THE BRUHAT ORDER ON THE INVOLUTIONS OF THE SYMMETRIC GROUP

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ABSTRACT. In this work we study the partially ordered set of the involutions of the symmetric group S_n with the order induced by the Bruhat order of S_n. We prove that this is a graded poset, with rank function given by the average of the number of inversions and the number of excedances, and that it is lexicographically shellable, hence Cohen-Macaulay, and Eulerian.

RÉSUMÉ. Dans cet article on étudie l'ensemble des involutions de S_n muni de l'ordre induit par l'ordre de Bruhat de S_n. On démontre que c'est un poset gradué, avec function rang donnée par la moyenne entre le nombre des éxcédances et le nombre des inversions, et qu'il est léxicografiquement shellable, donc Cohen-Macaulay, et Eulérien.

1. INTRODUCTION

It is well-known that the symmetric group S_n ordered by the Bruhat order encodes the cell decomposition of Schubert varieties (see, e.g., [8]). This partially ordered set has been studied extensively (see, e.g., [4], [5], [7], [11], [12] and [19]) and it is known that it is a graded poset, with rank function given by the number of inversions, and that it is lexicographically shellable, hence Cohen-Macaulay, and Eulerian.

In [14] and [15] a vast generalization of this partial order has been considered, in relation to the cell decomposition of certain symmetric varieties. In this work we study this partial order in a special case that is particularly attractive from a combinatorial point of view, namely that of the involutions of S_n with the order induced by the Bruhat order (see [14, §10]). Our main results are that this is a graded poset, with rank function given by the average of the number of inversions and the number of excedances, and that it is lexicographically shellable and Eulerian.

The organization of this work is as follows. In Section 2 we give basic definitions, notation and results that will be needed later. Sections 3 and 4 are devoted to the study of some new combinatorial concepts related to involutions, namely those of "suitable rise", "covering transformation" and "minimal covering transformation", that play a crucial role in the sequel. In Sections 5, 6 and 7 we prove our main results, namely that the poset is graded, lexicographically shellable and Eulerian.

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2. Preliminaries

In this section we recall some basic definitions, notation and results that will be used in the rest of this work.

We let $\mathbf{N} = \{1, 2, 3, ...\}$ and $\mathbf{Z}$ be the set of integers. For every $n \in \mathbf{N}$ we let $[n] = \{1, 2, ..., n\}$ and for every $n, m \in \mathbf{Z}$, with $n \leq m$, we let $[n, m] = \{n, n+1, ..., m\}$.

2.1. **Posets.** We follow [16, Chapter 3] for poset notation and terminology. In particular we denote by $\lhd$ the covering relation: $x \lhd y$ means that $x < y$ and there is no z such that $x < z < y$. The *Hasse diagram* of a finite poset P is the graph whose vertices are the elements of P, whose edges are the covering relations, and such that if $x < y$, then y is drawn "above" x. A poset is said to be *bounded* if it has a minimum and a maximum, denoted by $\hat{0}$ and $\hat{1}$ respectively. For a bounded poset P we denote by $\bar{P}$ the subposet $P \backslash \{\hat{0}, \hat{1}\}$. If $x, y \in P$, with $x \leq y$, we let $[x, y] = \{z \in P : x \leq z \leq y\}$, and we call it an *interval* of P. If $x, y \in P$, with $x < y$, a *chain* from x to y of *length* k is a $(k+1)$-tuple $(x_0, x_1, ..., x_k)$ such that $x = x_0 < x_1 < ... < x_k = y$, denoted simply by "$x_0 < x_1 < ... < x_k$". A chain $x_0 < x_1 < ... < x_k$ is said to be *saturated* if all the relations in it are covering relations, and in this case we denote it simply by "$x_0 \lhd x_1 \lhd ... \lhd x_k$". A poset is said to be *graded* of *rank* n if it is finite, bounded and if all maximal chains of P have the same length n. If P is a graded poset of rank n, then there is a unique *rank function* $\rho : P \to [0, n]$ such that $\rho(x) = 0$ if x is a minimal element of P, and $\rho(y) = \rho(x) + 1$ whenever y covers x in P.

If P is a graded poset and Q is a totally ordered set, an *edge-labeling* of P with values in Q is a function $\lambda : \{(x, y) \in P^2 : x \lhd y\} \to Q$. If λ is an edge-labeling of P, for every saturated chain $x_0 \lhd x_1 \lhd ... \lhd x_k$ we set

$$\lambda(x_0, x_1, ..., x_k) = (\lambda(x_0, x_1), \lambda(x_1, x_2), ..., \lambda(x_{k-1}, x_k)).$$

An edge-labeling λ of P is said to be an *EL-labeling* if for every $x, y \in P$, with $x < y$,

(i) there is exactly one saturated chain from x to y, say $x = x_0 \lhd x_1 \lhd ... \lhd x_k = y$, such that $\lambda(x_0, x_1, ..., x_k)$ is a non-decreasing sequence (i.e., $\lambda(x_0, x_1) \leq \lambda(x_1, x_2) \leq ... \leq \lambda(x_{k-1}, x_k)$);

(ii) any other saturated chain from x to y, say $x = y_0 \lhd y_1 \lhd ... \lhd y_k = y$, different from the previous one, is such that $\lambda(x_0, x_1, ..., x_k) <_L \lambda(y_0, y_1, ..., y_k)$, where $<_L$ denotes the lexicographic order ($(a_1, a_2, ..., a_k) <_L (b_1, b_2, ..., b_k)$ if and only if $a_i < b_i$, where $i = \min\{j \in [k] : a_j \neq b_j\}$).

A graded poset P is said to be *lexicographically shellable*, or *EL-shellable*, if it has an *EL*-labeling.

Connections between *EL*-shellable posets and shellable complexes, Cohen-Macaulay complexes and Cohen-Macaulay rings can be found, for example, in [1], [2], [3], [9], [10], [13] and [17]. Here we only recall some basic facts. The order complex $\Delta(P)$ of a poset P is the simplicial complex of all chains of P. A poset P is said to be *shellable* if $\Delta(P)$ is shellable, and *Cohen-Macaulay* if $\Delta(P)$ is Cohen-Macaulay (see, e.g., [3, Appendix], for the definitions of a shellable complex and of a Cohen-Macaulay complex). Furthermore, a poset is Cohen-Macaulay if and only if the Stanley-Reisner ring associated with it is Cohen-Macaulay (see, e.g., [13]). It is known that if a complex [a poset] is shellable, then it is Cohen-Macaulay (see [9, Remark 5.3]). Finally, Björner has proved the following (see [3, Theorem 2.3]).

Theorem 2.1. *Let P be a graded poset. If P is EL-shellable then P is shellable and hence Cohen-Macaulay.*

A graded poset P with rank function ρ is said to be *Eulerian* if

$$|\{z \in [x,y] : \rho(z) \text{ is even}\}| = |\{z \in [x,y] : \rho(z) \text{ is odd}\}|.$$

for every $x, y \in P$ such that $x \leq y$.

In an EL-shellable poset there is a necessary and sufficient condition for the poset to be Eulerian. We state it in the following form (see [3, Theorem 2.7] and [18, Theorem 1.2] for proofs of more general results).

Theorem 2.2. *Let P be a graded EL-shellable poset and let λ be an EL-labeling of P. Then, for every $x, y \in P$ such that $x \leq y$, the number $(-1)^{\rho(y)-\rho(x)} \mu(x,y)$ equals the number of saturated chains from x to y with decreasing labels, i.e. the number of saturated chains $x = x_0 \lhd x_1 \lhd \ldots \lhd x_k = y$ such that $\lambda(x_0, x_1) > \lambda(x_1, x_2) > \ldots > \lambda(x_{k-1}, x_k)$.*

In particular, P is Eulerian if and only if for every $x, y \in P$ such that $x \leq y$, there is exactly one saturated chain from x to y with decreasing labels.

Finally, we refer to [10] or [17] for the definition of a *Gorenstein* poset, just recalling the following result (see, e.g., [17, §8]).

Theorem 2.3. *Let P be a graded Cohen-Macaulay poset. Then P is Gorenstein if and only if the subposet of P induced by $P \backslash \{x \in \bar{P} : x$ is comparable with every $y \in P\}$ is Eulerian.*

2.2. **The Bruhat order on the symmetric group.** Given a set T we let $S(T)$ be the set of all bijections $\pi : T \to T$, and $S_n = S([n])$. If $\sigma \in S_n$ then we write $\sigma = \sigma_1\sigma_2 \ldots \sigma_n$, to mean that $\sigma(i) = \sigma_i$ for every $i \in [n]$. We also write σ in disjoint cycle form (see, e.g., [16, p. 17]), omitting to write the 1-cycles of σ. For example, if $\sigma = 364152$, then we also write $\sigma = (1,3,4)(2,6)$. Given $\sigma, \tau \in S_n$, we let $\sigma\tau = \sigma \circ \tau$ (composition of functions) so that, for example, $(1,2)(2,3) = (1,2,3)$.

Given $\sigma \in S_n$, the *diagram* of σ is a square of $n \times n$ cells, with the cell (i,j) (i.e., the cell in the i-th column and in the j-th row, with the convention that the first column is the leftmost one and the first row is the lowest one) filled with a dot if and only if $\sigma(i) = j$. The *diagonal* of the diagram is the set of cells $\{(i,i) : i \in [n]\}$.

Let $\sigma \in S_n$. As usual we denote by

$$Inv(\sigma) = \left\{(i,j) \in [n]^2 : i < j,\ \sigma(i) > \sigma(j)\right\}$$

the set of *inversions* of σ and by $inv(\sigma)$ their number.

The (*strong*) *Bruhat order* of S_n is the partial order relation on S_n, denoted by $\leq_B$, which is the transitive closure of the relation $\to$ defined by

$$\sigma \to \tau \quad \Leftrightarrow \quad \text{there is a transposition } (i,j) \text{ such that } \tau = \sigma(i,j) \text{ and } inv(\sigma) \le inv(\tau).$$

Let $\sigma \in S_n$. A *rise* of σ is a pair $(i,j) \in [n]^2$ such that $i < j$ and $\sigma(i) < \sigma(j)$. A rise (i,j) of σ is said to be *free* if there is no $k \in [n]$ such that $i < k < j$ and $\sigma(i) < \sigma(k) < \sigma(j)$. It is well-known that, if $\sigma, \tau \in S_n$, then τ covers σ in the Bruhat order if and only if $\tau = \sigma(i,j)$, for some free rise (i,j) of σ.

Let $\sigma \in S_n$. For every $(h,k) \in [n]^2$ we set

$$\sigma[h,k] = |\{i \in [h] : \sigma(i) \ge k\}|.$$

A fundamental characterization of the Bruhat order relation is the following (see, e.g., [11]).

Theorem 2.4. *Let $\sigma, \tau \in S_n$. Then $\sigma \le_B \tau$ if and only if $\sigma[h,k] \le \tau[h,k]$ for every $(h,k) \in [n]^2$.*

A consequence of Theorem 2.4 is the following: if $\sigma, \tau \in S_n$ then $\sigma \le_B \tau$ if and only if $\sigma^{-1} \le_B \tau^{-1}$.

We are interested in the set of involutions of S_n, which we denote by $Invol(n)$. Note that a permutation is an involution if and only if its diagram is symmetric with respect to the diagonal. We wish to study the poset $(Invol(n), \le_B)$ of the involutions with the order induced by the Bruhat order of S_n. We will denote simply by S_n and $Invol(n)$ the respective posets with the Bruhat order.

In Figures 1 and 2 are represented, respectively, the Hasse diagram of the poset S_4, with the involutions marked, and the Hasse diagram of the poset $Invol(4)$.

It is well-known that S_n is a graded poset, with rank function given by the number of inversions, and that it is Eulerian (see, e.g., [19, p. 395]). It has been also shown that S_n is EL-shellable (see, e.g., [7]). Our goal is to prove that similar results hold for the poset $Invol(n)$.

It should be mentioned that the set of fixed-point-free involutions of S_n, partially ordered by the Bruhat order, has been studied recently in [6].

3. Suitable rises and covering tranformations

In this section we introduce the concepts of "suitable rise" and "covering transformation" of an involution, which play a crucial role in the description of the covering relation in $Invol(n)$.

Let $\sigma \in S_n$. We denote by

$$I_f(\sigma) = Fix(\sigma) = \{i \in [n] : \sigma(i) = i\},$$
$$I_e(\sigma) = Exc(\sigma) = \{i \in [n] : \sigma(i) > i\},$$
$$I_d(\sigma) = Def(\sigma) = \{i \in [n] : \sigma(i) < i\},$$

respectively the sets of *fixed points*, of *excedances* and of *deficiencies* of σ. As usual we denote by $exc(\sigma)$ the number of excedances of σ.

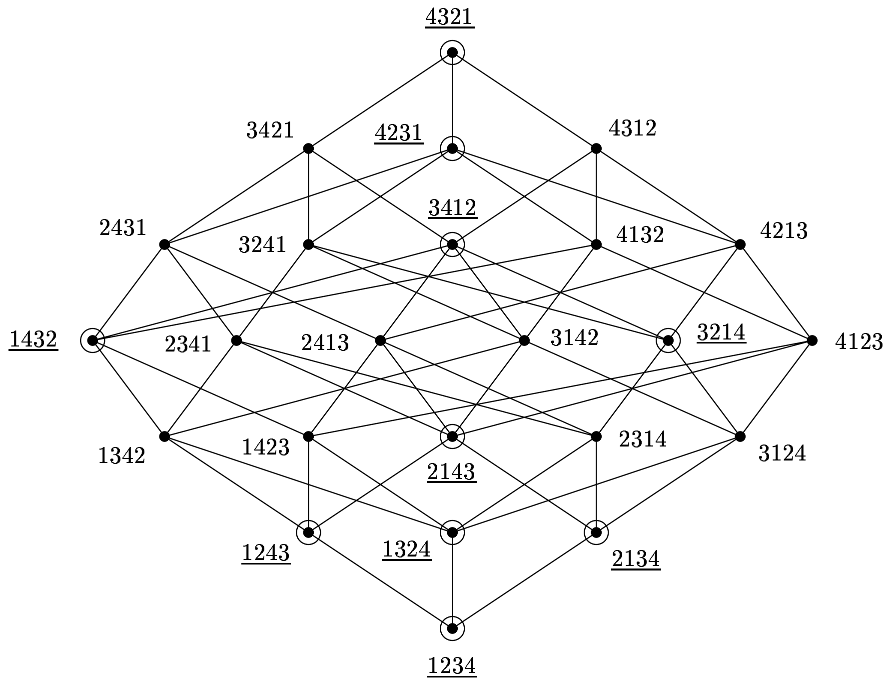


FIGURE 1. Hasse diagram of S_4

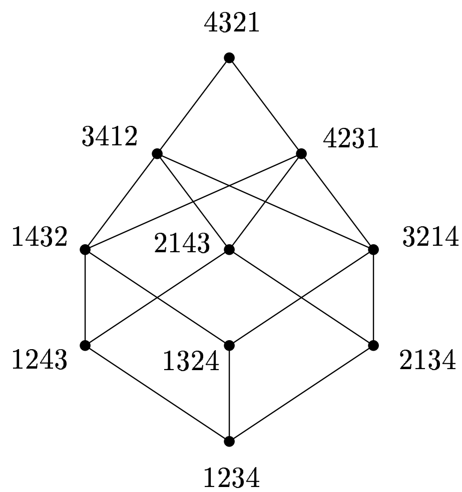


FIGURE 2. Hasse diagram of $Invol(4)$

The *type* of a rise (i,j) is the pair (a,b), where $a,b \in \{f,e,d\}$ are such that $i \in I_a(\sigma)$ and $j \in I_b(\sigma)$. We call a rise of type (a,b) also an *ab-rise*. Furthermore, we need to distinguish two kinds of *ee*-rises: an *ee*-rise (i,j) is *crossing* if $i < \sigma(i) < j < \sigma(j)$, *non-crossing* if $i < j < \sigma(i) < \sigma(j)$.

For example, if $\sigma = 321654$, the free rises of σ are $(1,4)$, $(1,5)$, $(1,6)$, $(2,4)$, $(2,5)$, $(2,6)$, $(3,4)$, $(3,5)$, $(3,6)$, whose types are, respectively, (e,e), (e,f), (e,d), (f,e), (f,f), (f,d), (d,e), (d,f), (d,d) (all nine possible types) and the ee-rise is crossing.

It is clear that if $\sigma \in Invol(n)$, then with every ee-rise of σ is associated a symmetric dd-rise, with every ef-rise a df-rise, with every fe-rise an fd-rise and with every ed-rise a de-rise. This justifies the following definition.

Definition 3.1. *Let $\sigma \in Invol(n)$. A rise (i,j) of σ is* suitable *if it is free and if its type is one of the following: (f,f), (f,e), (e,f), (e,e), (e,d).*

We now introduce the following "covering moves".

Definition 3.2. *Let $\sigma \in Invol(n)$. Let (i,j) be a suitable rise of σ. The* covering transformation *of σ with respect to (i,j), denoted by $ct_{(i,j)}(\sigma)$, is the involution obtained from σ by substituting the black dots with the white dots as described in Table 1, depending on the type of (i,j).*

Lemma 3.3. *Let $\sigma, \tau \in Invol(n)$, with $\sigma \leq_B \tau$. Let (i,j) be a suitable rise of σ, but not an ed-rise, such that $\sigma(i,j) \leq_B \tau$. Then*

$$ct_{(i,j)}(\sigma) \leq_B \tau.$$

Proof. In case 1, $ct_{(i,j)}(\sigma) = \sigma(i,j)$, and there is nothing to prove. In all other cases (2, 3, 4 and 5), we let $\sigma_1 = \sigma(i,j)$, $\sigma_2 = \sigma_1^{-1} = \sigma(\sigma(i), \sigma(j))$ and $\sigma_3 = ct_{(i,j)}(\sigma)$. We have $\sigma_1 \leq_B \tau$ and $\sigma_2 \leq_B \tau$, and we want to show that $\sigma_3 \leq_B \tau$. But in each of these four cases, for every $(h,k) \in [n]^2$, we have that

$$\sigma_3[h,k] = \begin{cases} \sigma_1[h,k], & \text{if } h > k, \\ \sigma_2[h,k], & \text{if } h \leq k. \end{cases}$$

So, in every case, $\sigma_3[h,k] \leq \tau[h,k]$ and this implies $\sigma_3 \leq_B \tau$. □

It remains to consider case 6 (ed-rise). Note that Lemma 3.3 does not hold in general if (i,j) is a free ed-rise. For example, if we consider $\sigma = 351624$, $\tau = 653421$ and the free ed-rise $(1,6)$ of σ, we have $\sigma \leq_B \tau$ and $\sigma(1,6) = 451623 \leq_B \tau$, but $ct_{(1,6)}(\sigma) = 456123 \not\leq_B \tau$.

4. Minimal covering transformation

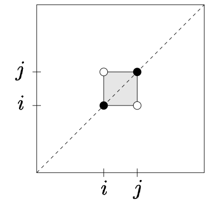
In this section we introduce the concept of "minimal covering transformation" of an involution, crucial in the proofs that the poset $Invol(n)$ is graded and EL-shellable.

Definition 4.1. *Let $\sigma, \tau \in Invol(n)$, with $\sigma < \tau$. The* difference index *of σ with respect to τ, denoted by $di_\tau(\sigma)$ (or simply di), is the minimal index on which σ and τ differ:*

$$di_\tau(\sigma) = \min\{i \in [n] : \sigma(i) \neq \tau(i)\}.$$

The covering index *of σ with respect to τ, denoted by $ci_\tau(\sigma)$ (or simply ci), is*

$$ci_\tau(\sigma) = \min\{j \in [di+1, n] : \sigma(j) \leq \tau(di) \text{ and } (di, j) \text{ is a free rise of } \sigma\}.$$

<table>
<tr><th>CASE</th><th>TYPE OF (i,j)</th><th>$ct_{(i,j)}(\sigma)$</th><th>MOVE</th></tr>
<tr><td>1</td><td>ff-rise</td><td>$\sigma(i,j)$</td><td>
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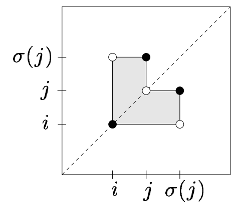
<tr><td>2</td><td>fe-rise</td><td>$\sigma(i,j,\sigma(j))$</td><td>
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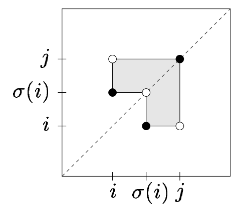
<tr><td>3</td><td>ef-rise</td><td>$\sigma(i,j,\sigma(i))$</td><td>
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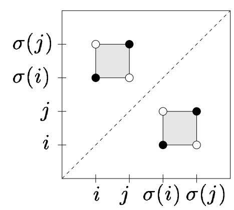
<tr><td>4</td><td>non-crossing
ee-rise</td><td>$\sigma(i,j)(\sigma(i),\sigma(j))$</td><td>
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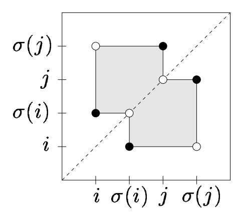
<tr><td>5</td><td>crossing
ee-rise</td><td>$\sigma(i,j,\sigma(j),\sigma(i))$</td><td>
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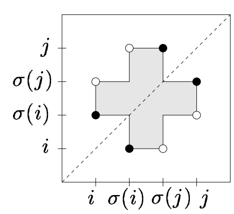
<tr><td>6</td><td>ed-rise</td><td>$\sigma(i,j)(\sigma(i),\sigma(j))$</td><td>
</td></tr>
</table>

TABLE 1. Covering transformation

Note that $ci_\tau(\sigma)$ is well defined since $k = \sigma^{-1}(\tau(di)) \in [di+1, n]$ is such that $\sigma(k) \leq \tau(di)$ and (di, k) is a rise (not necessarely free) of σ, and this implies the existence of at least one element in the set

$$\{j \in [di+1, n] : \sigma(j) \leq \tau(di) \text{ and } (di, j) \text{ is a free rise of } \sigma\}.$$

It is clear that (di, ci) is a free rise of σ, but more is true.

Proposition 4.2. *Let $\sigma, \tau \in Invol(n)$, with $\sigma <_B \tau$. Then (di, ci) is a suitable rise of σ.*

Sketch of the proof. It can be shown that (di, ci) is not an fd-rise. □

Proposition 4.2 allows us to give the following definitions.

Definition 4.3. *Let $\sigma, \tau \in Invol(n)$, with $\sigma <_B \tau$. The* minimal covering rise *of σ with respect to τ, denoted by $mcr_\tau(\sigma)$ (or simply mcr) is*

$$mcr_\tau(\sigma) = (di, ci).$$

The minimal covering transformation *of σ with respect to τ, denoted by $mct_\tau(\sigma)$ (or simply mct) is*

$$mct_\tau(\sigma) = ct_{mcr}(\sigma) = ct_{(di,ci)}(\sigma).$$

We can now give our main result concerning ed-rises.

Theorem 4.4. *Let $\sigma, \tau \in Invol(n)$, with $\sigma <_B \tau$. If $mcr_\tau(\sigma)$ is an ed-rise then*

$$mct_\tau(\sigma) \leq_B \tau.$$

Sketch of the proof. It can be proved that, for every $(h, k) \in [n]^2$, we have

$$mct_\tau(\sigma)[h, k] \leq \tau[h, k].$$

So the result follows by Theorem 2.4. □

Lemma 3.3 and Theorem 4.4 have the following consequence, which will be crucial in the proof that $Invol(n)$ is graded.

Corollary 4.5. *Let $\sigma, \tau \in Invol(n)$, with $\sigma <_B \tau$. Then*

$$mct_\tau(\sigma) \leq_B \tau.$$

5. $Invol(n)$ is graded

In this section we prove the first main result of this work, namely that $Invol(n)$ is a graded poset, and we determine explicitly its rank function. In order to do this we first give a characterization of the covering relation in $Invol(n)$, in terms of suitable rises and covering transformations.

Theorem 5.1. *Let $\sigma, \tau \in Invol(n)$. Then τ covers σ in $Invol(n)$ if and only if $\tau = ct_{(i,j)}(\sigma)$, for some suitable rise (i, j) of σ.*

Proof. It's easy to see that, if (i,j) is a suitable rise of σ, then the involution $ct_{(i,j)}(\sigma)$ covers σ in $Invol(n)$.

On the other hand, if $\sigma \neq ct_{(i,j)}(\sigma)$, for every suitable rise (i,j) of σ, by Corollary 4.5 we have

$$\sigma <_B mct_\tau(\sigma) <_B \tau,$$

thus τ does not cover σ in $Invol(n)$. □

As an example of application of Theorem 5.1, consider $\sigma = 321654 \in Invol(6)$. The suitable rises of σ are $(1,4)$, $(1,5)$, $(1,6)$, $(2,4)$ and $(2,5)$, and we have $ct_{(1,4)}(\sigma) = 623451$, $ct_{(1,5)}(\sigma) = 523614$, $ct_{(1,6)}(\sigma) = 426153$, $ct_{(2,4)}(\sigma) = 361452$ and $ct_{(2,5)}(\sigma) = 351624$. So

$$\{\tau \in Invol(n) : \sigma \lhd \tau \text{ in } Invol(n)\} = \{623451, 523614, 426153, 361452, 351624\}.$$

We can now state and prove the main result of this section.

Theorem 5.2. *The poset $Invol(n)$ is graded, with rank function ρ given by*

$$\rho(\sigma) = \frac{inv(\sigma) + exc(\sigma)}{2},$$

for every $\sigma \in Invol(n)$. In particular $Invol(n)$ has rank

$$\rho\left(Invol(n)\right) = \left\lfloor \frac{n^2}{4} \right\rfloor.$$

Proof. It suffices to observe that $\rho(ct_{(i,j)}(\sigma)) = \rho(\sigma) + 1$, for every $\sigma \in Invol(n)$ and for every suitable rise (i,j) of σ.

For the second part, note that the maximum of S_n, which is also the maximum of $Invol(n)$, has $\lfloor n/2 \rfloor$ excedances and $n\,(n-1)\,/2$ inversions. □

6. $Invol(n)$ IS EL-SHELLABLE

In this section we prove that the poset $Invol(n)$ is EL-shellable, defining a particular edge-labeling of $Invol(n)$, which we call "standard", and showing that it is an EL-labeling.

Theorem 5.1 allows us to give the following definition.

Definition 6.1. *The* standard *edge-labeling of $Invol(n)$, with values in the set $\{(i,j) \in [n]^2 : i < j\}$ (totally ordered by the lexicographic order), is defined in the following way: for every $\sigma, \tau \in Invol(n)$ such that τ covers σ in $Invol(n)$, if (i,j) is the suitable rise of σ such that $\tau = ct_{(i,j)}(\sigma)$, then we set*

$$\lambda(\sigma, \tau) = (i,j).$$

The standard edge-labeling of $Invol(n)$ is actually an EL-labeling, as we prove in the following.

Theorem 6.2. *The poset $Invol(n)$ is EL-shellable.*

Sketch of the proof. Let $\sigma, \tau \in Invol(n)$, with $\sigma <_B \tau$. It can be shown that the saturated chain $\sigma = \sigma_0 \lhd \sigma_1 \lhd \ldots \lhd \sigma_k = \tau$, defined by

$$\sigma_i = mct_\tau(\sigma_{i-1}),$$

for every $i \in [k]$, has a non-decreasing labeling, which is, among all saturated chains from σ to τ, the minimal in the lexicographic order, and that any other saturated chain from σ to τ, different from the previous one, has at least one decrease. □

As a consequence, by Theorem 2.1, we have the following.

Corollary 6.3. *The poset $Invol(n)$ is Cohen-Macaulay.*

7. $Invol(n)$ is Eulerian

In this section we prove that the poset $Invol(n)$ is Eulerian. In order to do this, we introduce some notions which somehow invert those introduced in §3 and §4.

Definition 7.1. *Let $\tau \in Invol(n)$. An inversion (i, j) of τ is* inv-suitable *if (i, j) is a suitable rise of some $\sigma \in Invol(n)$ and $ct_{(i,j)}(\sigma) = \tau$. We call such a σ (obviously unique) the* inverse covering transformation *of τ with respect to (i, j) and we denote it by $ict_{(i,j)}(\tau)$.*

Obviously $ict_{(i,j)}\left(ct_{(i,j)}(\sigma)\right) = \sigma$ and $ct_{(i,j)}\left(ict_{(i,j)}(\tau)\right) = \tau$. We summarize the action of the inverse covering transformation on the diagram of an involution in Table 2, with a notation similar to that used in Table 1.

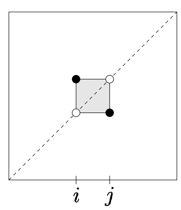
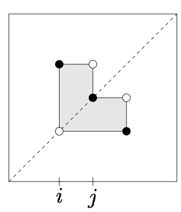
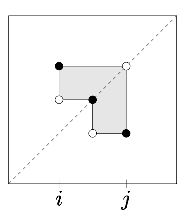
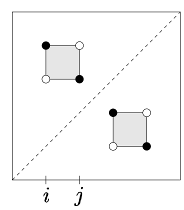
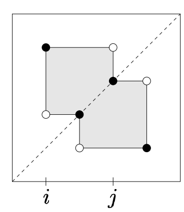
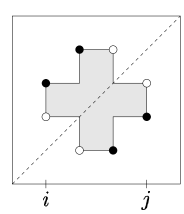
Case	Move
1	
2	
3	
4	
5	
6	

Table 2. Inverse covering transformation

Definition 7.2. *Let* $\sigma, \tau \in Invol(n)$, *with* $\sigma <_B \tau$. *The* minimal covering inversion *of* τ *with respect to* σ, *denoted by* $mci_\sigma(\tau)$, *is the minimal (in the lexicographic order) inv-suitable inversion* (i,j) *of* τ *such that* $\sigma \leq_B ict_{(i,j)}(\tau)$.

The minimal inverse covering tranformation *of* τ *with respect to* σ, *denoted by* $mict_\sigma(\tau)$, *is*

$$mict_\sigma(\tau) = ict_{mci_\sigma(\tau)}(\tau).$$

We can now prove that the condition of Theorem 2.2 holds for the standard labeling of $Invol(n)$, and thus that it is Eulerian.

Theorem 7.3. *Let* $\sigma, \tau \in Invol(n)$, *with* $\sigma <_B \tau$. *Let* λ *be the standard labeling of* $Invol(n)$. *Then there is exactly one saturated chain from* σ *to* τ *with decreasing labels.*

Sketch of the proof. It can be shown that the *descending* saturated chain $\tau = \sigma_0 \rhd \sigma_1 \rhd \ldots \rhd \sigma_k = \sigma$, defined by

$$\sigma_i = mict_\sigma(\sigma_{i-1}),$$

for every $i \in [k]$, has increasing labels (so the corresponding *ascending* chain will have decreasing labels), and that any other saturated chain from σ to τ, different from the previous one, has at least one increase. □

As a consequence, by Theorem 2.2, we have the following.

Corollary 7.4. *The poset* $Invol(n)$ *is Eulerian.*

Furthermore, by Theorem 2.3, we can conclude the following.

Corollary 7.5. *The poset* $Invol(n)$ *is Gorenstein.*

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