emb}_B"] \\ \mathrm{RC}(\hat{\lambda}, \hat{B}) \arrow[r, "\Phi_{\hat{B}}"] & \mathcal{P}(\hat{\lambda}, \hat{B}). \end{tikzcd}$$

The proof of this theorem will be given in a subsequent paper.

The definition of δ_s is given in such a way that Theorem 4.4 also holds for the spinor case. Let us define

$$\delta_s : \mathrm{RC}(\lambda, B) \to \bigcup_{\lambda^{-s}} \mathrm{RC}(\lambda^{-s}, B')$$

as follows. For $(\nu^\bullet, J^\bullet) \in \mathrm{RC}(\lambda, B)$ apply $\mathrm{emb}_{\mathrm{RC}}$ and then a sequence of points 1 and 2 of Theorem 4.1 to remove the last tensor factor. The claim is that this rigged configuration is in the image of $\mathrm{emb}_{\mathrm{RC}}$ so that one can apply $\mathrm{emb}_{\mathrm{RC}}^{-1}$. The result is $\delta_s(\nu^\bullet, J^\bullet)$.

Example 4.5. Let $(\nu^\bullet, J^\bullet) \in \mathrm{RC}(\lambda, B)$ with $\lambda = 2\Lambda_1 + \Lambda_4$ and $B = B^{5,1} \otimes B^{2,1} \otimes (B^{1,1})^{\otimes 3}$ of type $D_5^{(1)}$ given by

$$\begin{array}{ccc} 2 & \square\square & 1 \\ & & \end{array} \qquad \begin{array}{ccc} 0 & \square\square & 0 \\ 0 & \square & 0 \end{array} \qquad \begin{array}{ccc} 0 & \square\square & 0 \\ 0 & \square & 0 \end{array} \qquad \begin{array}{ccc} 0 & \square & 0 \\ & & \end{array} \qquad \begin{array}{ccc} 0 & \square\square & 0 \\ & & \end{array}$$

Then $\delta_s(\nu^\bullet, J^\bullet) \in \mathrm{RC}(\lambda^{-s}, B')$ with $\lambda^{-s} = \Lambda_1 + \Lambda_2$ and $B' = B^{2,1} \otimes (B^{1,1})^{\otimes 3}$ is

$$\begin{array}{ccc} 1 & \square\square & 1 \\ & & \end{array} \qquad \begin{array}{ccc} 0 & \square & 0 \\ & \square & 0 \end{array} \qquad \begin{array}{ccc} 0 & \square & 0 \\ & \square & 0 \end{array} \qquad \begin{array}{ccc} 0 & \square & 0 \\ & & \end{array} \qquad \begin{array}{ccc} 0 & \square & 0 \\ & & \end{array}$$

The details are given in Table 1. The first entry is $\mathrm{emb}_{\mathrm{RC}}(\nu^\bullet, J^\bullet)$. The next entries are obtained by acting with $\delta \circ \mathrm{tj}$. Acting with $\mathrm{emb}_{\mathrm{RC}}^{-1}$ on the last rigged configuration yields $\delta_s(\nu^\bullet, J^\bullet)$. In the last column of the table we recorded b of the algorithm δ. Hence the step in $B^{5,1}$ corresponding to this rigged configuration is $\overline{2}\overline{5}431$.

5. Statistics on paths and rigged configurations

A statistic can be defined on both paths and rigged configurations. In this section we will define the intrinsic energy function on paths and cocharge statistics on rigged configurations. The bijection Φ preserves the statistics.

In Section 2.3 we defined the combinatorial R-matrix $R : B_2 \otimes B_1 \to B_1 \otimes B_1 \otimes B_2$. In addition, there exists a function $H : B_2 \otimes B_1 \to \mathbb{Z}$ called the local energy function, that is unique up to a global additive constant. It is constant on J components and satisfies for all $b_2 \in B_2$ and $b_1 \in B_1$ with $R(b_2 \otimes b_1) = b_1' \otimes b_2'$

$$H(e_0(b_2 \otimes b_1)) = H(b_2 \otimes b_1) + \begin{cases} -1 & \text{if } \epsilon_0(b_2) > \varphi_0(b_1) \text{ and } \epsilon_0(b_1') > \varphi_0(b_2') \\ 1 & \text{if } \epsilon_0(b_2) \le \varphi_0(b_1) \text{ and } \epsilon_0(b_1') \le \varphi_0(b_2') \\ 0 & \text{otherwise.} \end{cases}$$

We shall normalize the local energy function by the condition $H(u(B_2) \otimes u(B_1)) = 0$.

For a crystal $B^{k,1}$ of type $D_n^{(1)}$, the intrinsic energy $D_{B^{k,1}} : B^{k,1} \to \mathbb{Z}$ is defined as follows. Let $b \in B^{k,1}$ which is in the classical component $B(\Lambda_{k-2j})$ (see (3.4)). Then $D_{B^{k,1}}(b) = j$.

$(\tilde\nu^\bullet,\tilde J^\bullet)^{(1)}$	$(\tilde\nu^\bullet,\tilde J^\bullet)^{(2)}$	$(\tilde\nu^\bullet,\tilde J^\bullet)^{(3)}$	$(\tilde\nu^\bullet,\tilde J^\bullet)^{(4)}$	$(\tilde\nu^\bullet,\tilde J^\bullet)^{(5)}$	b
$4\ \square\square\square\square\ 2$	$\begin{array}{lll} 0 & \square\square\square\square & 0 \\ 0 & \square\square & 0 \end{array}$	$\begin{array}{lll} 0 & \square\square\square\square & 0 \\ 0 & \square\square & 0 \end{array}$	$0\ \square\square\ 0$	$0\ \square\square\square\square\ 0$	
$3\ \square\square\square\square\ 2$	$\begin{array}{lll} 0 & \square\square\square & 0 \\ 0 & \square\square & 0 \end{array}$	$\begin{array}{lll} 0 & \square\square\square & 0 \\ 0 & \square\square & 0 \end{array}$	$1\ \square\square\ 0$	$0\ \square\square\square\ 0$	$\overline{2}$
$3\ \square\square\square\square\ 2$	$\begin{array}{lll} 0 & \square\square\square & 0 \\ 0 & \square\square & 0 \end{array}$	$\begin{array}{lll} 0 & \square\square\square & 0 \\ 1 & \square\square & 0 \end{array}$	$0\ \square\square\ 0$	$0\ \square\square\ 0$	$\overline{5}$
$3\ \square\square\square\square\ 2$	$\begin{array}{lll} 0 & \square\square\square & 0 \\ 1 & \square\square & 0 \end{array}$	$\begin{array}{lll} 0 & \square\square & 0 \\ & \square\square & 0 \end{array}$	$0\ \square\square\ 0$	$0\ \square\square\ 0$	4
$3\ \square\square\square\square\ 2$	$\begin{array}{lll} 0 & \square\square & 0 \\ & \square\square & 0 \end{array}$	$\begin{array}{lll} 0 & \square\square & 0 \\ & \square\square & 0 \end{array}$	$0\ \square\square\ 0$	$0\ \square\square\ 0$	3
$2\ \square\square\square\square\ 2$	$\begin{array}{lll} 0 & \square\square & 0 \\ & \square\square & 0 \end{array}$	$\begin{array}{lll} 0 & \square\square & 0 \\ & \square\square & 0 \end{array}$	$0\ \square\square\ 0$	$0\ \square\square\ 0$	1

TABLE 1. An example for δ_s

On the tensor product $B = B_L \otimes \cdots \otimes B_1$ of simple crystals there is an intrinsic energy function defined $D_B : B \to \mathbb{Z}$ (see for example [18, Section 2.5])

$$D_B = \sum_{1\le i<j\le L} H_i R_{i+1} R_{i+2} \cdots R_{j-1} + \sum_{j=1}^{L} D_{B_j} \pi_1 R_1 R_2 \cdots R_{j-1}. \tag{5.1}$$

Here R_i and H_i denote the combinatorial R-matrix and local energy function acting on the i-th and $(i+1)$-th tensor factors, respectively. π_1 is the projection onto the rightmost tensor factor.

Similarly, there is a statistic on the set of rigged configurations given by $cc(\nu^\bullet, J^\bullet) = cc(\nu^\bullet) + \sum_{a,i} |J^{(a,i)}|$ where $|J^{(a,i)}|$ is the size of the partition $J^{(a,i)}$ and

$$cc(\nu^\bullet) = \frac{1}{2} \sum_{a,b\in J} \sum_{j,k\ge 1} (\alpha_a \mid \alpha_b) \min(j,k) m_j^{(a)} m_k^{(b)}. \tag{5.2}$$

Let $\tilde\Phi = \Phi\circ\theta$ where $\theta : \mathrm{RC}(\lambda, B) \to \mathrm{RC}(\lambda, B)$ with $\theta(\nu^\bullet, J^\bullet) = (\nu^\bullet, \tilde J^\bullet)$ is the function which complements the riggings, meaning that $\tilde J^\bullet$ is obtained from $J^\bullet$ by complementing all partitions $J^{(a,i)}$ in the $m_i^{(a)} \times p_i^{(a)}$ rectangle.

Conjecture 5.1. *Let $B = B^{k_L,1} \otimes \cdots \otimes B^{k_1,1}$ be a crystal of type $D_n^{(1)}$ and λ a dominant integral weight. The bijection $\tilde\Phi : \mathrm{RC}(\lambda, B) \to \mathcal{P}(\lambda, B)$ preserves the statistics, that is $cc(\nu^\bullet, J^\bullet) = D(\tilde\Phi(\nu^\bullet, J^\bullet))$ for all $(\nu^\bullet, J^\bullet) \in \mathrm{RC}(\lambda, B)$.*

An immediate corollary of Theorem 4.1 and Conjecture 5.1 is the equality

$$\sum_{b\in\mathcal{P}(\lambda,B)} q^{D(b)} = \sum_{(\nu^\bullet,J^\bullet)\in\mathrm{RC}(\lambda,B)} q^{cc(\nu^\bullet,J^\bullet)}.$$

We expect to provide a proof of Conjecture 5.1 in a subsequent paper.

REFERENCES

[1] V. G. Drinfeld, *Hopf algebra and the Yang–Baxter equation*, Soviet. Math. Dokl. **32** (1985) 254–258.

[2] G. Hatayama, A. Kuniba, M. Okado, T. Takagi, Z. Tsuboi, *Paths, crystals and fermionic formulae*, Prog. Math. Phys. **23** (2002) 205–272, Birkhäuser Boston, Boston, MA.

[3] G. Hatayama, A. Kuniba, M. Okado, T. Takagi and Y. Yamada, *Remarks on fermionic formula*, Contemp. Math. **248** (1999) 243–291.

[4] M. Jimbo, *A q-difference analogue of $U(\mathcal{G})$ and the Yang–Baxter equation*, Lett. Math. Phys. **10** (1985) 63–69.

[5] M. Jimbo and T. Miwa, *On a duality of branching rules for affine Lie algebras*, Advanced Studies in Pure Math. **6** (1985) 17–65.

[6] M. Kashiwara, *Crystalizing the q-analogue of universal enveloping algebras*, Commun. Math. Phys. **133** (1990) 249–260.

[7] M. Kashiwara, *On crystal bases of the q-analogue of universal enveloping algebras*, Duke Math. J. **63** (1991) 465–516.

[8] M. Kashiwara, *On crystal bases*, in: Representations of groups (Banff, AB, 1994), 155–197, CMS Conf. Proc., 16, Amer. Math. Soc., Providence, RI, 1995.

[9] S.-J. Kang, M. Kashiwara, K. C. Misra, T. Miwa, T. Nakashima and A. Nakayashiki, *Perfect crystals of quantum affine Lie algebras*, Duke Math. J. **68** (1992) 499–607.

[10] M. Kashiwara and T. Nakashima, *Crystal graphs for representations of the q-analogue of classical Lie algebras*, J. Alg. **165** (1994) 295–345.

[11] Y. Koga, *Level one perfect crystals for $B_n^{(1)}$, $C_n^{(1)}$ and $D_n^{(1)}$*, J. Alg. **217** (1999) 312–334.

[12] R. Kedem, T. R. Klassen, B. M. McCoy, E. Melzer, *Fermionic quasi-particle representations for characters of $(G^{(1)})_1 \times (G^{(1)})_1/(G^{(1)})_2$*, Phys. Lett. B **304** (1993), no. 3-4, 263–270.

[13] R. Kedem, T. R. Klassen, B. M. McCoy, E. Melzer, *Fermionic sum representations for conformal field theory characters*, Phys. Lett. B **307** (1993), no. 1-2, 68–76.

[14] A. N. Kirillov and N. Y. Reshetikhin, *The Bethe ansatz and the combinatorics of Young tableaux*, (Russian) Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **155** (1986), Differentsialnaya Geometriya, Gruppy Li i Mekh. VIII, 65–115, 194; translation in J. Soviet Math. **41** (1988), no. 2, 925–955.

[15] A. N. Kirillov, A. Schilling and M. Shimozono, *A bijection between Littlewood-Richardson tableaux and rigged configurations*, Selecta Mathematica (N.S.) **8** (2002) 67–135.

[16] M. Okado, A. Schilling and M. Shimozono, *Virtual crystals and fermionic formulas of type $D_{n+1}^{(2)}$, $A_{2n}^{(2)}$, and $C_n^{(1)}$*, Representation Theory **7** (2003) 101–163.

[17] M. Okado, A. Schilling and M. Shimozono, *A crystal to rigged configuration bijection for nonexceptional affine algebras*, preprint math.QA/0203163.

[18] M. Okado, A. Schilling and M. Shimozono, *Virtual crystals and Kleber's algorithm*, Commun. Math. Phys., to appear (math.QA/0209082).

[19] M. Shimozono, *Affine type A crystal structure on tensor products of rectangles, Demazure characters, and nilpotent varieties*, J. Algebraic Combin. **15** (2002), no. 2, 151–187.

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