Symmetric Functions in Noncommuting Variables

by

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Abstract

Consider the algebra $\mathbb{Q}\langle\langle x_1, x_2, \ldots\rangle\rangle$ of formal power series in countably many noncommuting variables over the rationals. The subalgebra $\Pi(x_1, x_2, \ldots)$ of symmetric functions in noncommuting variables consists of all elements invariant under permutation of the variables and of bounded degree. We develop a theory of such functions analogous to the ordinary theory of symmetric functions. In particular, there are analogs of the monomial, power sum, elementary, complete homogeneous, and Schur symmetric functions. We also investigate their properties.

Résumé

Soit $\mathbb{Q}\langle\langle x_1, x_2, \ldots\rangle\rangle$ l'algèbre des séries formelles à coefficients rationelles en variables noncommutatives. La soualgèbre $\Pi(x_1, x_2, \ldots)$ des fonctions symétriques consiste de tous les éléments qui sont invariables sous permutation des variables et de degré borné. Nous développons une théorie des fonctions en variables noncommutatives parallèle à la théorie des fonctions symétriques ordinaires. En particulier, il y a des analogues des fonctions symétriques monômes, sommes de puissances, élémentaires, homogènes complètes, et de Schur. Nous recherchons leurs propriétés aussi.

1 Introduction

Let $\mathbb{Q}[[x_1, x_2, \ldots]] = \mathbb{Q}[[\mathbf{x}]]$ be the algebra of formal power series in a countably infinite set of commuting variables x_i with coefficients in the rational numbers. There is an action of elements g of the symmetric group $\mathfrak{S}_n$ on this algebra given by

$$gf(x_1, x_2, \ldots) = f(x_{g1}, x_{g2}, \ldots) \tag{1}$$

where $gi = i$ for $i > n$. We say that f is *symmetric* if it is invariant under the action of $\mathfrak{S}_n$ for all $n \geq 1$. The *algebra of symmetric functions*, $\Lambda = \Lambda(\mathbf{x})$, consists of all symmetric f that are also of bounded degree. This algebra has a long, venerable history and is also of interest in combinatorics, algebraic geometry, and representation theory. For more information in this regard, see the texts of Fulton [4], Macdonald [13], Sagan [20], or Stanley [25].

Now consider the algebra $\mathbb{Q}\langle\langle x_1, x_2, \ldots\rangle\rangle$ where the x_i do not commute. Define the *algebra of symmetric functions in noncommuting variables*, $\Pi = \Pi(\mathbf{x})$, to be the subalgebra consisting of all elements invariant under the action (1) and of bounded degree. (This is not to be confused with the algebra of noncommutative symmetric functions of Gelfand et. al. [8] or the partially commutative symmetric functions studied by Lascoux and Schützenberger [11] as well as by Fomin and Greene [3].) This algebra was first studied by M. C. Wolf [28] in 1936. Her goal was to provide an analogue of the fundamental theorem of symmetric functions in this context.

The concept then lay dormant for over 30 years until Bergman and Cohn generalized Wolf's result [1]. Most recently, Gebhard and Sagan [7] revived these ideas as a tool for studying Stanley's chromatic symmetric function [23, 24].

The aim of this work is to give the first systematic study of the properties of $\Pi(\mathbf{x})$. We define analogues of all of the standard bases for $\Lambda(\mathbf{x})$, including the Schur functions. We then study the corresponding basis change equations and inner products. For the Schur functions, which are defined combinatorially via tableaux, we derive analogues Jacobi-Trudi determinants and Robinson-Schensted-Knuth correspondence.

2 Basic definitions

If n is a positive integer, then we define $[n] = \{1, 2, \ldots, n\}$. We will let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ denote an *integer partition of* n which is just a weakly decreasing sequence of positive integers, called *parts*, summing to n. In this case we write $\lambda \vdash n$ and denote the number of parts or *length* of λ by $l = l(\lambda)$. We will also use the notation

$$\lambda = (1^{l_1}, 2^{l_2}, \ldots, n^{l_n}) \tag{2}$$

to mean that i appears in λ with multiplicity l_i, $1 \leq i \leq n$. The bases of the symmetric function algebra $\Lambda(\mathbf{x})$ are indexed by partitions. We use the notation m_λ, p_λ, e_λ, and h_λ for the elements of the monomial, power sum, elementary, and complete homogeneous symmetric functions bases, respectively.

Analogously, define a *set partition* π *of* $[n]$ to be a family of sets, called *blocks*, whose disjoint union is $[n]$. Here we write $\pi = B_1/B_2/\ldots/B_l \vdash [n]$ where the B_i are the blocks and also define *length* $l(\pi)$ as the number of blocks. There is a natural mapping from set partitions to integer partitions given by

$$\lambda(B_1/B_2/\ldots/B_l) = (|B_1|, |B_2|, \ldots, |B_l|)$$

where we assume that the blocks of the set partition have been listed in weakly decreasing order of size. If $\lambda(\pi) = \lambda$ then we say that the integer partition λ is the *type* of the set partition π. We also write $\pi \in \lambda$ for this relation.

To obtain analogues of the bases of Λ in this setting, take a set partition π of $[n]$. It will be helpful to think of the elements of $[n]$ as indexing the positions in a monomial $x_{i_1} x_{i_2} \cdots x_{i_n}$. This makes sense because the variables do not commute. Now define the *monomial symmetric function in noncommuting variables*, m_π, by

$$m_\pi = \sum_{i_1, i_2, \ldots, i_n} x_{i_1} x_{i_2} \cdots x_{i_n} \text{ where } i_j = i_k \text{ iff } j, k \text{ are in the same block in } \pi.$$

For example,

$$m_{13/24} = x_1x_2x_1x_2 + x_2x_1x_2x_1 + x_1x_3x_1x_3 + x_3x_1x_3x_1 + x_2x_3x_2x_3 + x_3x_2x_3x_2 + \cdots$$

Note that these functions are exactly those gotten by symmetrizing a monomial and so are invariant under the action defined previously. It follows easily that they form a basis for $\Pi(\mathbf{x})$.

We now define the *power sum function in noncommuting variables*, p_π, by

$$p_\pi = \sum_{i_1, i_2, \ldots, i_n} x_{i_1} x_{i_2} \cdots x_{i_n} \text{ where } i_j = i_k \text{ if } j, k \text{ are in the same block in } \pi.$$

To illustrate,

$$p_{13/24} = x_1x_2x_1x_2 + x_2x_1x_2x_1 + x_1^4 + x_2^4 + \cdots = m_{13/24} + m_{1234}.$$

It is not clear why these functions form a basis for $\Pi(\mathbf{x})$ or why they deserve the power sum moniker. We will establish the former fact in the next section and the latter one now.

Define the *elementary symmetric functions in noncommuting variables* to be

$$e_\pi = \sum_{i_1,i_2,\ldots,i_n} x_{i_1}x_{i_2}\cdots x_{i_n} \text{ where } i_j \neq i_k \text{ if } j,k \text{ are in the same block in } \pi.$$

By way of example

$$\begin{aligned} e_{13/24} &= x_1x_1x_2x_2 + x_2x_2x_1x_1 + x_1x_2x_2x_1 + x_2x_1x_2x_1 + \cdots \\ &= m_{12/34} + m_{14/23} + m_{12/3/4} + m_{14/2/3} + m_{1/23/4} + m_{1/2/34} + m_{1/2/3/4}. \end{aligned}$$

To define the analogue of the complete homogeneous symmetric functions, it will be useful to introduce another way of looking at the previous definitions. Any two sets D, R and a function $f : D \to R$ determine a *kernel*, $\ker f \vdash D$, whose blocks are the nonempty sets among the $f^{-1}(r)$ for $r \in R$. If our function looks like $f : [n] \to \mathbf{x}$, then there is a corresponding *monomial*

$$M_f = f(1)f(2)\cdots f(n).$$

Directly from these definitions it follows that

$$m_\pi = \sum_{\ker f = \pi} M_f.$$

Using our running example, if $\pi = 13/24$ then the functions with $\ker f = \pi$ are exactly those of the form $f(1) = f(3) = x_i$ and $f(2) = f(4) = x_j$ where $i \neq j$. This f is the one that gives rise to the monomial $M_f = x_ix_jx_ix_j$ in the sum for $m_{13/24}$.

Now define

$$h_\pi = \sum_{(f,L)} M_f \tag{3}$$

where $f : [n] \to \mathbf{x}$ and L is a linear ordering of the elements of each block of $\ker f \wedge \pi$. Continuing with our usual example partition,

$$\begin{aligned} h_{13/24} = m_{1/2/3/4} + m_{12/3/4} + 2m_{13/2/4} + m_{14/2/3} + m_{1/23/4} + 2m_{1/24/3} + m_{1/2/34} \\ + m_{12/34} + 4m_{13/24} + m_{14/23} + 2m_{123/4} + 2m_{124/3} + 2m_{134/2} + 2m_{1/234} + 4m_{1234}. \end{aligned}$$

To exhibit the relationship between the functions we have defined and the corresponding ordinary symmetric functions, consider the *forgetful* or *projection* map

$$\rho : \mathbb{Q}\langle\langle \mathbf{x} \rangle\rangle \to \mathbb{Q}[[\mathbf{x}]]$$

which merely lets the variables commute. We also need the notation

$$\lambda! = \lambda_1!\lambda_2!\cdots\lambda_l! \quad \text{and} \quad \lambda^! = l_1!l_2!\cdots l_n!$$

where the l_i are the multiplicities in (2). We extend these conventions to set partitions by letting $\pi! = \lambda(\pi)!$ and similarly for the "exponential" factorial. Note that given λ,

$$\binom{n}{\lambda} := \text{number of } \pi \text{ of type } \lambda = \frac{n!}{\lambda!\lambda^!}. \tag{4}$$

An analogue of the next proposition was proved by Doubilet [2] for his set partition analogues of ordinary symmetric functions and a similar proof can be given in the noncommuting case.

Theorem 2.1 *The images of our noncommuting functions under the forgetful map are*

$$\rho(m_\pi) = \pi^! m_{\lambda(\pi)},\ \ \rho(p_\pi) = p_{\lambda(\pi)},\ \ \rho(e_\pi) = \pi! e_{\lambda(\pi)},\ \ \rho(h_\pi) = \pi! h_{\lambda(\pi)}. \qquad \blacksquare$$

We end this section by defining a second action of $\mathfrak{S}_n$ that is also interesting. Since our variables do not commute, we can define an action on places (rather than variables). Explicitly, consider the vector space $\Pi^n(\mathbf{x})$ of elements of $\Pi(\mathbf{x})$ which are homogeneous of degree n. Given a monomial of that degree, we define

$$g \circ (x_{i_1} x_{i_2} \cdots x_{i_n}) = x_{i_{g1}} x_{i_{g2}} \cdots x_{i_{gn}} \tag{5}$$

and extend linearly. It is easy to see that if b_π is a basis element for any of our four bases, then $g \circ b_\pi = b_{g\pi}$ where g acts on set partitions in the usual manner.

3 Change of basis and inner products

We already know that the m_π with $\pi \vdash [n]$ form a basis for $\Pi^n(\mathbf{x})$. So to prove that any other equicardinal set of functions forms a basis, it suffices to show that they span the m_π. Doubilet [2] has obtained such basis exchange formulas as well as the other results in this section in a formal setting that includes ours as a special case. But we replicate his theorems to present them in standard notation. We also note that our proofs sometimes extend, simplify, and combinatorialize his.

Expressing each symmetric function in terms of m_π is easy to do directly from the definitions. These involve certain sums over the lattice Π_n of partitions of $[n]$, so the inverse formulas can be obtained using its Möbius functions μ [19]. Because of space limitations we will only state the results for the monomial and power sum bases.

Theorem 3.1 *We have the following change of basis formulae:*

$$p_\pi = \sum_{\sigma \geq \pi} m_\sigma \quad \text{and} \quad m_\pi = \sum_{\sigma \geq \pi} \mu(\pi, \sigma) p_\sigma. \qquad \blacksquare$$

Define the *sign of* λ, $(-1)^\lambda$, to be the sign of any permutation of cycle type λ and the *sign of* π to be $(-1)^\pi = (-1)^{\lambda(\pi)}$. As an application of the various change-of-basis equations, one can derive the properties of an analogue of the involution $\omega : \Lambda(\mathbf{x}) \to \Lambda(\mathbf{x})$ defined by linearly extending

$$\omega(p_\lambda) = (-1)^\lambda p_\lambda.$$

Define a map on $\Pi(\mathbf{x})$, which we will also call ω, by

$$\omega(p_\pi) = (-1)^\pi p_\pi$$

for all set partitions π and linear extension.

Theorem 3.2 *The map $\omega : \Pi(\mathbf{x}) \to \Pi(\mathbf{x})$ is an involution which commutes with the projection map ρ (where we use the standard involution in $\Lambda(\mathbf{x})$).* $\blacksquare$

We will now introduced a partial right inverse $\tilde{\rho}$ for the projection map ρ and an inner product for which $\tilde{\rho}$ is an isometry. Define the *lifting map*

$$\tilde{\rho} : \Lambda(\mathbf{x}) \to \Pi(\mathbf{x})$$

by linearly extending

$$\tilde{\rho}(m_\lambda) = \frac{\lambda!}{n!} \sum_{\pi \in \lambda} m_\pi. \tag{6}$$

The next propostion follows easily from equation (4) and Theorem 2.1.

Proposition 3.3 *The map $\rho\tilde{\rho}$ is the identity on $\Lambda(\mathbf{x})$.* ■

The standard inner product on $\Lambda(\mathbf{x})$ is defined by

$$\langle m_\lambda, h_\mu \rangle = \delta_{\lambda,\mu}$$

and bilinear extension. Analogously, in $\Pi(\mathbf{x})$ consider the bilinear form defined by extending

$$\langle m_\pi, h_\sigma \rangle = n!\delta_{\pi,\sigma}$$

where $\pi \vdash [n]$. It follows directly from the definition that this form respects the grading of $\Pi(x)$ in the sense that if f, g are two homogeneous symmetric functions with $\langle f, g \rangle \neq 0$ then f and g have the same degree. We collect some of our results about $\langle \cdot, \cdot \rangle$ in the next theorem.

Theorem 3.4 *The bilinear form $\langle \cdot, \cdot \rangle$ has the following properties.*

(i) It is symmetric and positive definite and so an inner product.

(ii) It is invariant under the action (5).

(iii) It makes the map $\tilde{\rho} : \Lambda(\mathbf{x}) \to \Pi(\mathbf{x})$ an isometry. ■

4 MacMahon symmetric functions

We have delayed defining Schur functions in noncommuting variables because to do so it is best to introduce another piece of machinery, namely the MacMahon symmetric functions [14]. The connection between functions in noncommuting variables and MacMahon symmetric functions was first pointed out by Rosas [17, 18].

Consider n sets of variables

$$\dot{\mathbf{x}} = \{\dot{x}_1, \dot{x}_2, \ldots\}, \quad \ddot{\mathbf{x}} = \{\ddot{x}_1, \ddot{x}_2, \ldots\}, \quad \ldots, \quad \mathbf{x^{(n)}} = \{x_1^{(n)}, x_2^{(n)}, \ldots\}.$$

Let $g \in \mathfrak{S}_m$ act on $f(\dot{\mathbf{x}}, \ddot{\mathbf{x}}, \ldots, \mathbf{x^{(n)}}) \in \mathbb{Q}[[(\dot{\mathbf{x}}, \ddot{\mathbf{x}}, \ldots, \mathbf{x^{(n)}}]]$ diagonally, i.e.,

$$gf(\dot{x}_1, \ddot{x}_1, \ldots, \dot{x}_2, \ddot{x}_2, \ldots) = f(\dot{x}_{g1}, \ddot{x}_{g1}, \ldots, \dot{x}_{g2}, \ddot{x}_{g2}, \ldots).$$

We consider a function *symmetric* if it is left invariant by all $g \in \mathfrak{S}_m$ for all m.

Consider the monomial

$$M = \dot{x}_1^{a_1} \ddot{x}_1^{b_1} \cdots x_1^{(n)c_1} \dot{x}_2^{a_2} \ddot{x}_2^{b_2} \cdots x_2^{(n)c_2} \cdots .$$

Letting $\lambda^i = [a_i, b_i, \ldots, c_i]$ be the exponent sequence of the variables of subscript i, we define the *multiexponent of M* to be

$$\vec{\lambda} = \{\lambda^1, \lambda^2, \ldots\} = \{[a_1, b_1, \ldots, c_1], [a_2, b_2, \ldots, c_2], \ldots\}.$$

By summing up the vectors which make up the parts of $\vec{\lambda}$ we get the *multidegree of M*

$$\vec{m} = [m_1, m_2, \ldots, m_n] = [a_1, b_1, \ldots, c_1] + [a_2, b_2, \ldots, c_2] + \cdots .$$

In this situation we write $\vec{\lambda} \vdash \vec{m}$ and $\vec{m} \vdash m$ where $m = \sum_i m_i$.

Given $\vec{\lambda}$, there is an associated *monomial MacMahon symmetric function* defined by

$$m_{\vec{\lambda}} = \text{ sum of all the monomials with multiexponent } \vec{\lambda}.$$

By way of example

$$m_{[2,1],[3,0]} = \dot{x}_1^2 \ddot{x}_1 \dot{x}_2^3 + \dot{x}_1^3 \dot{x}_2^2 \ddot{x}_2 + \cdots$$

Note that we drop the curly brackets about $\vec{\lambda}$ for readability. Again, these functions are exactly the ones gotten by acting on a monomial with all possible permutations. We now define the *algebra of MacMahon symmetric functions*, $\mathcal{M} = \mathcal{M}(\dot{\mathbf{x}}, \ddot{\mathbf{x}}, \ldots, \mathbf{x}^{(\mathbf{n})})$, to be the span of the $m_{\vec{\lambda}}$ as $\vec{\lambda}$ runs over all vector partitions where each part is a vector of n nonnegative integers. As before, the $m_{\vec{\lambda}}$ are independent and so give a basis for $\mathcal{M}$.

For analogues of the other 3 bases of Λ, call a basis $b_{\vec{\lambda}}$ of $\mathcal{M}$ *multiplicative* if it is constructed by first defining $b_{[a,b,\ldots,c]}$, i.e., for a vector partition with only one part, and then letting

$$b_{\vec{\lambda}} = b_{\lambda^1} b_{\lambda^2} \cdots b_{\lambda^l}.$$

Note that it is sometimes easiest to define $b_{[a,b,\ldots,c]}$ in terms of its generating function. We now declare the *power sum, elementary, and complete homogeneous MacMahon symmetric functions* to be multiplicative with

$$\begin{aligned}
p_{[a,b,\ldots,c]} &= m_{[a,b,\ldots,c]} \\
\sum_{a,b,\ldots,c} e_{[a,b,\ldots,c]} q^a r^b \cdots s^c &= \prod_{i\geq 1} \left(1 + \dot{x}_i q + \ddot{x}_i r + \cdots + x_i^{(n)} s\right) \\
\sum_{a,b,\ldots,c} h_{[a,b,\ldots,c]} q^a r^b \cdots s^c &= \prod_{i\geq 1} \frac{1}{1 - \dot{x}_i q - \ddot{x}_i r - \cdots - x_i^{(n)} s}.
\end{aligned}$$

To see the connection with noncommutative symmetric functions, let $[1^n]$ denote the vector of n ones. Now consider the subspace $\mathcal{M}_{[1^n]}$ of $\mathcal{M}$ spanned by all the $m_{\vec{\lambda}}$ where $\vec{\lambda} \vdash [1^n]$. There is a map

$$\Phi : \bigoplus_{n\geq 0} \mathcal{M}_{[1^n]} \to \Pi$$

given by

$$\dot{x}_i \ddot{x}_j \cdots x_k^{(n)} \stackrel{\Phi}{\mapsto} x_i x_j \cdots x_k$$

and linear extension.

Theorem 4.1 ([17]) *The map Φ is an isomorphism of vector spaces. Furthermore, for each basis we have discussed*

$$b_{\lambda^1, \lambda^2, \ldots, \lambda^l} \stackrel{\Phi}{\mapsto} b_{B_1/B_2/\ldots/B_l}$$

where $b = m$, p, e, or h and λ^i is the characteristic vector of B_i. ■

By way of illustration

$$b_{[1,0,1,0],[0,1,0,1]} \stackrel{\Phi}{\mapsto} b_{13/24}$$

for any of our bases.

5 Schur functions

We will now give a combinatorial definition of a Schur function in the setting of MacMahon symmetric functions. This will give, via the map Φ, such a function in noncommuting variables.

Let $\lambda \vdash m$ and let T be a semistandard Young tableau of shape λ, denoted $\lambda(T) = \lambda$. We will write all our shapes in English notation. Now let $\vec{m}$ be a nonnegative integer vector of n components whose entries sum to m. Define a *dotted Young tableaux* $\dot{T}$ *of multidegree* $\vec{m} = [m_1, m_2, \ldots, m_n]$ to be any array obtained from T by putting single dots on m_1 entries of T, double dots on m_2 entries of T, etc. Now define the corresponding *MacMahon Schur function* to be

$$S_\lambda^{\vec{m}} = \sum_{\lambda(\dot{T})=\lambda} M_{\dot{T}} \quad \text{where} \quad M_{\dot{T}} = \prod_{i^{(j)} \in \dot{T}} x_i^{(j)}$$

and elements $i^{(j)}$ of $\dot{T}$ appear in the product with multiplicity. For example, if $\lambda = (3,1)$ and $\vec{m} = [2,2]$, then the coefficient of $\dot{x}_1^2 \ddot{x}_1 \ddot{x}_2$ in $S_\lambda^{\vec{m}}$ is 3, corresponding to the three dotted tableaux

$$T_1 = \begin{array}{ccc} \dot{1} & \dot{1} & \ddot{1} \\ \ddot{2} & & \end{array}, \quad T_2 = \begin{array}{ccc} \dot{1} & \ddot{1} & \dot{1} \\ \ddot{2} & & \end{array}, \quad T_3 = \begin{array}{ccc} \ddot{1} & \dot{1} & \dot{1} \\ \ddot{2} & & \end{array}.$$

It will also be useful to generalize the notion of having multidegree $\vec{m}$ to any multiset M of elements from the alphabet $A = \{\dot{1}, \dot{2}, \ldots\} \uplus \{\ddot{1}, \ddot{2}, \ldots\} \uplus \cdots \uplus \{1^{(n)}, 2^{(n)}, \ldots\}$, possibly with additional structure, having m_k elements with k dots. In this case we write $\vec{m}(M) = \vec{m}$.

Using an analogue of an involution of Knuth [10], one can show that $S_\lambda^{\vec{m}}$ is indeed a MacMahon symmetric function.

Theorem 5.1 *Function $S_\lambda^{\vec{m}}$ is invariant under the diagonal action of the symmetric groups.* ■

If $\vec{m} = (1, 1, \ldots, 1)$ then we will write S_λ for $S_\lambda^{\vec{m}}$ and make no distinction between S_λ and its image under the map Φ. The latter will cause no problems because we will never be multiplying these functions. Note also that if $\vec{m}$ has only one component then $S_\lambda^{\vec{m}} = s_\lambda$, the ordinary Schur function.

The following result shows how the S_λ relate to the previous concepts introduced for $\Pi(\mathbf{x})$. In it, $\lambda \trianglerighteq \mu$ means that λ dominates μ and $K_{\lambda,\mu}$ is a Kostka number.

Theorem 5.2 *The functions S_λ have the following properties.*

(i) In terms of the monomial symmetric functions

$$S_\lambda = \sum_{\mu \trianglelefteq \lambda} \mu! K_{\lambda,\mu} \sum_{\sigma \in \mu} m_\sigma. \tag{7}$$

(ii) The S_λ are independent.

(iii) In terms of the forgetful map $\rho(S_\lambda) = n! s_\lambda$.

(iv) The S_λ are in the image of $\tilde{\rho}$ and $\tilde{\rho}(n! s_\lambda) = S_\lambda$.

(v) In terms of the involution $\omega(S_\lambda) = S_{\lambda'}$ where λ' is the conjugate of λ.

(vi) In terms of the inner product $\langle S_\lambda, S_\mu \rangle = n!^2 \delta_{\lambda,\mu}$. ■

6 Jacobi-Trudi determinants and the Robinson-Schensted-Knuth map

Using a variation of the lattic-path approach introduced by Lindström [12] and popularized by Gessel and Viennot [9], one can prove analogues of the Jacobi-Trudi determinants in this setting.

Theorem 6.1 *Given a partition λ and vector $\vec{m}$ with $\lambda, \vec{m} \vdash m$, we have*

$$S_\lambda^{\vec{m}} = \det\left(\sum_{\vec{q} \,\vdash \lambda_i - i + j} h_{\vec{q}} \right)$$

with the convention that if the product of two monomials in the determinant does not have multidegree $\vec{m}$ then that product is zero. Also, with the same convention,

$$S_{\lambda'}^{\vec{m}} = \det\left(\sum_{\vec{q} \,\vdash \lambda_i - i + j} e_{\vec{q}} \right). \qquad \blacksquare$$

Note that if $\vec{q} = [q]$ has a single component, then $\vec{q} \vdash \lambda_i - i + j$ forces $q = \lambda_i - i + j$. So the sums in each entry of the determinants reduce to a single term and we recover the ordinary form of Jacobi-Trudi.

One can also generalize the famous Robinson-Schensted-Knuth bijection [10, 16, 21] to tableaux of arbitrary multidegree. First, however, we will need some definitions.

A *biword of length n over the alphabet A* is a $2 \times n$ array β of elements of A such that if the dots are removed then the columns are ordered lexicographically with the top row taking precedence. The lower and upper rows of β are denoted $\check{\beta}$ and $\hat{\beta}$, respectively. Viewing $\check{\beta}$ and $\hat{\beta}$ as multisets, the multidegree of β is the pair

$$\vec{m}(\beta) = (\vec{m}(\check{\beta}), \vec{m}(\hat{\beta})).$$

We now define a map $\beta \overset{\text{R–S–K}}{\mapsto} (\dot{T}, \dot{U})$ whose image is all pairs of dotted semistandard Young Tableaux of the same shape. Peform the normal Robinson-Schensted-Knuth algorithm on β (see [20] for an exposition) by merely ignoring the dots and just having them "come along for the ride." For example, if

$$\beta = \begin{array}{cccccc} \dot{1} & \dot{2} & \ddot{2} & \dot{2} & \ddot{3} & \dot{4} \\ \dot{2} & \ddot{1} & \ddot{3} & \dot{3} & \ddot{2} & \dot{1} \end{array}$$

then the sequence of tableux built by the algorithm is

$$\begin{array}{l} \dot{2} \end{array}, \quad \begin{array}{l} \ddot{1} \\ \dot{2} \end{array}, \quad \begin{array}{ll} \ddot{1} & \ddot{3} \\ \dot{2} \end{array}, \quad \begin{array}{lll} \ddot{1} & \ddot{3} & \dot{3} \\ \dot{2} \end{array}, \quad \begin{array}{lll} \ddot{1} & \ddot{2} & \dot{3} \\ \dot{2} & \ddot{3} \end{array}, \quad \begin{array}{lll} \ddot{1} & \dot{1} & \dot{3} \\ \dot{2} & \ddot{2} \\ \ddot{3} \end{array} \quad = \dot{T},$$

$$\begin{array}{l} \dot{1} \end{array}, \quad \begin{array}{l} \dot{1} \\ \dot{2} \end{array}, \quad \begin{array}{ll} \dot{1} & \ddot{2} \\ \dot{2} \end{array}, \quad \begin{array}{lll} \dot{1} & \ddot{2} & \dot{2} \\ \dot{2} \end{array}, \quad \begin{array}{lll} \dot{1} & \ddot{2} & \dot{2} \\ \dot{2} & \ddot{3} \end{array}, \quad \begin{array}{lll} \dot{1} & \ddot{2} & \dot{2} \\ \dot{2} & \ddot{3} \\ \ddot{4} \end{array} \quad = \dot{U}.$$

The next theorem follows directly from the definitions and the analogous result for the ordinary Robinson-Schensted-Knuth map.

Theorem 6.2 *The map*

$$\beta \overset{\text{R–S–K}}{\mapsto} (\dot{T}, \dot{U})$$

is a bijection between biwords and pairs of dotted semistandard Young tableaux of the same shape such that

$$\vec{m}(\beta) = (\vec{m}(\dot{T}), \vec{m}(\dot{U})). \qquad \blacksquare$$

Because this analogue is so like the original, most of the properties of the ordinary Robinson-Schensted-Knuth correspondence carry over into this setting with little change. By way of illustration, here is the corresponding Cauchy identity [20] which follows directly by turning each side of the previous bijection into a generating function. Note that for $\hat{\beta}$ and $\dot{U}$ we are using a second set of variables $\dot{\mathbf{y}}, \ddot{\mathbf{y}}, \ldots, \mathbf{y}^{(\mathbf{n})}$.

Theorem 6.3 *We have*

$$\sum_{m\geq 0}\sum_{\lambda,\vec{m},\vec{p}\ \vdash m} S_\lambda^{\vec{m}}(\dot{\mathbf{x}},\ldots,\mathbf{x}^{(\mathbf{n})})S_\lambda^{\vec{p}}(\dot{\mathbf{y}},\ldots,\mathbf{y}^{(\mathbf{n})}) = \prod_{i,j\geq 1}\frac{1}{1-\sum_{k,l=1}^{n} x_i^{(k)}y_j^{(l)}}. \qquad \blacksquare$$

7 Comments and questions

We now list various open problems raised by this research in the hopes that the reader will be tempted to tackle some of them.

(I) One can compute specializations for symmetric functions in noncommuting variables or, more generally, for MacMahon symmetric functions. This has been done by Rosas [18].

(II) Jacobi also showed that the Schur functions could be expressed as a quotient of two determinants called alternants. In fact, this was the expression Schur himself used [22] when defining the functions which bear his name. This definition has the advantage that it becomes immediately apparent that s_λ is symmetric since it is the quotient of two skew-symmetric determinants. However, we have not been able to find an analogue of this result in the case of noncommuting variables, much less for general MacMahon Schur functions.

(III) One of the other important properties of s_λ is that, when expanded in terms of the power sum basis, the coefficients are essentially the character values of the irreducible representation of the symmetric group corresponding to λ. The function itself is the character of the polynomial representation of the general linear group corresponding to λ. It would be very interesting to have some connection between the S_λ, or even the $S_\lambda^{\vec{m}}$, and representation theory.

(IV) The reader will have noticed that we only defined Schur functions in noncommuting variables for integer, rather than set, partitions. This is because we have not been able to come up with a completely satisfactory definition in the set case.

One possible approach would be to define functions S_π such that

$$S_\lambda = \sum_{\pi\in\lambda} S_\pi \tag{8}$$

as follows. For each m_σ in the expansion (7) we have two possibilities. If $\sigma \leq \pi$ for some $\pi \in \lambda$ then we uniformly distribute the coefficient of m_σ among all S_π with $\pi \geq \sigma$. If $\sigma \not\leq \pi$ for all $\pi \in \lambda$, then we uniformly distribute the coefficient of m_σ among all S_π with $\pi \in \lambda$. More explicitly, if $\lambda(\pi) = \lambda$ then define

$$S_\pi = \sum_{\mu \trianglelefteq \lambda} \mu! K_{\lambda,\mu} \sum_{\sigma\in\mu} c_\sigma m_\sigma$$

where

$$c_\sigma = \begin{cases} 1/|\{\tau \geq \sigma\ :\ \tau \in \lambda\}| & \text{if } \sigma \leq \pi, \\ 1/|\{\tau :\ \tau\in\lambda\}| & \text{if } \sigma \not\leq \tau \text{ for all } \tau \in \lambda, \\ 0 & \text{else.} \end{cases}$$

It is easy to see directly from the definition that we have (8). It is also straightforward to verify that the S_π are a basis for $\Pi(\mathbf{x})$. However, there are difficulties with many of the other desired properties for such a function. For example, the S_π are not orthogonal, although one can show that $\langle S_\pi, S_\lambda \rangle = 0$ if $\pi \not\leq \lambda$. And we have no idea what the Jacobi-Trudi determinants or Robinson-Schensted-Knuth map look like for these functions. Perhaps there is even another definition of the S_π that will be better for developing such analogues. We should also note that Doubilet [2] also posed the problem of finding an analogue of the Schur basis indexed by set partitions in his setting.

(V) As mentioned in the introduction, one of the motivations for introducing noncommuting variables is to study Stanley's chromatic symmetric function [7, 23, 24]. Let $G = (V, E)$ be a graph and let $P_G(n)$ denote Whitney's chromatic polynomial of G [27], i.e., the number of proper colorings of G from a set with n colors. This is a polynomial in n having many interesting properties as well as connections with Möbius functions of partially ordered sets, hyperplane arrangements, etc.

Stanley introduced a related chromatic symmetric function. Let the variables $\mathbf{x}$ commute and suppose G has vertices $V = \{v_1, v_2, \ldots, v_n\}$. Then define

$$X_G = X_G(\mathbf{x}) = \sum_c x_{c(v_1)} x_{c(v_2)} \cdots x_{c(v_n)} \tag{9}$$

where the sum is over all proper colorings of G using the positive integers as colors. It is easy to see that X_G is a symmetric function and that if we set the first n variables equal to 1 and the rest equal to 0 then we recover $P_G(n)$. Stanley was able to prove various results for X_G generalizing those known for the chromatic polynomial as well as theorems that have no analogue in the earlier setting.

One thing that was missing from Stanley's development was a version of the Deletion-Contraction Law which is a useful tool for proving theorems about $P_G(n)$ by induction. Gebhard and Sagan showed that if one does not allow the variables to commute then one can use (9) to define an analogue of $P_G(n)$ in noncommuting variables, $Y_G(\mathbf{x})$. Of course, $X_G = \rho(Y_G)$. But more importantly, Y_G satisfies a Deletion-Contraction Law. (It should be noted that Noble and Welsh [15] also have a version of this Law for X_G itself in the category of vertex-weighted graphs.) This permits the derivation of a number of Stanley's results by straightforward induction.

It is also possible to use this inductive approach to make progress on the $(\mathbf{3}+\mathbf{1})$-free Conjecture of Stanley and Stembridge [26]. Call a poset (partially ordered set) $(\mathbf{a}+\mathbf{b})$-*free* if it has no induced subposet isomorphic to the dijoint union of an a-element chain and a b-element chain. Any poset P has a corresponding *incomparability graph*, $G(P)$, whose vertices are the elements of P with an edge between two vertices if they are not comparable in P.

Conjecture 7.1 (Stanley-Stembridge) *Let P be a $(\mathbf{3}+\mathbf{1})$-free poset and suppose*

$$X_{G(P)} = \sum_\lambda c_\lambda e_\lambda$$

for certain coefficients c_λ. Then $c_\lambda \geq 0$ for all λ.

This conclusion is summarized by saying that $X_{G(P)}$ is *e-positive.*

There is quite a bit of evidence for this conjecture. Stembridge has verified it by computer for all 884 $(\mathbf{3}+\mathbf{1})$-free posets having at most 7 elements. Gasharov [5] has shown that the analogous, but weaker, result that $X_{G(P)}$ is s-positive (for the Schur function basis). Gebhard and Sagan focused on the case where one keeps the elementary symmetric functions, but specializes to the case where P is both $(\mathbf{3}+\mathbf{1})$-free and $(\mathbf{2}+\mathbf{2})$-free. These $G(P)$ are called *indifference graphs* and have another characterization which is suitable for induction. Specifically, the indifference graphs are precisely those obtained by using $V = \{1, 2, \ldots, n\}$ together with a collection of intervals $I_1, I_2, \ldots, I_l \subseteq V$ and putting a complete graph on the vertices of each I_j.

Theorem 7.2 (Gebhard-Sagan) *If G is an indifference graph with $|I_j \cap I_k| \leq 1$ for all j, k then X_G is e-positive.* ■

It would be wonderful to use the theory of symmetric functions in noncommuting variables to extend this result to all indifference graphs.

Acknowledgments. We would like to thank the Isaac Newton Institute for Mathematical Sciences for support and Timothy Chow, Ira Gessel, and Larry Smith for helpful discussions.

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