# QUASI-INVARIANT AND SUPER-COINVARIANT POLYNOMIALS FOR THE GENERALIZED SYMMETRIC GROUP

J.-C. AVAL

ABSTRACT. The aim of this work is to extend the study of super-coinvariant polynomials, introduced in [2, 3], to the case of the generalized symmetric group $G_{n,m}$, defined as the wreath product $C_m \wr \mathcal{S}_n$ of the symmetric group by the cyclic group. We define a quasi-symmetrizing action of $G_{n,m}$ on $\mathbb{Q}[x_1,\ldots,x_n]$, analogous to those defined in [12] in the case of $\mathcal{S}_n$. The polynomials invariant under this action are called quasi-invariant, and we define super-coinvariant polynomials as polynomials orthogonal, with respect to a given scalar product, to the quasi-invariant polynomials with no constant term. Our main result is the description of a Gröbner basis for the ideal generated by quasi-invariant polynomials, from which we dedece that the dimension of the space of super-coinvariant polynomials is equal to $m^n\, C_n$ where $C_n$ is the $n$-th Catalan number.

RÉSUMÉ. Le but de ce travail est d'étendre l'étude des polynômes super-coinvariants (définis dans [2]), au cas du groupe symétrique généralisé $G_{n,m}$, défini comme le produit en couronne $C_m \wr \mathcal{S}_n$ du groupe symétrique par le groupe cyclique. Nous définissons ici une action quasi-symétrisante de $G_{n,m}$ sur $\mathbb{Q}[x_1,\ldots,x_n]$, analogue à celle définie dans [12] dans le cas de $\mathcal{S}_n$. Les polynômes invariants sous cette action sont dits quasi-invariants, et les polynômes super-coinvariants sont les polynômes orthogonaux aux polynômes quasi-invariants sans terme constant (pour un certain produit scalaire). Notre résultat principal est l'obtention d'une base de Gröbner pour l'idéal engendré par les polynômes quasi-invariants. Nous en déduisons alors que la dimension de l'espace des polynômes super-coinvariants est $m^n\, C_n$ où $C_n$ est le $n$-ième nombre de Catalan.

## 1. INTRODUCTION

Let $X$ denote the alphabet in $n$ variables $(x_1,\ldots,x_n)$ and $\mathbb{C}[X]$ denote the space of polynomials with complex coefficients in the alphabet $X$. Let $G_{n,m} = C_m \wr \mathcal{S}_n$ denote the wreath product of the symmetric group $\mathcal{S}_n$ by the cyclic group $C_m$. This group is sometimes known as the *generalized symmetric group* (*cf.* [17]). It may be seen as the group of $n \times n$ matrices in which each row and each column has exactly one non-zero entry (pseudo-permutation matrices), and such that the non-zero entries are $m$-th roots of unity. The order of $G_{n,m}$ is $m^n\, n!$. When $m = 1$, $G_{n,m}$ reduces to the symmetric group $\mathcal{S}_n$, and when $m = 2$, $G_{n,m}$ is the hyperoctahedral group $B_n$, *i.e.* the group of signed permutations, which is the Weyl group of type $B$ (see [14]

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for example for further details). The group $G_{n,m}$ acts classically on $\mathbb{C}[X]$ by the rule

$$\forall g \in G_{n,m},\ \forall P \in \mathbb{C}[X],\ g.P(X) = P(X.{}^tg), \tag{1.1}$$

where $g$ is the transpose of the matrix $g$ and $X$ is considered as a row vector. Let

$$Inv_{n,m} = \{P \in \mathbb{C}[X] \ / \ \forall g \in G_{n,m},\ g.P = P\}$$

denote the set of $G_{n,m}$-invariant polynomials. Let us denote by $Inv^+_{n,m}$ the set of such polynomials with no constant term. We consider the following scalar product on $\mathbb{C}[X]$:

$$\langle P, Q\rangle = P(\partial X)Q(X) \mid_{X=0} \tag{1.2}$$

where $\partial X$ stands for $(\partial x_1, \ldots, \partial x_n)$ and $X = 0$ stands for $x_1 = \cdots = x_n = 0$. The space of $G_{n,m}$-coinvariant polynomials is then defined by

$$\begin{aligned} Cov_{n,m} &= \{P \in \mathbb{C}[X] \ / \ \forall Q \in Inv_{n,m},\ Q(\partial X)P = 0\} \\ &= \langle Inv^+_{n,m}\rangle^{\perp} \ \simeq \ \mathbb{C}[X]/\langle Inv^+_{n,m}\rangle \end{aligned}$$

where $\langle S\rangle$ denotes the ideal generated by a subset $S$ of $\mathbb{C}[X]$.

A classical result of Chevalley [6] states the following equality:

$$\dim Cov_{n,m} = |G_{n,m}| = m^n\, n! \tag{1.3}$$

which reduces when $m = 1$ to the theorem of Artin [1] that the dimension of the harmonic space $\mathbf{H}_n = Cov_{n,1}$ (*cf.* [9]) is $n!$.

Our aim is to give an analogous result in the case of quasi-symmetrizing action. The ring *Qsym* of quasi-symmetric functions was introduced by Gessel [11] as a source of generating functions for $P$-partitions [18] and appears in more and more combinatorial contexts [5, 18, 19]. Malvenuto and Reutenauer [16] proved a graded Hopf duality between *QSym* and the Solomon descent algebras and Gelfand *et. al.* [10] defined the graded Hopf algebra *NC* of non-commutative symmetric functions and identified it with the Solomon descent algebra.

In [2, 3], Aval *et. al.* investigated the space $\mathbf{SH}_n$ of super-coinvariant polynomials for the symmetric group, defined as the orthogonal (with respect to (1.2)) of the ideal generated by quasi-symmetric polynomials with no constant term, and proved that its dimension as a vector space equals the $n$-th Catalan number:

$$\dim \mathbf{SH}_n = C_n = \frac{1}{n+1}\binom{2n}{n}. \tag{1.4}$$

Our main result is a generalization of the previous equation in the case of super-coinvariant polynomials for the group $G_{n,m}$.

In Section 2, we define and study a "quasi-symmetrizing" action of $G_{n,m}$ on $\mathbb{C}[X]$. We also introduce invariant polynomials under this action, which are called quasi-invariant, and polynomials orthogonal to quasi-invariant polynomials, which are called super-coinvariant. The Section 3 is devoted to the proof of our main result (Theorem 2.4), which gives the dimension of the space $SCov_{n,m}$ of super-coinvariant polynomials for $G_{n,m}$: we construct an explicit basis for $SCov_{n,m}$ from which we deduce its Hilbert series.

## 2. A quasi-symmetrizing action of $G_{n,m}$

We use vector notation for monomials. More precisely, for $\nu = (\nu_1, \dots, \nu_n) \in \mathbb{N}^n$, we denote $X^\nu$ the monomial

$$x_1^{\nu_1} x_2^{\nu_2} \cdots x_n^{\nu_n}. \tag{2.1}$$

For a polynomial $P \in \mathbb{Q}[X]$, we further denote $[X^\nu]\, P(X)$ as the coefficient of the monomial $X^\nu$ in $P(X)$.

Our first task is to define a quasi-symmetrizing action of the group $G_{n,m}$ on $\mathbb{C}[X]$, which reduces to the quasi-symmetrizing action of Hivert (*cf.* [12]) in the case $n = 1$. This is done as follows. Let $A \subset X$ be a subalphabet of $X$ with $l$ variables and $K = (k_1, \dots, k_l)$ be a vector of positive ($> 0$) integers. If $B$ is a vector whose entries are distinct variables $x_i$ multiplied by roots of unity, the vector $(B)_<$ is obtained by ordering the elements in $B$ with respect to the variable order. Now the quasi-symmetrizing action of $g \in G_{n,m}$ is given by

$$g \bullet A^K = w(g)^{c(K)} (A.{}^t|g|)_<{}^K \tag{2.2}$$

where $w(g)$ is the weight of $g$, *i.e.* the product of its non-zero entries, $|g|$ is the matrix obtained by taking the modules of the entries of $g$, and the oefficient $c(K)$ is defined as follows:

$$c(K) = \begin{cases} 0 & \text{if } \forall i,\ k_i \equiv 0\ [m] \\ 1 & \text{if not.} \end{cases}$$

**Example 2.1.** If $m = 3$ and $n = 3$, and we denote by $j$ the complex number $j = e^{\frac{2i\pi}{3}}$, then for example

$$\begin{aligned} &\begin{pmatrix} 0 & 0 & j \\ 1 & 0 & 0 \\ 0 & j & 0 \end{pmatrix} \bullet (x_1^2\, x_2) \\ = \ & (j^2)^1 \left[ \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} . (x_1, x_2) \right]_<^{(2,1)} \\ = \ & j^2 (x_3, x_1)_<{}^{(2,1)} \\ = \ & j^2 (x_1, x_3)^{(2,1)} \\ = \ & j^2\, x_1^2\, x_3. \end{aligned}$$

It is clear that this defines an action of the generalized symmetric group $G_{n,m}$ on $\mathbb{C}[X]$, which reduces to Hivert's quasi-symmetrizing action (*cf.* [12], Proposition 3.4) in the case $m = 1$.

Let us now study its invariant and coinvariant polynomials. We need to recall some definitions.

A *composition* $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k)$ of a positive integer $d$ is an ordered list of positive integers ($> 0$) whose sum is $d$. For a *vector* $\nu \in \mathbb{N}^n$, let $c(\nu)$ represent the composition obtained by erasing zeros (if any) in $\nu$. A polynomial $P \in \mathbb{Q}[X]$ is said to be *quasi-symmetric* if and only if, for any $\nu$ and $\mu$ in $\mathbb{N}^n$, we have

$$[X^\nu]P(X) = [X^\mu]P(X)$$

whenever $c(\nu) = c(\mu)$. The space of quasi-symmetric polynomials in $n$ variables is denoted by $Qsym_n$.

The polynomials invariant under the action (2.2) of $G_{n,m}$ are said to be *quasi-invariant* and the space of quasi-invariant polynomials is denoted by $QInv_{n,m}$, *i.e.*

$$P \in QInv_{n,m} \Leftrightarrow \forall g \in G_{n,m},\ g \bullet P = P.$$

Let us recall (*cf.* [12], Proposition 3.15) that $QInv_{n,1} = QSym_n$. The following proposition gives a characterization of $QInv_{n,m}$.

**Proposition 2.2.** *One has*

$$P \in QInv_{n,m} \Leftrightarrow \exists Q \in QSym_n \ /\ P(X) = Q(X^m)$$

*where* $Q(X^m) = Q(x_1^m, \ldots, x_n^m)$.

*Proof.* Let $P$ be an element of $QInv_{n,m}$. Let us denote by $\zeta$ the $m$-th root of unity $\zeta = e^{\frac{2i\pi}{m}}$ and by $g_j$ the element of $G_{n,m}$ whose matrix is

$$\begin{pmatrix} \zeta & & 0 & \\ & 1 & & \\ 0 & & \ddots & \\ & & & 1 \end{pmatrix}$$

with the $\zeta$ in place $j$. Then we observe that the identities

$$\forall j = 1, \ldots, n,\ \frac{1}{m}(P + g_j \bullet P + g_j^2 \bullet P + \cdots + g_j^{m-1} \bullet P) = P$$

imply that every exponents appearing in $P$ are multiples of $m$. Thus there exists a polynomial $Q \in \mathbb{C}[X]$ such that $P(X) = Q(X^m)$. To conclude, we note that $\mathcal{S}_n \subset G_{n,m}$ implies that $P$ is quasi-symmetric, whence $Q$ is also quasi-symmetric.

The reverse implication is obvious. □

Let us now define *super-coinvariant* polynomials:

$$\begin{aligned} SCov_{n,m} &= \{P \in \mathbb{C}[X] \ /\ \forall Q \in QInv_{n,m},\ Q(\partial X)P = 0\} \\ &= \langle QInv_{n,m}^+\rangle^\perp \ \simeq\ \mathbb{C}[X]/\langle QInv_{n,m}^+\rangle \end{aligned}$$

with the scalar product defined in (1.2). This is the natural analogous to $Cov_n$ in the case of quasi-symmetrizing actions and $SCov_{n,m}$ reduces to the space of super-harmonic polynomials $\mathbf{SH}_n$ (*cf.* [3]) when $m = 1$.

**Remark 2.3.** It is clear that any polynomial invariant under (2.2) is also invariant under (1.1), *i.e.* $Inv_{n,m} \subset QInv_{n,m}$. By taking the orthogonal, this implies that $SCov_{n,m} \subset Cov_{n,m}$. These observations somewhat justify the terminology.

Our main result is the following theorem which is a generalization of equality (1.4).

**Theorem 2.4.** *The dimension of the space $Scov_{n,m}$ is given by*

$$\dim SCov_{n,m} = m^n\, C_n = m^n\, \frac{1}{n+1}\binom{2n}{n}. \tag{2.3}$$

**Remark 2.5.** In the case of the hyperoctahedral group $B_n = G_{n,2}$, C.-O. Chow [7] defined a class $BQSym(x_0, X)$ of quasi-symmetric functions of type $B$ in the alphabet $(x_0, X)$. His approach is quite different from ours. In particular, one has the equality:

$$BQSYm(x_0, X) = QSym(X) + QSym(x_0, X).$$

In the study of the coinvariant polynomials, it is not difficult to prove that the quotient $\mathbb{C}[x_0, X]/\langle BQSym^+\rangle$ is isomorphic to the quotient $\mathbb{C}[X]/\langle QSym^+\rangle$ studied in [3]. To see this, we observe that if $\mathcal{G}$ is the Gröbner basis of $\langle QSym^+\rangle$ constructed in [3] (see also the next section), then the set $\{x_0, \mathcal{G}\}$ is a Gröbner basis (any syzygy is reducible thanks to Buchberger's first criterion, *cf.* [8]).

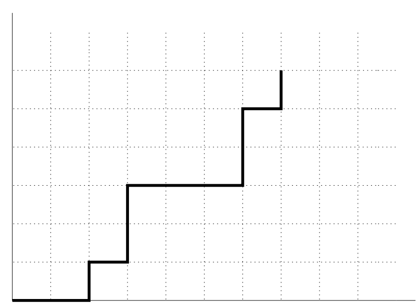
The next section is devoted to give a proof of Theorem 2.4 by constructing an explicit basis for the quotient $\mathbb{C}[X]/\langle QInv^+_{n,m}\rangle$.

## 3. Proof of the main theorem

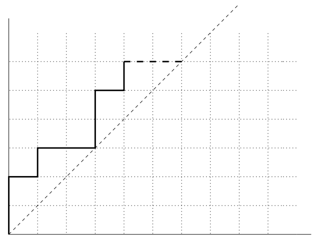
Our task is here to construct an explicit monomial basis for the quotient space $\mathbb{C}[X]/\langle QInv^+_{n,m}\rangle$. Let us first recall (*cf.* [3]) the following bijection which associates to any vector $\nu \in \mathbb{N}^n$ a path $\pi(\nu)$ in the $\mathbb{N}\times\mathbb{N}$ plane with steps going north or east as follows. If $\nu = (\nu_1, \dots, \nu_n)$, the path $\pi(\nu)$ is

$$(0,0) \to (\nu_1, 0) \to (\nu_1, 1) \to (\nu_1+\nu_2, 1) \to (\nu_1+\nu_2, 2) \to \cdots$$
$$\to (\nu_1+\cdots+\nu_n, n-1) \to (\nu_1+\cdots+\nu_n, n).$$

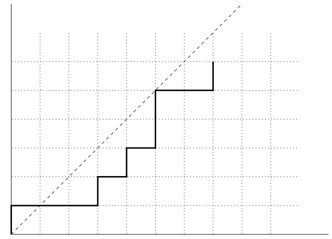
For example the path associated to $\nu = (2,1,0,3,0,1)$ is

$\pi(\nu) =$ 

We distinguish two kinds of paths, thus two kinds of vectors, with respect to their "behavior" regarding the diagonal $y = x$. If the path remains above the diagonal, we call it a *Dyck path*, and say that the corresponding vector is *Dyck*. If not, we say that the path (or equivalently the associated vector) is *transdiagonal*. For example $\eta = (0,0,1,2,0,1)$ is Dyck and $\varepsilon = (0,3,1,1,0,2)$ is transdiagonal.



η



ε

We then have the following result which generalizes Theorem 4.1 of [3] and which clearly implies the Theorem 2.4.

**Theorem 3.1.** *The set of monomials*

$$\mathcal{B}_{n,m} = \{(X_n)^{m\,\eta+\alpha} /\ \pi(\eta) \text{ is a Dyck path},\ 0 \leq \alpha_i < m\}$$

*is a basis for the quotient* $\mathbb{C}[X_n]/\langle QInv^+_{n,m}\rangle$.

To prove this result, the goal is here to construct a Gröbner basis for the ideal $\mathcal{J}_{n,m} = \langle QInv^+_{n,m}\rangle$. We shall use results of [2, 3].

Recall that the *lexicographic order* on monomials is

$$X^\nu >_{\text{lex}} X^\mu \qquad \text{iff} \qquad \nu >_{\text{lex}} \mu, \tag{3.1}$$

if and only if the first non-zero part of the vector $\nu - \mu$ is positive.

For any subset $\mathcal{S}$ of $\mathbb{Q}[X]$ and for any positive integer $m$, let us introduce $\mathcal{S}^m = \{P(X^m)\ ,\ P \in \mathcal{S}\}$. If we denote by $G(I)$ the unique reduced monic Gröbner basis (*cf.* [8]) of an ideal $I$, then the simple but crucial fact in our context is the following.

**Proposition 3.2.** *With the previous notations,*

$$G(\langle \mathcal{S}^m \rangle) = G(\langle \mathcal{S} \rangle)^m. \tag{3.2}$$

*Proof.* This is a direct consequence of Buchberger's criterion. Indeed, if for every pair $g$, $g'$ in $G(\langle S\rangle)$, the syzygy

$$S(g, g')$$

reduces to zero, then the syzygy

$$S(g(X^m), g'(X^m))$$

also reduces to zero in $G(\langle S^m\rangle)$ by exactly the same computation. □

Let us recall that in [2] is constructed a family $\mathcal{G}$ of polynomials $G_\varepsilon$ indexed by transdiagonal vectors $\varepsilon$. This family is constructed by using recursive relations of the fundamental quasi-symmetric functions and one of its property (*cf.* [2]) says that the leading monomial of $G_\varepsilon$ is: $LM(G_\varepsilon) = X^\varepsilon$. Since $\mathcal{G}$ is a Gröbner basis of $\mathcal{J}_{n,1}$, the following result is a consequence of Propositions 2.2 and 3.2.

**Proposition 3.3.** *The set* $\mathcal{G}^m$ *is a Gröbner basis of the ideal* $\mathcal{J}_{n,m}$.

To conclude the proof of Theorem 3.1, it is sufficient to observe that the monomials not divisible by a leading monomial of an element of $\mathcal{G}^m$, *i.e.* by a $X^{m\varepsilon}$ for $\varepsilon$ transdiagonal, are precisely the monomials appearing in the set $\mathcal{B}_{n,m}$.

As a corollary of Theorem 3.1, one gets an explicit formula for the Hilbert series of $SCov_{n,m}$. For $k \in \mathbb{N}$, let $SCov_{n,m}^{(k)}$ denote the projection

$$SCov_{n,m}^{(k)} = SCov_{n,m} \cap \mathbb{Q}^{(k)}[X] \tag{3.3}$$

where $\mathbb{Q}^{(k)}[X]$ is the vector space of homogeneous polynomials of degree $k$ together with zero.

Let us denote by $F_{n,m}(t)$ the Hilbert series of $SCov_{n,m}$, *i.e.*

$$F_{n,m}(t) = \sum_{k\geq 0} \dim SCov_{n,m}^{(k)}\, t^k. \tag{3.4}$$

Let us recall that in [3] is given an explicit formula for $F_{n,1}$:

$$F_{n,1}(t) = F_n(t) = \sum_{k=0}^{n-1} \frac{n-k}{n+k} \binom{n+k}{k} t^k \tag{3.5}$$

using the number of Dyck paths with a given number of factors (*cf.* [13]).

The Theorem 3.1 then implies the

**Corollary 3.4.** *With the notations of (3.5), the Hilbert series of $SCov_{n,m}$ is given by*

$$F_{n,m}(t) = \frac{1-t^m}{1-t} F_n(t^m)$$

*from which one deduces the close formula*

$$\sum_n F_{n,m}(t)\, x^n = \frac{(1-t) - \sqrt{(1-t)(1-t-4t^m x(1-t^m))} - 2x(1-t^m)}{(1-t)(2t^m-1) - x(1-t^m)}.$$

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(Jean-Christophe Aval) LaBRI, Université Bordeaux 1, 351 cours de la Libération, 33405 Talence cedex, FRANCE

*E-mail address*: aval@labri.fr