HILBERT SERIES OF INVARIANT ALGEBRAS FOR CLASSICAL WEYL GROUPS

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ABSTRACT. We introduce several new Mahonian statistics on the classical Weyl groups of type B and D and use them to study the invariant algebras of two natural actions of these groups on the ring of polynomials.
RÉSUMÉ. On introduit des nouvelles statistiques de MacMahon sur les groupes de Weyl classiques et on les utilise pour étudier les algebres invariantes de deux actions naturelles de ces groupes sur l'anneau des polynômes.

1. INTRODUCTION

Let W be a classical Weyl group, i.e. W is either the symmetric group A_{n-1}, the hyperoctahedral group B_n, or the even-signed permutation group D_n. Consider the natural diagonal and tensor actions of W and W^t, respectively, on the polynomial ring $\mathbf{C}[x_1, \ldots, x_n]^{\otimes t}$ and denote by DIA and TIA the corresponding invariant algebras. Let $\mathcal{Z}_W(\bar{q})$ be the quotient of the Hilbert series of DIA and TIA.

A well known result due to MacMahon [17] asserts that the major index is equidistributed with the length function on the symmetric group. The Euler-Mahonian distribution of descent number and major index was extensively studied (see e.g. [8, 11, 12]) and its generating function is known as Carlitz's identity. Although its nature is combinatorial, the major index has also important algebraic properties. It is known that, if $W = A_{n-1}$, then $\mathcal{Z}_{A_{n-1}}(\bar{q})$ is a polynomial with non-negative integer coefficients, which admits an explicit simple formula in terms of the major index [13]. Moreover, Garsia and Stanton provide a descent basis for the coinvariant algebra of type A whose elements are monomials of degree uqual to the major index of the indexing permutation [14]. The problem of generalizing these results to the hyperoctahedral group has been open for many years. Several authors have defined analogues of the major index for B_n (see, e.g., [9, 18, 19]) but none of these is Mahonian, (i.e. equidistributed with length). Finally in a recent paper [1] Adin and Roichman introduced the flag-major index ($fmaj$) on the hyperoctahedral group. They show that it is Mahonian and find a formula for $\mathcal{Z}_{B_n}(\bar{q})$ by means of this new statistic. In [2] the previous two authors and Brenti give a generalization to B_n of Carlitz's identity. The flag-major index has been further studied in [3]; it plays a crucial role in representation theory, more precisely in the decomposition of the coinvariant algebra into irreducible modules. In [4] the first of the present authors defines the D-flag major index ($fmaj_D$) for the even-signed permutation group, and proves that it is Mahonian. Moreover, he defines a pair of Euler-Mahonian statistics that allows a generalization of Carlitz's identity to D_n. Neither similar formula for $\mathcal{Z}_{D_n}(\bar{q})$ nor other algebraic properties have been found so far.

The purpose of this work is to introduce a new "major" statistic on D_n from which one can generalize all the combinatorial and algebraic properties known for type A and B. In particular we would like to find an explicit formula for $\mathcal{Z}_{D_n}(\bar{q})$ using this new statistic. Toward this end, we define two new Mahonian statistics ned_D and $Dmaj$. The latter $Dmaj$, defined in a combinatorial way, has the analogous algebraic meaning for D_n, as the major index for S_n, and $fmaj$ for B_n; namely, it allows us to find an explicit formula for $\mathcal{Z}_{D_n}(\bar{q})$ which implies, in particular, that this series (as in types A and B) is actually a polynomial with non-negative integer coefficients. To prove the results we introduce suitable even and odd t-partite partitions. These are related with the t-partite partitions introduced by Gordon in [15], and further studied by Garsia and Gessel in [13] where applications to the permutation enumerations are shown. Using similar ideas, we define the Mahonian statistic ned_B on B_n that allows a new and simpler proof of the Adin-Roichman formula for $\mathcal{Z}_{B_n}(\bar{q})$ (see, [5]). Finally, we define a new descent number $Ddes$ on D_n so that the pair $(Ddes, Dmaj)$ gives a generalization to D_n of Carlitz's identity. In a upcoming paper [6] we show that $Dmaj$ and $Ddes$ play an important role in the decomposition in irreducible submodules of the coinvariant algebra of type D.

The organization of this extended abstract is as follows. In §2 we introduce some preliminaries and notation. In particular we present some combinatorial properties of classical Weyl groups, we define their actions on the polynomial rings, and we give some results on t-partite partitions. In §3 we define several new combinatorial tools that are needed in the rest of our work and we state some of their fundamental properties: we introduce the concept of parity of a partition and the new statistics ned_B, ned_D and $Dmaj$. In §4 we collect some combinatorial properties of $Dmaj$. We show that it's equidistributed with length and that, together with $Ddes$, satisfies the Carlitz's identity for D_n. §5 is devoted to our main result, i.e. we find an explicit formula for the polynomial $\mathcal{Z}_{D_n}(\bar{q})$ using $Dmaj$.

2. Notation, Definitions and Preliminaries

2.1. **Classical Weyl Groups.** In this section we give some definitions, notation and results that will be used in the rest of this work. We let $\mathbf{P} := \{1,2,3,\dots\}$, $\mathbf{N} := \mathbf{P} \cup \{0\}$, $\mathbf{Z}$ be the ring of integers and $\mathbf{C}$ be the field of complex numbers; for $a \in \mathbf{N}$ we let $[a] := \{1, 2, \dots, a\}$ (where $[0] := \emptyset$). Given $a, b \in \mathbf{N}$ we let $[a, b) := \{i \in \mathbf{N} : min(a, b) \leq i < max(a, b)\}$ and similarly for $n, m \in \mathbf{Z}$, $n \leq m$, $[n, m] := \{n, n+1, \dots, m\}$. Given $n, m \in \mathbf{Z}$, by $n \equiv m$ we mean $n \equiv m \pmod 2$. For a set A we denote its cardinality by $|A|$ and the set of all its subsets by 2^A. If $A \subseteq [n]$ its complementary set $[n] \setminus A$ will be denoted by $\mathcal{C}_n(A)$. Given two sets A and B we denote by $A \triangle B$ their symmetric difference $(A \cup B) \setminus (A \cap B)$.
We always consider the linear order on $\mathbf{Z}$

$$-1 \prec -2 \prec \cdots \prec -n \prec \cdots \prec 0 \prec 1 \prec 2 \prec \cdots \prec n \prec \cdots$$

instead of the usual ordering. Given a sequence $\sigma = (\sigma_1, \dots, \sigma_n) \in \mathbf{Z}^n$ we let $Neg(\sigma) := \{i \in [n] : \sigma_i < 0\}$ and $neg(\sigma) := |Neg(\sigma)|$. We say that $(i, j) \in [n] \times [n]$ is an *inversion* of σ if $i < j$ and $\sigma_i \succ \sigma_j$ and that $i \in [n-1]$ is a *descent* of σ if $\sigma_i \succ \sigma_{i+1}$. We denote by $Inv(\sigma)$ and $Des(\sigma)$ the set of inversions and the set of descents of σ and by $inv(\sigma)$ and $des(\sigma)$ their cardinalities, respectively. We also let

$$maj(\sigma) := \sum_{i \in Des(\sigma)} i$$

and call it the *major index* of σ.

Given a set A we let $S(A)$ be the set of all bijections $\tau : A \to A$, and $S_n := S([n])$. If $\sigma \in S_n$ then we write $\sigma = \sigma_1 \dots \sigma_n$ to mean that $\sigma(i) = \sigma_i$, for $i = 1, \dots, n$. Given $\sigma, \tau \in S_n$ we let $\sigma\tau := \sigma \circ \tau$ (composition of functions) so that, for example, $1423 \cdot 2134 = 4123$.
Given a variable q and a commutative ring R we denote by $R[q]$ (respectively, $R[[q]]$) the ring of polynomials (respectively, formal power series) in q with coefficients in R. For $i \in \mathbf{N}$ we let, as customary, $[i]_q := 1 + q + q^2 + \ldots + q^{i-1}$ (so $[0]_q = 0$).
For $n \in \mathbf{P}$ we let

$$A_n(t, q) := \sum_{\sigma \in S_n} t^{des(\sigma)} q^{maj(\sigma)},$$

and $A_0(t,q) := 1$. For example, $A_3(t,q) = 1 + 2tq^2 + 2tq + t^2q^3$. The following result is due to Carlitz, and we refer the reader to [8] for its proof, (see also [3] for a refinement).

Theorem 1. *Let $n \in \mathbf{P}$. Then*

$$\sum_{r\geq 0}[r+1]_q^n t^r = \frac{A_n(t,q)}{\prod_{i=0}^{n}(1-tq^i)}$$

in $\mathbf{Z}[q][[t]]$.

Let B_n be the group of all bijections β of the set $[-n,n]\setminus\{0\}$ onto itself such that

$$\beta(-i) = -\beta(i)$$

for all $i \in [-n,n]\setminus\{0\}$, with composition as the group operation. If $\beta \in B_n$ then, following [7], we write $\beta = [\beta_1,\ldots,\beta_n]$ to mean that $\beta(i) = \beta_i$, for $i = 1,\ldots,n$, and call this the *window* notation of β. Because of this notation the group B_n is often called the group of all *signed permutations* on $[n]$ or the *hyperoctahedral group* of rank n.

We find it convenient to introduce this *pair* notation: for each $\sigma \in S_n$ and $H \subseteq [n]$, we let $(\sigma, H) := [\beta_1,\ldots,\beta_n]$ be the signed permutation defined as follows:

$$\beta_i := \begin{cases} -\sigma_i, & \text{if } i \in H, \\ \sigma_i, & \text{if } i \notin H. \end{cases}$$

Note that in this notation we have

$$(\sigma,H)^{-1} = (\sigma^{-1},\sigma(H)) \tag{1}$$

and

$$(\sigma,H)(\tau,K) = (\sigma\tau, K\triangle\tau^{-1}(H)) \tag{2}$$

For example, if $(\sigma,H) = (43512,\{1,2,5\}) = [-4,-3,5,1,-2] \in B_5$ then $(\sigma,H)^{-1} = (45213,\{2,3,4\}) = [4,-5,-2,-1,3]$ and if $(\tau,K) = (21345,\{2,5\})$ then $(\sigma,H)(\tau,K) = (34512,\{1\})$.

Following [1] and [2] we define the *flag-major index* of $\beta \in B_n$ by

$$fmaj(\beta) := 2maj(\beta) + neg(\beta) \tag{3}$$

and the *flag descent number* of β by

$$fdes(\beta) := 2des(\beta) + \eta(\beta), \tag{4}$$

where

$$\eta(\beta) := \begin{cases} 1, & \text{if } \beta(1) < 0, \\ 0, & \text{otherwise.} \end{cases}$$

For example, if $\beta = [-4,-3,5,1,-2] \in B_5$ then $fmaj(\beta) = 2\cdot 8 + 3 = 19$ and $fdes(\beta) = 2\cdot 3 + 1 = 7$.

It's known that $fmaj$ is equidistributed with length on B_n, (see [1] where $fmaj$ is denoted $flag - major$).

The pair of statistics $(fdes, fmaj)$ gives a generalization of Carlitz's identity (Theorem 1) to B_n. More precisely we have the following theorem due to Adin, Brenti and Roichman [2] (see also [3] for a refinement).

Theorem 2. *Let $n \in \mathbf{P}$. Then*

$$\sum_{r\geq 0}[r+1]_q^n t^r = \frac{\sum_{\beta\in B_n} t^{fdes(\beta)}q^{fmaj(\beta)}}{(1-t)\prod_{i=1}^{n}(1-t^2q^{2i})}$$

in $\mathbf{Z}[q][[t]]$.

We denote by D_n the subgroup of B_n consisting of all the signed permutations having an even number of negative entries in their window notation, more precisely

$$D_n := \{\gamma \in B_n : neg(\gamma) \equiv 0\}.$$

As for B_n we introduce a *pair* notation: for each $\sigma \in S_n$ and $K \subseteq [n-1]$ we let $(\sigma, K)_D := [\gamma_1, \ldots, \gamma_n]$ be the unique even-signed permutation γ such that $|\gamma_i| = \sigma_i$ for all $i \in [n]$ and $K \cup \{n\} \supseteq Neg(\gamma) \supseteq K$. More precisely

$$\gamma_i := \begin{cases} -\sigma_i, & \text{if } i \in K, \\ \sigma_i, & \text{if } i \notin K \cup \{n\}, \\ (-1)^{|K|}\sigma_n, & \text{if } i = n. \end{cases}$$

For example $(54312, \{1,3,4\})_D = [-5, 4, -3, -1, -2] \in D_5$. We will usually omit the index D in the pair notation of D_n when there is no risk of confusion with the pair notation of B_n.
Following [4], for every $\gamma \in D_n$ we define the *D-negative multiset*

$$DDes(\gamma) := Des(\gamma) \biguplus \{-\gamma(i) - 1 \ : \ i \in Neg(\gamma)\} \setminus \{0\},$$

$$ddes(\gamma) := |DDes(\gamma)|$$

and

$$dmaj(\gamma) := \sum_{i \in DDes(\gamma)} i.$$

For example, if $\gamma = [-5, 4, -3, -1, -2] \in D_5$ then $DDes(\gamma) = \{1, 2^2, 3, 4\}$, $ddes(\gamma) = 5$ and $dmaj(\gamma) = 12$.
The pair of statistics $(ddes, dmaj)$ gives a generalization of Carlitz's identity to D_n. More precisely, we have the following theorem, (see [4]).

Theorem 3. *Let $n \in \mathbf{P}$. Then*

$$\sum_{r \geq 0} [r+1]_q^n t^r = \frac{\sum_{\gamma \in D_n} t^{ddes(\gamma)} q^{dmaj(\gamma)}}{(1-t)(1-tq^n) \prod_{i=1}^{n-1} (1 - t^2 q^{2i})}$$

in $\mathbf{Z}[q][[t]]$.

2.2. **Group Actions on Polynomial Rings.** Let W be a classical Weyl group, i.e $W = S_n, B_n$ or D_n. There is a natural action of W on the polynomial ring $\mathbf{P}_n := \mathbf{C}[x_1, \ldots, x_n]$, $\varphi : W \to \mathrm{Aut}(\mathbf{P}_n)$ defined on the generators by

$$\varphi(w) : x_i \mapsto \frac{w(i)}{\mid w(i) \mid} x_{|w(i)|},$$

for all $w \in W$ and extended uniquely to an algebra homomorphism. This action gives rise to two actions on the tensor power $\mathbf{P}_n^{\otimes t} := \mathbf{P}_n \otimes \cdots \otimes \mathbf{P}_n$ (t-times): the natural *tensor action* φ_T of $W^t := W \times \cdots \times W$ (t-times), and the *diagonal action* of W on $\mathbf{P}_n^{\otimes t}$, $\varphi_D := \varphi_T \circ d$ defined using the diagonal embedding $d : W \hookrightarrow W^t$, $w \mapsto (w, \ldots, w)$.
The *tensor invariant algebra*

$$\mathrm{TIA} := \{\bar{p} \in \mathbf{P}_n^{\otimes t} : \varphi_T(\bar{w})\bar{p} = \bar{p} \ \text{ for all } \bar{w} \in W^t\}$$

is a subalgebra of the *diagonal invariant algebra*

$$\mathrm{DIA} := \{\bar{p} \in \mathbf{P}_n^{\otimes t} : \varphi_D(w)\bar{p} = \bar{p} \ \text{ for all } w \in W\}.$$

These two algebras are naturally multigraded and hence we can consider the corresponding Hilbert series

$$F_D(\bar{q}) := \sum_{n_1, \ldots, n_t} \dim_{\mathbf{C}}(\mathrm{DIA}_{n_1, \ldots, n_t}) q_1^{n_1} \cdots q_t^{n_t},$$

$$F_T(\bar{q}) := \sum_{n_1,\dots,n_t} \dim_{\mathbf{C}}(\mathrm{TIA}_{n_1,\dots,n_t}) q_1^{n_1} \cdots q_t^{n_t},$$

where $\mathrm{DIA}_{n_1,\dots,n_t}$ and $\mathrm{TIA}_{n_1,\dots,n_t}$ are the homogeneous components of multi-degree $(n_1,\dots,n_t)$ in DIA and TIA respectively and $\bar{q} = (q_1,\dots,q_t)$.
We denote the quotient series by

$$\mathcal{Z}_W(\bar{q}) := \frac{F_D(\bar{q})}{F_T(\bar{q})} \in \mathbf{Z}[[\bar{q}]].$$

2.3. **t-Partite Partitions.** In this section we recall the language of t-partite partitions which was originally defined by Gordon [15] as well as some results of Garsia and Gessel [13] that we use in the rest of this work.

Let $\mathcal{F}_n$ be the set of all functions $f : [n] \to \mathbf{N}$. For $f \in \mathcal{F}_n$ we let

$$|f| := \sum_{i=1}^{n} f(i),$$

and we denote $\mathcal{F}_{n,t} := (\mathcal{F}_n)^t$. Moreover, for $f = (f_1,\dots,f_t) \in \mathcal{F}_{n,t}$, we define

$$\alpha_j(f) := \sum_{i=1}^{t} f_i(j),$$

and we let $\mathcal{F}^e_{n,t} := \{f \in \mathcal{F}_{n,t} \ : \ \alpha_j(f) \equiv 0 \text{ for all } j \in [n]\}$ and $\mathcal{F}^o_{n,t} := \{f \in \mathcal{F}_{n,t} \ : \ \alpha_j(f) \equiv 1 \text{ for all } j \in [n]\}$.

A *t-partite partition with n parts* is a sequence $f = (f_1,\dots,f_t) \in \mathcal{F}_{n,t}$,

$$f = \begin{pmatrix} f_1(1) & f_1(2) & \dots & f_1(n) \\ f_2(1) & f_2(2) & \dots & f_2(n) \\ \vdots & \vdots & & \vdots \\ f_t(1) & f_t(2) & \dots & f_t(n) \end{pmatrix}$$

satisfying the following condition:
for $i_o \in [t]$ and $j \in [n]$, if $f_i(j) = f_i(j+1)$ for all $i < i_0$, then $f_{i_0}(j) \geq f_{i_0}(j+1)$.
Note, in particular, that for $i_0 = 1$ this implies that

$$f_1(1) \geq f_1(2) \geq \ldots \geq f_1(n) \geq 0,$$

so f_1 is a partition with at most n parts.
We denote the set of all the t-partite partitions with n parts by $\mathcal{B}_{n,t}$. In particular, $\mathcal{B}_{n,1}$ is the set of all integer partitions with at most n parts.
For example, if $n = 5$ and $t = 2$, then $f = (f_1, f_2)$ with $f_1 = (4,4,4,3,3)$ and $f_2 = (3,3,2,5,4)$ is a bipartite partition with 5 parts.

Given a permutation $\sigma = \sigma_1 \cdots \sigma_n$ we say that the partition $\lambda = (\lambda_1,\dots,\lambda_n)$ is *σ-compatible* if, for all $i \in [n-1]$,

$$\lambda_i - \lambda_{i+1} \geq \varepsilon_i(\sigma) := \begin{cases} 1, & \text{if } \sigma_i > \sigma_{i+1}, \\ 0, & \text{otherwise.} \end{cases} \tag{5}$$

We also set $\varepsilon_n(\sigma) := 0$. Clearly, a partition λ is σ-compatible if and only if it is of the form

$$\lambda_i = p_i + p_{i+1} + \cdots + p_n$$

with $p_i \geq \varepsilon_i(\sigma)$ for all i. We let $\mathcal{P}(\sigma)$ be the set of all σ-compatible partitions.
For example, if $\sigma = 15342$ then $\lambda = (6,6,4,4,3) \in \mathcal{P}(\sigma)$.
The following theorems are due to Garsia and Gessel (see [13, Theorems 2.1 and 2.2 and Remark 2.2]).

Theorem 4. *There exists a bijection between $\mathcal{B}_{n,t}$ and the set $\mathcal{P}_{n,t}$ of the $2t$-tuples*

$$(\sigma_1,\dots,\sigma_t,\lambda^{(1)},\dots,\lambda^{(t)})$$

where $\sigma_i \in S_n$, $\lambda^{(i)} \in \mathcal{P}(\sigma_i)$ for all $i \in [t]$ and $\sigma_t \cdots \sigma_2\sigma_1 = e$. This bijection is given by

$$\Omega(\sigma_1, \ldots, \sigma_t, \lambda^{(1)}, \ldots, \lambda^{(t)}) := \begin{pmatrix} \lambda_1^{(1)} & \lambda_2^{(1)} & \ldots & \lambda_n^{(1)} \\ \lambda_{\sigma_1(1)}^{(2)} & \lambda_{\sigma_1(2)}^{(2)} & \ldots & \lambda_{\sigma_1(n)}^{(2)} \\ \vdots & \vdots & & \vdots \\ \lambda_{\sigma_{t-1}\cdots\sigma_1(1)}^{(t)} & \lambda_{\sigma_{t-1}\cdots\sigma_1(2)}^{(t)} & \ldots & \lambda_{\sigma_{t-1}\cdots\sigma_1(n)}^{(t)} \end{pmatrix}.$$

We let

$$\mathcal{B}_{n,t}^e := \{f \in \mathcal{B}_{n,t} \ : \ \alpha_j(f) \equiv 0 \ \text{ for all } j \in [n]\}$$

and

$$\mathcal{B}_{n,t}^o := \{f \in \mathcal{B}_{n,t} \ : \ \alpha_j(f) \equiv 1 \ \text{ for all } j \in [n]\}$$

the sets of all the *even* and *odd* t-partite partitions with n parts, respectively. Moreover we let $\mathcal{P}_{n,t}^e$ the set

$$\{(\sigma_1, \ldots, \sigma_t, \lambda^{(1)}, \ldots, \lambda^{(t)}) \in \mathcal{P}_{n,t} \ : \ \lambda_i^{(1)} + \lambda_{\sigma_1(i)}^{(2)} + \cdots + \lambda_{\sigma_{t-1}\cdots\sigma_1(i)}^{(t)} \equiv 0 \ \text{ for all } i \in [n]\}$$

and similarly $\mathcal{P}_{n,t}^o$

$$\{(\sigma_1, \ldots, \sigma_t, \lambda^{(1)}, \ldots, \lambda^{(t)}) \in \mathcal{P}_{n,t} \ : \ \lambda_i^{(1)} + \lambda_{\sigma_1(i)}^{(2)} + \cdots + \lambda_{\sigma_{t-1}\cdots\sigma_1(i)}^{(t)} \equiv 1 \ \text{ for all } i \in [n]\}.$$

It's clear that, by restriction, the map Ω of Theorem 4 gives rise to two bijections $\mathcal{B}_{n,t}^e \leftrightarrow \mathcal{P}_{n,t}^e$ and $\mathcal{B}_{n,t}^o \leftrightarrow \mathcal{P}_{n,t}^o$.

Theorem 5. *Let $W = S_n$ and $t \in \mathbf{N}$. Then*

$$\mathcal{Z}_{S_n}(\bar{q}) = \sum_{\sigma_1,\ldots,\sigma_t} \prod_{i=1}^{t} q_i^{maj(\sigma_i)},$$

where the sum is over all t-tuples $(\sigma_1, \ldots, \sigma_t)$ of permutations in S_n such that $\sigma_t\sigma_{t-1} \cdots \sigma_1 = e$.

The following is the corresponding result of Theorem 5 for B_n and it is due to Adin and Roichman [1].

Theorem 6. *Let $n, t \in \mathbf{N}$. Then*

$$\mathcal{Z}_{B_n}(\bar{q}) = \sum_{\beta_1,\ldots,\beta_t} \prod_{i=1}^{t} q_i^{fmaj(\beta_i)},$$

where the sum is over all the signed permutation $\beta_1, \ldots, \beta_t \in B_n$ such that $\beta_t \cdots \beta_1 = e$.

3. New Statistics on B_n and D_n

3.1. **Bijections and Parity Sets.** We define a bijection $\varphi_n : 2^{[n]} \to 2^{[n]}$, for every $n \in \mathbf{N}$, in the following inductive way: for $n \geq 1$,

$$\varphi_n(H) := \begin{cases} \mathcal{C}_n\varphi_{n-1}(H), & \text{if } H \subseteq [n-1], \\ \varphi_{n-1}(H \setminus \{n\}), & \text{if } H \not\subseteq [n-1], \end{cases}$$

and $\varphi_0(\emptyset) := \emptyset$.

For example, let $n = 4$ and $H = \{2\}$, then,

$$\begin{aligned} \varphi_4(\{2\}) &= \mathcal{C}_4\varphi_3(\{2\}) = \mathcal{C}_4\mathcal{C}_3\varphi_2(\{2\}) = \mathcal{C}_4\mathcal{C}_3\varphi_1(\emptyset) = \mathcal{C}_4\mathcal{C}_3\mathcal{C}_1\varphi_0(\emptyset) \\ &= \mathcal{C}_4\mathcal{C}_3(\{1\}) = \mathcal{C}_4(\{2,3\}) = \{1,4\}. \end{aligned}$$

Our goal is to understand the action of a permutation σ on $\varphi_n(H)$. For this it's useful to introduce the following concept. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n, 0, 0, \ldots) \in \mathcal{B}_{n,1}$. We define the *parity set* of λ to be

$$H(\lambda) := \{i \in [n] \ : \ \lambda_i - \lambda_{i+1} \equiv 0\}.$$

Let $\sigma \in S_n$ and $H \subseteq [n]$. Let $\lambda \in \mathcal{B}_{n,1}$ be such that $H = H(\lambda)$. Then we define

$$H^\sigma := H(\mu),$$

where μ is any partition in $\mathcal{B}_{n,1}$ such that $\lambda_i + \mu_{\sigma(i)} \equiv 0$ for all $i \in [n]$. Note that the definition of H^σ doesn't depend on λ and μ but only on H and σ.
Observe that the following statements are equivalent:

i) $(H(\lambda))^\sigma = H(\mu)$;
ii) $\lambda_i + \mu_{\sigma(i)} \equiv 0$ for all $i \in [n]$.

For example, suppose $n = 4$ and $\sigma = 4312$. Let $\lambda_i = p_i + \cdots + p_n$ and $\mu_i = r_i + \cdots + r_n$, for $i = 1, \dots, n$. The condition $\lambda_i + \mu_{\sigma(i)} \equiv 0$ for all $i \in [n]$ is equivalent to the following system of congruences:

$$\begin{cases} p_1 + p_2 + p_3 + p_4 \equiv r_4 \\ p_2 + p_3 + p_4 \equiv r_3 + r_4 \\ p_3 + p_4 \equiv r_1 + r_2 + r_3 + r_4 \\ p_4 \equiv r_2 + r_3 + r_4. \end{cases}$$

If $H = \{1, 3\}$ is the parity set of λ then p_1, p_3 are even, and p_2, p_4 are odd. All these conditions force r_3, r_4 to be even and r_1, r_2 to be odd, hence $H^\sigma = \{3, 4\}$.

The following is the main technical result of this section.

Lemma 1. *Let $n \in \mathbf{N}$. Then for all $H \subseteq [n]$ and $\sigma \in S_n$ we have*

$$\sigma\varphi_n(H) = \varphi_n(H^\sigma).$$

Note that Lemma 1 implies that $(\sigma, H) \mapsto H^\sigma$ is a left action of S_n on $2^{[n]}$.

Let $p : 2^{[n]} \to 2^{[n-1]}$ be the following projection of sets

$$H \mapsto \begin{cases} H, & \text{if } n \notin H, \\ \mathcal{C}_n(H), & \text{if } n \in H. \end{cases}$$

Let $\sigma \in S_n$, $H \subseteq [n]$ and $\lambda \in \mathcal{B}_{n,1}$ be such that $H(\lambda) = H$. We define

$$\overline{H^\sigma} := H(\mu)$$

where $\mu \in \mathcal{B}_{n,1}$ is such that $\lambda_i + \mu_{\sigma(i)} \equiv 1$ for all $i \in [n]$.

Lemma 2. *Let $\sigma \in S_n$ and $H \subseteq [n]$. Then*

$$\overline{H^\sigma} = H^\sigma \triangle \{n\} = (H \triangle \{n\})^\sigma.$$

Lemma 3. *Let $\sigma \in S_n$ and $K \subseteq [n-1]$. Then*

$$\varphi_{n-1}(K^\sigma \setminus \{n\}) = p\left(\sigma\varphi_{n-1}(K)\right).$$

3.2. **Generalization to the Multivariable Case.** In this section we generalize the definitions and results given in §3.1 to the multivariable case.
Let $n, t \in \mathbf{N}, \sigma_1, \dots, \sigma_t \in S_n$ and $H_1, \dots, H_t \subseteq [n]$. Let $\lambda^{(1)}, \dots, \lambda^{(t)} \in \mathcal{B}_{n,1}$ be such that the parity set $H(\lambda^{(i)}) = H_i$ for all $i \in [t]$. Then we define

$$(H_1, \dots, H_t)^{(\sigma_1, \dots, \sigma_t)} := H(\mu),$$

where the partition $\mu \in \mathcal{B}_{n,1}$ is such that for all $j \in [n]$, $\lambda^{(1)}_j + \lambda^{(2)}_{\sigma_1(j)} + \cdots + \lambda^{(t)}_{\sigma_{t-1}\cdots\sigma_1(j)} + \mu_{\sigma_t\cdots\sigma_1(j)} \equiv 0$. Note that, as for the one-dimensional case, the definition of $(H_1, \dots, H_t)^{(\sigma_1, \dots, \sigma_t)}$ doesn't depend on the $\lambda^{(i)}$'s and μ but only on the H_i's and σ_i's.
Observe that the following conditions are equivalent:

i) $(H(\lambda^{(1)}), \dots, H(\lambda^{(t)}))^{(\sigma_1, \dots, \sigma_t)} = H(\mu)$;
ii) $\lambda^{(1)}_j + \lambda^{(2)}_{\sigma_1(j)} + \cdots + \lambda^{(t)}_{\sigma_{t-1}\cdots\sigma_1(j)} + \mu_{\sigma_t\cdots\sigma_1(j)} \equiv 0$.

The following two technical lemmas are needed to prove the main result (Theorem 18).

Lemma 4. *Let $n, t \in \mathbf{N}$, $\sigma_i \in S_n$ and $H_i \subseteq [n]$ for all $i \in [t]$. Then*

$$(H_1, \ldots, H_t)^{(\sigma_1, \ldots, \sigma_t)} = \mathcal{C}_n^{t+1}\left(H_1^{\sigma_t \cdots \sigma_1} \triangle H_2^{\sigma_t \cdots \sigma_2} \triangle \cdots \triangle H_t^{\sigma_t}\right).$$

The next result says that the bijection φ_n is "almost" distributive with respect to the symmetric difference of sets.

Lemma 5. *Let $n \in \mathbf{N}$. Then for all $H_1, \ldots, H_t \subseteq [n]$ we have:*

$$\varphi_n(H_1) \triangle \cdots \triangle \varphi_n(H_t) = \varphi_n \mathcal{C}_n^{t+1}\left(H_1 \triangle \cdots \triangle H_t\right).$$

The following is the generalization of Lemma 1.

Corollary 7. *Let $n \in \mathbf{N}$. Then for all $H_1, \cdots, H_t \in [n]$ and $\sigma_1, \ldots, \sigma_t \in S_n$ we have*

$$\sigma_t \cdots \sigma_1 \varphi_n(H_1) \triangle \sigma_t \cdots \sigma_2 \varphi_n(H_2) \triangle \cdots \triangle \sigma_t \varphi_n(H_t) = \varphi_n\left((H_1, \ldots, H_t)^{(\sigma_1, \ldots, \sigma_t)}\right).$$

Let $\sigma_1, \ldots, \sigma_t \in S_n$ and $H_1, \ldots, H_t \subseteq [n]$. Moreover, let $\lambda^{(1)}, \ldots, \lambda^{(t)} \in \mathcal{B}_{n,1}$ be such that $H(\lambda^{(i)}) = H_i$ for all $i \in [n]$. Then we define

$$\overline{(H_1, \ldots, H_t)^{(\sigma_1, \ldots, \sigma_t)}} = H(\mu)$$

where $\mu \in \mathcal{B}_{n,1}$ is such that $\lambda_j^{(1)} + \lambda_{\sigma_1(j)}^{(2)} + \cdots + \lambda_{\sigma_{t-1} \cdots \sigma_1(j)}^{(t)} + \mu_{\sigma_t \cdots \sigma_1(j)} \equiv 1$ for all $j \in [n]$.

The following two results are natural generalizations of Lemmas 2 and 3.

Lemma 6. *Let $\sigma_1, \ldots, \sigma_t \in S_n$ and $H_1, \ldots, H_t \subseteq [n]$. Then for all $i \in [t]$ we have:*

$$\begin{aligned} \overline{(H_1, \ldots, H_t)^{(\sigma_1, \ldots, \sigma_t)}} &= (H_1, \ldots, H_t)^{(\sigma_1, \ldots, \sigma_t)} \triangle \{n\} \\ &= (H_1, \ldots, H_{i-1}, H_i \triangle \{n\}, H_{i+1}, \ldots, H_t)^{(\sigma_1, \ldots, \sigma_t)}. \end{aligned}$$

Lemma 7. *Let $\sigma_1, \ldots, \sigma_{t-1} \in S_n$ and $K_1, \ldots, K_{t-1} \subseteq [n-1]$. Then*

$$\varphi_{n-1}\left((K_1, \ldots, K_{t-1})^{(\sigma_1, \ldots, \sigma_{t-1})} \setminus \{n\}\right) = \pi\left(\sigma_{t-1} \cdots \sigma_1 \varphi_{n-1}(K_1) \triangle \cdots \triangle \sigma_{t-1} \varphi_{n-1}(K_{t-1})\right).$$

3.3. **The Statistics *ned* and *Dmaj*.** In this section we introduce the fundamental statistics ned and $Dmaj$ and study some of their basic properties. For every $\beta \in B_n$ we define $\bar{\beta} \in B_{n-1}$ by deleting the last entry of β and scaling the others as follows

$$\bar{\beta}(i) := \begin{cases} \beta(i), & \text{if } |\beta(i)| < |\beta(n)|, \\ \beta(i) - 1, & \text{if } \beta(i) > 0 \text{ and } |\beta(i)| > |\beta(n)|, \\ \beta(i) + 1, & \text{if } \beta(i) < 0 \text{ and } |\beta(i)| > |\beta(n)|. \end{cases}$$

For example, if $\beta = [-4, -3, 5, 1, -2] \in B_5$ then $\bar{\beta} = [-3, -2, 4, 1]$.
We let B_n^+ the set of the signed permutations $\beta \in B_n$ such that $\beta(n) > 0$.

Lemma 8. *Let $\beta \in B_n^+$. Then*

$$maj(-\beta) = maj(\beta) + neg(\beta),$$

where $-\beta := [-\beta(1), \ldots, -\beta(n)]$.

Corollary 8. *Let $\beta \in B_n^+$. Then*

$$fmaj(-\beta) = fmaj(\beta) + n.$$

Recall the definitions of $\varepsilon_i(\sigma)$ given in (5) and of φ_n given in §3.1. We are ready to introduce two new fundamental statistics for this work.

Definition. For $(\sigma, H) \in B_n$ we let

$$ned_B(\sigma, H) := \sum_{i \in H} 2i\varepsilon_i(\sigma) + \sum_{i \in \mathcal{C}_n(H)} i. \tag{6}$$

For $(\sigma, K)_D \in D_n$ we let

$$ned_D(\sigma, K) := \sum_{i \in K} 2i\varepsilon_i(\sigma) + \sum_{i \in \mathcal{C}_{n-1}(K)} i. \tag{7}$$

For example, if $\beta = [-2, 4, -3, -1] = (2431, \{1, 3, 4\}) \in B_4$ then $ned_B(\beta) = 2\cdot 3 + 2 = 8$ and if $\gamma = [2, 4, -3, -1] = (2431, \{3\}) \in D_4$ then $ned_D(\gamma) = 2 \cdot 3 + 1 + 2 = 9$. The main property of ned_B is the following one.

Theorem 9. *For every $(\sigma, H) \in B_n$*

$$ned_B(\sigma, H) = fmaj(\sigma, \varphi_n(H)). \tag{8}$$

Corollary 10. *Let $n \in \mathbf{P}$. Then*

$$\sum_{\beta \in B_n} q^{ned_B(\beta)} = \sum_{\beta \in B_n} q^{fmaj(\beta)}.$$

The following statistic is fundamental for this work and its definition is naturally suggested by Theorem 9. We will see in §4 and in §5 that it's Mahonian and, moreover, that it plays the same algebraic role for D_n, as maj for S_n and $fmaj$ for B_n, in the Hilbert series of DIA and TIA defined in §2.
Let $\gamma \in D_n$, we define

$$Dmaj(\gamma) := fmaj([\gamma_1, \ldots, \gamma_{n-1}, |\gamma_n|]).$$

For example, if $\gamma = [-2, 3, -1, -5, -4]$, then $Dmaj(\gamma) = fmaj([-2, 3, -1, -5, 4]) = 2 \cdot 2 + 3 = 7$. Note that $Dmaj((\sigma, K)_D) = fmaj((\sigma, K))$.
The next result follows immediately from Theorem 9.

Corollary 11. *Let $(\sigma, K) \in D_n$, then*

$$ned_D(\sigma, K) = Dmaj(\sigma, \varphi_{n-1}(K)).$$

4. Combinatorial Properties of $Dmaj$

In this section we show that the $Dmaj$ satisfies the fundamental combinatorial properties of a candidate for a major index for D_n. In fact, one can show through an explicit bijection that the D-major index is equidistributed with the Mahonian statistic D-flag major index $fmaj_D$ defined in [4]. This fact clearly implies the following proposition.

Proposition 12. *Let $n \in \mathbf{P}$. Then*

$$\sum_{\gamma \in D_n} q^{Dmaj(\gamma)} = \sum_{\gamma \in D_n} q^{\ell(\gamma)}.$$

For $\gamma \in D_n$ we define the *D-descent number* by

$$Ddes(\gamma) := fdes([\gamma_1, \ldots, \gamma_{n-1}, |\gamma_n|]).$$

For example, if $\gamma = [-2, -1, 4, 5, -6, -3] \in D_6$ then $Ddes(\gamma) = fdes([-2, -1, 4, 5, -6, 3]) = 2 \cdot 2 + 1 = 5$.
The pair of statistics $Dmaj$ and $Ddes$ satisfy a generalization of Carlitz's identity.

Theorem 13. *Let $n \in \mathbf{P}$. Then*

$$\sum_{r \geq 0} [r+1]_q^n t^r = \frac{\sum_{\gamma \in D_n} t^{Ddes(\gamma)} q^{Dmaj(\gamma)}}{(1-t)(1-tq^n)\prod_{i=1}^{n-1}(1-t^2q^{2i})}$$

in $\mathbf{Z}[q][[t]]$.

5. The Main Results

In this section we use the combinatorial tools developed in §3 to find a closed formula for $\mathcal{Z}_{D_n}(\bar{q})$ in terms of the statistic $Dmaj$.

5.1. **A Basis for TIA and DIA for D_n.** Let $W = D_n$. The tensor invariant algebra TIA is $(\mathbf{P}_n^{D_n})^{\otimes t}$, and $\mathbf{P}_n^{D_n}$ is freely generated (as an algebra) by the $n-1$ elementary symmetric functions $e_j(x_1^2, \ldots, x_n^2)$ for $j \in [n-1]$ and the monomial $x_1 \cdots x_n$ (see, e.g., [16, §3]). Hence

$$F_T(\bar{q}) = \prod_{i=1}^{t} \Big(\frac{1}{(1-q_i^n)} \prod_{j=1}^{n-1} \frac{1}{(1-q_i^{2j})} \Big).$$

A linear basis for $\mathbf{P}_n^{\otimes t}$ consists of all tensor monomials

$$\bar{x}^f := \bigotimes_{i=1}^{t} \prod_{j=1}^{n} x_j^{f_i(j)}$$

where $f = (f_1, \ldots, f_t) \in \mathcal{F}_{n,t}$. The canonical projection $\pi : \mathbf{P}_n^{\otimes t} \to \mathrm{DIA}$ is defined by

$$\pi(\bar{p}) := \sum_{\gamma \in D_n} \varphi_D(\gamma)(\bar{p})$$

so that

$$\mathrm{DIA} = < \{\pi(\bar{x}^f) : \ f \in \mathcal{F}_{n,t}\} > .$$

Lemma 9. *For $f \in \mathcal{F}_{n,t}$,*

$$\pi(\bar{x}^f) \neq 0 \Longleftrightarrow f \in \mathcal{F}_{n,t}^e \cup \mathcal{F}_{n,t}^o,$$

where $\mathcal{F}_{n,t}^e$ and $\mathcal{F}_{n,t}^o$ are defined in §2.3.

Clearly $\mathcal{B}_{n,t}^e \cup \mathcal{B}_{n,t}^o$ is a complete system of representatives for the orbits of all $f \in \mathcal{F}_{n,t}^e \cup \mathcal{F}_{n,t}^o$, under the action of the symmetric group. Hence we have

Proposition 14. *The set*

$$\{\pi(\bar{x}^f) : f \in \mathcal{B}_{n,t}^e \cup \mathcal{B}_{n,t}^o\}$$

is a homogeneous basis for DIA.

Corollary 15. *The Hilbert series for* DIA *is*

$$F_D(\bar{q}) = \sum_{f \in \mathcal{B}_{n,t}^e \cup \mathcal{B}_{n,t}^o} q_1^{|f_1|} \cdots q_t^{|f_t|}.$$

5.2. **The Polynomial $\mathcal{Z}_{D_n}(q_1, q_2)$.** We define an involution $\alpha : D_n \to D_n$ by

$$(9) \qquad (\sigma, K) \quad \mapsto \quad (\sigma^{-1}, p(\sigma(K))),$$

where p is the projection defined in §3.1.
For example, $\alpha(4213, \{1,3\}) = (3241, p(\{1,4\}) = (3241, \{2,3\})$.

We are now ready to state the following

Theorem 16. *Let $n \in \mathbf{N}$. Then*

$$\mathcal{Z}_{D_n}(q_1, q_2) = \sum_{\gamma \in D_n} q_1^{Dmaj(\gamma)} q_2^{Dmaj(\alpha(\gamma))}.$$

Example. Consider the case $n = 2$. One may easily check that $\alpha(\gamma) = \gamma$ for all $\gamma \in D_2$ and hence

$$\begin{aligned} \mathcal{Z}_{D_2}(q_1, q_2) &= (q_1q_2)^{Dmaj(1,2)} + (q_1q_2)^{Dmaj(2,1)} + (q_1q_2)^{Dmaj(-1,-2)} + (q_1q_2)^{Dmaj(-2,-1)} \\ &= (1 + q_1q_2)^2. \end{aligned}$$

We denote by ι the inversion in D_n so that $\iota(\gamma) := \gamma^{-1}$. The next lemma says that it is possible to "substitute" α with ι in Theorem 16.

Lemma 10. *α and ι are conjugate in $S(D_n)$.*

Corollary 17. *There exists a function $M : D_n \to \mathbf{N}$, equidistributed with length, such that*

$$\mathcal{Z}_{D_n}(q_1, q_2) = \sum_{\gamma \in D_n} q_1^{M(\gamma)} q_2^{M(\gamma^{-1})}.$$

5.3. **The Polynomial $\mathcal{Z}_{D_n}(\bar{q})$.** In this section we provide an explicit simple formula for the polynomial $\mathcal{Z}_{D_n}(\bar{q})$ in terms of the $Dmaj$.

We denote by $\alpha : D_n^{t-1} \to D_n$ the map

$$((\sigma_1, K_1), \ldots, (\sigma_{t-1}, K_{t-1})) \quad \mapsto \quad \left((\sigma_{t-1} \cdots \sigma_1)^{-1}, p\left(\sigma_{t-1} \cdots \sigma_1(K_1) \triangle \cdots \triangle \sigma_{t-1}(K_{t-1})\right)\right).$$

For example,

$$\begin{aligned} \alpha\left((4231, \{1,3\}), (2143, \{3\})\right) &= (2413, p(3142(\{1,3\}) \triangle 2143(\{3\})) \\ &= (2413, p(\{3\})) = (2413, \{3\}). \end{aligned}$$

Note that this is consistent with the definition of α given in (9).

Theorem 18. *Let $n \in \mathbf{N}$. Then*

$$\mathcal{Z}_{D_n}(\bar{q}) = \sum_{\gamma_1, \ldots, \gamma_t} \prod_{i=1}^{t} q_i^{Dmaj(\gamma_i)},$$

where the sum runs through all $\gamma_1, \ldots, \gamma_t \in D_n$ such that $\gamma_t = \alpha(\gamma_1, \ldots, \gamma_{t-1})$.

5.4. **The case n odd.** If n is odd the formula appearing in Theorem 18 can be slightly improved. In particular we define one more statistic, $Dmaj^o$, that allows us to obtain a formula for $\mathcal{Z}_{D_n}(\bar{q})$ similar to the corresponding ones for S_n and B_n appearing in Theorem 5 and Theorem 6. Consider the set $S_n \times 2^{[n-1]}$ with the binary operation

$$(\sigma, H) * (\tau, K) := \left(\sigma\tau, p(K \triangle \tau^{-1}(H))\right).$$

Proposition 19. *Let $n > 1$. Then $\Delta_n = \left(S_n \times 2^{[n-1]}, *\right)$ is a group and it is isomorphic to D_n if and only if n is odd.*

Let $n \in \mathbf{N}$ be odd. Then we let

$$Dmaj^o := Dmaj \circ \Phi,$$

where we identify Δ_n with D_n through the pair notation and $\Phi : D_n \to \Delta_n$ is defined by

$$\gamma \quad \mapsto \quad (|\gamma|, p(Neg(\gamma))).$$

Corollary 20. *Let $n \in \mathbf{N}$. Then*

$$\mathcal{Z}_{D_{2n+1}}(\bar{q}) = \sum_{\gamma_1, \ldots, \gamma_t} \prod_{i=1}^{t} q_i^{Dmaj^o(\gamma_i)}.$$

where the sum is over all $\gamma_1, \ldots, \gamma_t \in D_{2n+1}$ such that $\gamma_t \cdots \gamma_1 = e$.

Proposition 19 implies that, if n is even, there is no $\Phi \in S(D_n)$ such that $\alpha(\Phi(\gamma_1), \ldots, \Phi(\gamma_2)) = \Phi(\gamma_t \cdots \gamma_1)^{-1}$ that would imply the corresponding result of Corollary 20. Nevertheless, we know that this result holds for $t = 2$ (Corollary 17) but we haven't been able to define a nice statistic, $Dmaj^e$, that works in this case, or to understand if it exists for $t > 2$. We therefore propose the following

Problem. Let $n \in \mathbf{N}$ be even. Is there a statistic $Dmaj^e : D_n \to \mathbf{N}$, necessarily equidistributed with length on D_n, such that

$$\mathcal{Z}_{D_n}(\bar{q}) = \sum_{\gamma_1, \ldots, \gamma_t} \prod_{i=1}^{t} q_i^{Dmaj^e(\gamma_i)}$$

with $\gamma_t \cdots \gamma_1 = e$?

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References

[1] R. M. Adin and Y. Roichman, *The Flag Major Index and Group Actions on Polynomial Rings*, European J. Combin., **22** (2001), 431-446.

[2] R. M. Adin, F. Brenti and Y. Roichman, *Descent Numbers and Major Indices for the Hyperoctahedral Group*, Adv. in Appl. Math., **27** (2001), 210-224.

[3] R. M. Adin, F. Brenti and Y. Roichman, *Descent Representations and Multivariate Statistics*, Trans. Amer. Math. Soc., to appear.

[4] R. Biagioli, *Major and Descent Statistics for the even-signed Permutation Group*, Adv. in Appl. Math., to appear.

[5] R. Biagioli and F. Caselli, *Invariant algebras and major indices for classical Weyl groups*, Proc. Lond. Math. Soc., to appear.

[6] R. Biagioli and F. Caselli, *A monomial basis for the coinvariant algebra of type D*, pre-print.

[7] F. Brenti, *q-Eulerian Polynomials arising from Coxeter Groups*, European J. Combin., **15** (1994), 417-441.

[8] L. Carlitz, *A Combinatorial Property of q-Eulerian Numbers*, Amer. Math. Montly, **82** (1975), 51-54.

[9] R. J. Clarke and D. Foata, *Eulerian Calculus. I. Univariable Statistics*, European J. Combin., **15** (1994), 345-362.

[10] D. Foata, *On the Netto Inversion Number of a Sequence*, Proc. Amer. Math. Soc.,**19** (1968), 236-240.

[11] D. Foata, *Distributions eulériennes et mahoniennes sur le groupes des permutations*, in Higher Combinatorics, 27-49, D. Reidel, Amsterdam 1997 (Proc. NATO Adv. Study Inst.).

[12] A. Garsia, *On the "maj" and "inv" q-analogues of Eulerian polynomials*, Linear and Multilinear Algebra, **8** (1979/80), 21-34.

[13] A. Garsia and I. Gessel, *Permutation Statistics and Partitions*, Adv. Math., **31** (1979), 288-305.

[14] A. Garsia and D. Stanton, *Group actions of Stanley-Reisner rings and invariants of permutation groups*, Adv. Math., **51** (1984), 107-201.

[15] B. Gordon, *Two Theorems on Multipartite Partitions*, J. London Math. Soc., **38** (1963), 459-464.

[16] J. E. Humphreys, *Reflection Groups and Coxeter Groups*, Cambridge Stud. Adv. Math., no.29, Cambridge Univ. Press, Cambridge, 1990.

[17] P. A. MacMahon, *Combinatory Analysis*, vol.1, Cambridge Univ. Press, London, 1915.

[18] V. Reiner, *Signed Permutation Statistics*, European J. Combin., **14** (1993), 553-567.

[19] E. Steingrimsson, *Permutation Statistics of Indexed Permutations*, European J. Combin., **15** (1994), 187-205.

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