OVERPARTITIONS

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ABSTRACT. We undertake a study of a generalization of partitions, called overpartitions. These objects are natural combinatorial structures associated with the q-binomial theorem, Heine's transformation, and Lebesgue'sidentity [21, 19]. We begin by studying bijectively the equivalence of the objects generated by two q-hypergeometric summations. One of the correspondences will lead to a family of generating functions for column-restricted Frobenius partitions. Moreover, it will be seen to imply the q-Chu-Vandermonde identityand a limiting case of Ramanujan's ${}_1\psi_1$ summation [18]. The full ${}_1\psi_1$ can be also explainedcombinatorially using overpartitions. A graphical representation can be utilized to give straightforward proofs of identities like the Rogers-Fine identity. We link overpartitions with divisor series, Theta series, Bailey chains and partial theta series. In particular we prove a theorem that is similar in flavor to results obtained by Andrews, Dyson and Hickerson [10, Thm. 4] for partitions into distinct parts. We give an analogue for overpartitions of Andrews' generalization of the Rogers-Ramanujan identities [6].

RÉSUMÉ. Nous étudions une généralisation des partages : les surpartages. Ces objets ont été utilisés pour donner des interprétations naturelles su théorème q-binomial, de la transformation de Heine et de l'identité de Lebesgue [21, 19]. Nous proposons deux bijections qui permettent de définir ces objets. Une des correspondances implique l'identité q-Chu-Vandermonde et la moitié de la somme ${}_1\psi_1$ de Ramanujan [18]. La somme ${}_1\psi_1$ peut aussi être expliquée combinatoirement grce aux surpartages. Nous utilisons une représentation graphique pour donner une preuve directe del'identité Rogers-Fine. Nous relions les overpartitions aux séries de diviseurs, aux séries Théta, aux chaines de Bailey et aux séries Théta partielles. Par exemple, nous obtenons un résultat similaire aux résultats de Andrews, Dyson and Hickerson [10, Thm. 4] pour les partitions en parts distinctes. Nous donnons aussi un analogue de la généralisation de Andrews des identités de Rogers-Ramanujan [6].

1. INTRODUCTION

A partition of n is a non-increasing sequence of natural numbers whose sum is n. The desire to discover and prove theorems about partitions has been a driving force behind the recent renaissance of basic hypergeometric series. However, it is still not clear how to interpret most q-series identities in a natural way as statements about partitions, and even fewer are deducible using combinatorial properties of partitions. While some progress has been made by considering parts in different congruence classes (see [2], for example) or by studying statistics on partitions ([1, 24], for instance), it seems to be most fruitful to employ the perspective of certain direct products of partitions which we call overpartitions.

An overpartition of n is a non-increasing sequence of natural numbers whose sum is n in which the first occurrence (equivalently, the final occurrence) of a number may be overlined.

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We denote the number of overpartitions of n by $\overline{p}(n)$. Since the overlined parts form a partition into distinct parts and the non-overlined parts form an ordinary partition, we have the generating function

$$\sum_{n=0}^{\infty} \overline{p}(n)q^n = \prod_{n=1}^{\infty} \frac{1+q^n}{1-q^n} = 1 + 2q + 4q^2 + 8q^3 + 14q^4 + ... \tag{1.1}$$

For example, the 14 overpartitions of 4 are

$$(4), (\overline{4}), (3,1), (\overline{3},1), (3,\overline{1}), (\overline{3},\overline{1}), (2,2), (\overline{2},2), (2,1,1),$$
$$(\overline{2},1,1), (2,\overline{1},1), (\overline{2},\overline{1},1), (1,1,1,1), (\overline{1},1,1,1).$$

These objects are natural combinatorial structures associated with the q-binomial theorem, Heine's transformation, and Lebesgue's identity (see [21] for a summary with references). In [19], they formed the basis for an algorithmic approach to the combinatorics of basic hypergeometric series. It should come as no surprise, then, that the theory of basic hypergeometric series contains a wealth of information about overpartitions and that many theorems and techniques for ordinary partitions have analogues for overpartitions. The following pages are intended as an introduction to the structure of overpartitions revealed by q-series identities.

We begin in the next section by studying bijectively the equivalence of the objects generated by the summations in

$$\sum_{n=0}^{\infty} \frac{(-a;q)_n q^n}{(q;q)_n} = \frac{(-aq;q)_\infty}{(q;q)_\infty} \tag{1.2}$$

and

$$\sum_{n=0}^{\infty} \frac{(-1/a;q)_n a^n q^{n(n+1)/2}}{(q;q)_n^2} = \frac{(-aq;q)_\infty}{(q;q)_\infty}. \tag{1.3}$$

Here and throughout we employ the standard q-series notation

$$(a_1, ...a_j;q)_\infty = \prod_{k=0}^{\infty}(1-a_1q^k)\cdots(1-a_jq^k),\ (a_1,...,a_j;q)_n = \frac{(a_1,...,a_j;q)_\infty}{(a_1q^n,...,a_jq^n;q)_\infty}.$$

In addition to providing another useful representation of overpartitions, the bijection which proves (1.2) and (1.3) will lead to a family of generating functions for column-restricted Frobenius partitions (see §2 for the definition).

Theorem 1.1. *Let $F_S(n)$ denote the number of Frobenius partitions of n with a partition into distinct positive parts in the bottom row and an overpartition into non-negative parts in the top row, with the restriction that non-overlined parts can only occur above parts from the set S. Then for any set of non-negative integers S we have*

$$\sum_{n=0}^{\infty} F_S(n)q^n = \frac{\prod_{s\in S}(1+q^s)}{(q;q)_\infty}.$$

Moreover, the correspondence will be seen to imply the q-Chu-Vandermonde identity

$$\sum_{k=0}^{n} \frac{(-1/a;q)_k c^k a^k q^{k(k+1)/2}}{(cq;q)_k} \begin{bmatrix} n \\ k \end{bmatrix} = \frac{(-acq;q)_n}{(cq;q)_n}, \tag{1.4}$$

as well as "half" of Ramanujan's ${}_1\psi_1$ summation [18, p.239, II.29],

$$\frac{(-bq;q)_\infty}{(q;q)_\infty} \sum_{n=-\infty}^{\infty} \frac{z^n q^{n(n+1)/2}}{(-bq;q)_n} = \frac{(-1/z,-zq;q)_\infty}{(b/z;q)_\infty}. \tag{1.5}$$

Here we have used the q-binomial coefficient

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}. \tag{1.6}$$

The full ${}_1\psi_1$,

$$\frac{(-aq;q)_\infty(-bq;q)_\infty}{(q;q)_\infty(abq;q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-a^{-1};q)_n (zqa)^n}{(-bq;q)_n} = \frac{(-zq;q)_\infty(-z^{-1};q)_\infty}{(bz^{-1};q)_\infty(azq;q)_\infty} \tag{1.7}$$

where $|b| < |z| < |\frac{1}{aq}|$ and $|q| < 1$, can be established combinatorially using overpartitions. The argument, which also appears in [15, 14], will be presented in §3.

In §4 we look at a graphical representation for overpartitions and its implications. This representation can be utilized to give straightforward proofs of identities like the Rogers-Fine identity,

$$\sum_{n=0}^{\infty} \frac{(-a;q)_n (tq)^n}{(bq;q)_n} = \sum_{n=0}^{\infty} \frac{(-a;q)_n(-atq/b;q)_n(1+atq^{2n+1})(bt)^n q^{n^2+n}}{(bq;q)_n(tq;q)_{n+1}}. \tag{1.8}$$

The remainder of the paper is devoted to the deduction of facts about overpartitions from identities and transformations in the theory of basic hypergeometric series. For instance, it will be natural in this context to define the *rank* of an overpartition as one less than the largest part minus the number of overlined parts. The following theorem is similar in flavor to results obtained by Andrews, Dyson and Hickerson [10, Thm. 4] for partitions into distinct parts.

Theorem 1.2. *Let $D(n)$ denote the number of overpartitions with even rank minus the number with odd rank. Then $D(n)$ is equal to 0 if and only if $n \equiv 2 \pmod 4$. On the other hand, $D(n) = 2k$ has infinitely many solutions for any $k \in \mathbb{Z}$.*

As another example, we give an analogue for overpartitions of Andrews' generalization of the Rogers-Ramanujan identities [6]. The combinatorial interpretation is in terms of a Durfee square dissection of the associated partition of a partition into distinct non-negative parts (see §3.4 for the definitions).

Theorem 1.3. *The number of overpartitions with parts not divisible by k is equal to the number of overpartitions whose Frobenius representation has a top row with at most $k-2$ Durfee squares in its associated partition.*

These and a variety of other theorems on overpartitions are established in §5. Due to space limitations, some proofs are omitted. We refer the reader to [16] for a more complete discussion.

2. The two representations

From the definition and the generating function (1.1) it follows that overpartitions can be viewed through a number of different lenses. For instance, the number of overpartitions of n is the number of partitions of n in which one part of each odd size may be tagged, or the number of partitions λ of n weighted by $2^{\mu(\lambda)}$, where $\mu(\lambda)$ denotes the number of different part sizes occurring in the partition. From the perspective of q-series, the most natural representations correspond to (1.2) and (1.3). Theorem 2.2 below will describe the correspondence between these two representations, but first we recall the bijection which establishes the generating function for the number of overpartitions with exactly k parts.

Proposition 2.1. *Let $\overline{p}_{k,l,m}(n)$ denote the number of overpartitions of n into k parts with l overlined parts and rank m (as defined in the introduction). Then*

$$\sum_{l,m,n=0}^{\infty} \overline{p}_{k,l,m}(n) a^l b^m z^k q^n = \frac{(-a;q)_k (zq)^k}{(bq;q)_k}. \tag{2.1}$$

Proof. The function $(zq)^k/(bq;q)_k$ generates a partition λ into k positive (non-overlined) parts, where the exponent on z keeps track of the number of parts and the exponent on b records the largest part minus 1. Note that since there are not yet any overlined parts, this is the same as the rank. Now $(-a;q)_k$ generates a partition $\mu = (\mu_1, \cdots, \mu_j)$ into distinct non-negative parts less than k, with the exponent on a tracking the number of parts. For each of these μ_i beginning with the largest, we add 1 to the first μ_i parts of λ, and then put a bar over the $(\mu_i + 1)$th part of λ. Here the parts of λ are written in non-increasing order. This operation leaves the rank invariant and counts one overlined part for each part of μ. For example, if $k = 5$, $\lambda = (8, 4, 4, 2, 1)$, and $\mu = (4, 3, 0)$, then we have

$$\begin{aligned} ((8,4,4,2,1),(4,3,0)) &\iff ((9,5,5,3,\overline{1}),(3,0)) \\ &\iff ((10,6,6,\overline{3},\overline{1}),(0)) \\ &\iff (\overline{10},6,6,\overline{3},\overline{1}) \end{aligned}$$

The result is obviously an overpartition and the process is easily inverted. □

The mapping above (with $b = 1$) was considered in [19], where it was noted that by summing over the non-negative integers k, we obtain the q-binomial theorem

$$\sum_{k=0}^{\infty} \frac{(-a;q)_k (zq)^k}{(q;q)_k} = \frac{(-azq;q)_\infty}{(zq;q)_\infty}. \tag{2.2}$$

The second representation of an overpartition is in terms of combinatorial objects called Frobenius partitions [9]. A Frobenius partition of n is a two-rowed array $\begin{pmatrix} a_1, a_2, ..., a_k \\ b_1, b_2, ..., b_k \end{pmatrix}$ where $\sum a_i$ is a partition into non-negative parts taken from a set A, $\sum b_i$ is a partition into positive parts taken from a set B, and $\sum(a_i + b_i) = n$. We write $p_{A,B}(n)$ for the number of such Frobenius partitions of n.

Theorem 2.2. *Let $\mathcal{O}$ be the set of overpartitions into non-negative parts and Q the set of partitions into distinct positive parts. There is a one-to-one correspondence between overpartitions λ of n and Frobenius partitions ν counted by $p_{\mathcal{O},Q}(n)$ in which the number of overlined parts in λ is equal to the number of non-overlined parts in the bottom row of ν.*

Proof. From Proposition 2.1 one deduces that the equality of the series in equations (1.2) and (1.3) is equivalent to the statement of the theorem. The bijection below explicitly gives the correspondence. In the case where the overpartition has no non-overlined parts, it reduces to the usual mapping between a partition and its Frobenius symbol [9].

We use the notion of a *hook*. Given a positive integer a and a non-negative integer b, $h(a, b)$ is the hook that corresponds to the partition $(a, 1, \ldots, 1)$ where there are b ones. Combining a hook $h(a, b)$ and a partition α is possible if and only if $a > \alpha_1$ and $b \geq l(\alpha)$, where $l(\alpha)$ denotes the number of parts of α. The result of the union is $\beta = h(a, b) \cup \alpha$ with $\beta_1 = a$, $l(\beta) = b + 1$ and $\beta_i = \alpha_{i-1} + 1$ for $i > 1$.

Now take a Frobenius partition ν counted by $p_{\mathcal{O},Q}(n)$ and initialize α and β to the empty object, ϵ. Beginning with the rightmost column of ν, we proceed to the left, building α into an ordinary partition and β into a partition into distinct parts. At the i^{th} column, if a_i is overlined, then we add the hook $h(b_i, a_i)$ to α. Otherwise, we add the part a_i to α' (the conjugate of α) and the part b_i to β. Joining the parts of α together with the parts of β gives the overpartition λ. An example is given below.

ν	α	β
$\begin{pmatrix} 6, \overline{4}, 4, 3, \overline{1} \\ 8, 6, 5, 3, 1 \end{pmatrix}$	ϵ	ϵ
$\begin{pmatrix} 6, \overline{4}, 4, 3 \\ 8, 6, 5, 3 \end{pmatrix}$	$(1, 1)$	ϵ
$\begin{pmatrix} 6, \overline{4}, 4 \\ 8, 6, 5 \end{pmatrix}$	$(2, 2, 1)$	(3)
$\begin{pmatrix} 6, \overline{4} \\ 8, 6 \end{pmatrix}$	$(3, 3, 2, 1)$	$(5, 3)$
$\begin{pmatrix} 6 \\ 8 \end{pmatrix}$	$(6, 4, 4, 3, 2)$	$(5, 3)$
ϵ	$(7, 5, 5, 4, 3, 1)$	$(8, 5, 3)$

We get $\lambda = (\bar{8}, 7, 6, \bar{5}, 4, 4, \bar{3}, 3, 1)$. The reverse bijection can be easily described. □

Remarks. Notice that if λ has parts at most n, then the corresponding Frobenius partition has a bottom row with parts at most n. Since the term $q^{k(k+1)/2} \begin{bmatrix} n \\ k \end{bmatrix}$ is the generating function for a partition into exactly k distinct parts $\leq n$, the bijection implies (1.4). Also, the coefficient of z^0 on the right side of (1.5) can be interpreted as the generating function for Frobenius partitions counted by $p_{\mathcal{O},Q}$ where the power of b tracks the number of non-overlined parts in the bottom row. By the bijection above, this generating function is $(-bq; q)_\infty/(q; q)_\infty$. Arguing as in [15], this is sufficient to prove the identity in full generality.

Proof of Theorem 1.1. This follows easily from the bijection described above by noticing that the overlined parts of the overpartition, which are the parts of β, are always mapped to a position under a non-overlined part in the bottom row of the corresponding Frobenius partition. □

3. Overpartitions and Ramanujan's ${}_1\psi_1$ summation

The full ${}_1\psi_1$ can be established combinatorially, but that bijection does not reduce to the one in the previous section [15, 14]. We give here a sketch of that proof.

Using the q-binomial theorem, the coefficient of $[z^m]$ of the right handside of (1.7) is

$$A_m(a,b,q) = \sum_{n=-m}^{\infty} b^n (aq)^{n+m} \frac{(-1/a;q)_{n+m}(-1/b;q)_n}{(q;q)_{n+m}(q;q)_n}.$$

This shows that A_m is the generating funtion of P_m, the set of unbalanced Frobenius partitions where the length of the top row plus m is equal to the length of the bottom row. We first show that

$$A_0(a,b,q) = \frac{(-aq;q)_\infty(-bq;q)_\infty}{(q;q)_\infty(abq;q)_\infty},$$

by exhibiting a bijection between P_0 and 4-tuples of partitions.

Given a Frobenius overpartition $\alpha = \begin{pmatrix} a_1, a_2, ..., a_n \\ b_1, b_2, ..., b_n \end{pmatrix}$, transform α into another two-rowed array $\beta = \begin{pmatrix} c_1, c_2, ..., c_p \\ d_1, d_2, ..., d_p \end{pmatrix}$ as follows:

- (i) Let $c_1, ..., c_p$ be $a_1, ..., a_n$ except $\bar{k}$ is inserted if $0 \leq k < a_1$ but $\bar{k}$ does not occur in the top row of α.
- (ii) Let $d_1, ..., d_p$ be the $-\bar{k}$'s from (i) written in increasing order, followed by the overlined parts of the bottom row of α, and written in increasing order, followed by the non-overlined parts from row 2 of α, written in non-increasing order.

For example, if

$$\alpha = \begin{pmatrix} 5, \bar{3}, 3, 3, 1, \bar{0} \\ \bar{4}, 4, 1, 1, 1, 1 \end{pmatrix} \quad \text{then} \quad \beta = \begin{pmatrix} 5, & \bar{4}, & \bar{3}, & 3, & 3, & \bar{2}, & \bar{1}, & 1, & \bar{0} \\ -\bar{4}, & -\bar{2}, & -\bar{1}, & \bar{4}, & 4, & 1, & 1, & 1, & 1 \end{pmatrix}.$$

Now map β to a 4 - tuple $\pi = (\pi_1, \pi_2, \pi_3, \pi_4)$ by adding, for all i, a part of size $d_i + c_i$ to

$$\begin{cases} \pi_1 & \text{if } c_i \text{ and } d_i \text{ are overlined} \\ \pi_2 & \text{if neither } c_i \text{ nor } d_i \text{ are overlined} \\ \pi_3 & \text{if } d_i \text{ but not } c_i \text{ is overlined} \\ \pi_4 & \text{if } c_i \text{ but not } d_i \text{ is overlined} \end{cases}$$

In our example, we obtain

$$\pi_1 = (2,2), \quad \pi_2 = (7,2), \quad \pi_3 = (7,1), \quad \pi_4 = (3,2,1)$$

It is easy to see that π_1 and π_2 are ordinary partitions and that π_3 and π_4 are partitions into distinct parts. This mapping is uniquely reversible.

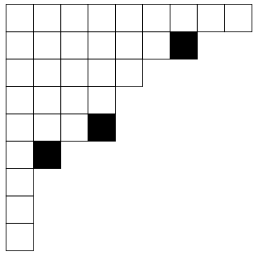

FIGURE 1. Overpartition

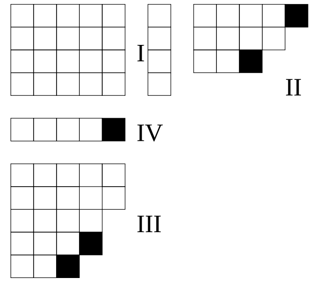

FIGURE 2. Rogers-Fine

To finish the proof we need a combinatorial interpretation of the following for $m > 0$:

$$A_{m-1}(a,b,q)(aq+q^m) = A_m(a,b,q)(1+bq^m);\;\; A_{-m+1}(a,b,q)(b+q^m) = A_{-m}(a,b,q)(1+aq^m). \tag{3.1}$$

To prove the first half of (3.1), we need to define the bijection

$$\Psi_m : P_{m-1} \times \{0,1\} \;\rightarrow P_m \times \{0,1\}.$$

Starting with $\beta \in P_{m-1}$ and $a \in \{0,1\}$. We set σ equal to β and have four cases :

- $a = 0$ then add a part 1 in the bottom row and set $b = 0$.
- $a = 1$ and there is a part 0 in the top row then delete it and set $b = 1$.
- $a = 1$ and if there is a part $\bar{0}$ in the top row then delete it decrease the elements of the top row by 1 and increase the elements of the bottom row by 1.
- $a = 1$ and the smallest part on the top row is positive then decrease the elements of the top row by 1 and increase the elements of the bottom row by 1, add a part 1 on the bottom row and set $b = 0$.

Example. Starting with $\beta = \begin{pmatrix} \bar{6},5,5,\bar{4},4,1,\bar{0} \\ \bar{3},3,3,\bar{2},\bar{1},1,1 \end{pmatrix}$ and $a = 1$. Applying Ψ_1 gives : $\sigma = \begin{pmatrix} \bar{5}, & 4, & 4, & \bar{3}, & 3, & 0 & \\ \bar{4}, & 4, & 4, & \bar{3}, & \bar{2}, & 2, & 2 \end{pmatrix}$; $b = 0$. The reverse is easy to describe. Using the same kind of arguments, we can also prove combinatorially the second half of (3.1). It is worth noting that this was was one of the open problems in [21].

4. A GRAPHICAL REPRESENTATION

If we think of an overpartition as a partition in which the *final* occurrence of a part may be overlined, then such a partition corresponds to an ordinary Ferrers diagram in which the corners may be colored. Conjugating, or reading the columns of the diagram, gives another overpartition. For example, the overpartition $(9,\overline{7},5,\overline{4},4,\overline{2},1,1,1)$ is represented on Fig. 1 and has conjugate $(9,\overline{6},\overline{5},5,3,\overline{2},2,1,1)$. See Fig. 2.

Observe that conjugating maps the largest part minus the number of overlined parts to the number of non-overlined parts. This number is the rank if the largest part is overlined and one more than the rank otherwise, which proves an overpartition-theoretic analogue of a theorem of Fine on partitions into distinct parts [17, p. 47, Eq. (24.6)] :

Proposition 4.1. *For $n, m \geq 1$, the number of overpartitions of n with rank m or $m+1$ is equal to twice the number of overpartitions of n with exactly $m+1$ non-overlined parts.*

As with ordinary partitions, there are many natural statistics associated with such a graph, and one may readily write down q-series identities by counting overpartitions in different ways. We highlight this for the Durfee rectangle:

Proof of (1.8). Proposition 2.1 tells us that the coefficient of $[q^n t^k a^l b^m]$ is the number of overpartitions of n with k parts, l overlined parts and rank m. We shall prove that the coefficient of $[q^n t^k a^l b^m]$ in $\frac{(-a;q)_d(-atq/b;q)_d(1+atq^{2d+1})(bt)^d q^{d^2+d}}{(bq;q)_d(tq;q)_{d+1}}$. is the number of overpartitions of n with Durfee rectangle size d, k parts, l overlined parts and rank m. Let us first recall that the Durfee rectangle is the largest $(d+1)\times d$ rectangle that can be placed on the Ferrers diagram [3]. For example the size of the Durfee rectangle of the overpartition on Fig. 1 is 3 and of the overpartition on Fig. 2 is 4.

We will interpret the formula "piece by piece" :

$$\frac{\overbrace{(-a;q)_d}^{II}\,\overbrace{(-atq/b;q)_d}^{III}\,\overbrace{(1+atq^{2d+1})}^{IV}\,\overbrace{(bt)^d q^{d^2+d}}^{I}}{\underbrace{(bq;q)_d}_{II}\,\underbrace{(tq;q)_{d+1}}_{III}}.$$

We construct the overpartition as follows :

- Piece I : the Durfee rectangle $(d+1)\times d$. It is obviously an overpartition of d^2+d into d parts, with rank d and 0 overlined parts.
- Piece II : an overpartition into d non-negative parts. This overpartition is put at the right of the Durfee rectangle. The number of overlined parts (resp. rank) of that overpartition is then added to the number of overlined parts (resp. rank) of the Durfee rectangle.
- Piece III : an overpartition into parts at most $d+1$ where each part increases the number of parts by 1 and the part $d+1$, if it occurs, can not be overlined. This overpartition is put under the Durfee rectangle. Each overlined part decreases the rank by 1 and each part increases the number of parts by 1.
- Piece IV: Allows to put an overlined part of size $d+1$ under the Durfee rectangle and increase the first d parts by 1.

Let us give an example with $d = 4$ that is illustrated on Fig. 2. We start with the overpartition $\pi = \epsilon$.

- Piece I : $(5,5,5,5)$. $\pi = (5,5,5,5)$
- Piece II : $(\overline{5},4,\overline{3},0)$. $\pi = (\overline{10},9,\overline{8},5)$.
- Piece III : $(5,5,\overline{4},4,\overline{3})$. $\pi = (\overline{10},9,\overline{8},5,5,5,\overline{4},4,\overline{3})$.
- Piece IV : (9). $\pi = (\overline{11},10,\overline{9},6,\overline{5},5,5,\overline{4},4,\overline{3})$.

5. Overpartitions and q-series

Armed with the interpretation of certain finite products in terms of overpartitions, we can now have a field day reading off theorems from q-series identities.

5.1. **Overpartitions and divisor functions.** By observing that

$$1/(1 - zq^n) = (zq;q)_{n-1}/(zq;q)_n$$

is a generating function for overpartitions, we can easily make connections with divisor series. Even the simplest cases reveal what is surprising behavior for such elementary combinatorial functions.

Theorem 5.1. *Let n have the factorization $2^x p_1^{y_1} ... p_j^{y_j}$, where the p_i are distinct odd primes. Then the number of overpartitions of n with even rank minus the number with odd rank is equal to*

$$2(1 - x)(y_1 + 1) \cdots (y_j + 1). \tag{5.1}$$

Proof. From Proposition (2.1) we have that

$$\sum_{n=0}^{\infty} \frac{(-1;q)_n q^n}{(zq;q)_n} = \sum_{m,n=0}^{\infty} \overline{p}(m,n) z^m q^n, \tag{5.2}$$

where $\overline{p}(m,n)$ denotes the number of overpartitions of n with rank m. Set $z = -1$ and observe that $\sum \frac{2q^n}{1+q^n}$ is the generating function for twice the number of odd divisors minus twice the number of even divisors of a natural number, which is in turn expressed by (5.1). ☐

Proof of Theorem 1.2. This is an obvious corollary, as the expression in (5.1) can be made into any even integer, and is 0 only when $x = 1$. ☐

Another simple case relates the ordinary divisor function to the *co-rank* of an overpartition. The co-rank is the number of overlined parts less than the largest part.

Theorem 5.2. *Let n have the factorization above and let $D_1(n)$ be the number of overpartitions with even co-rank minus the number with odd co-rank. Then*

$$D_1(n) = 2(1 + x)(y_1 + 1) \cdots (y_j + 1). \tag{5.3}$$

It is worth pointing out that the above theorems can be deduced using a simple involution on overpartitions. Let m be the smallest part size. If $\bar{m}$ is in the partition, then take off the overline; otherwise overline the first part m. This involution changes the parity of the rank and of the co-rank except when the partition has only one part size. The theorems follow.

We close with an analogue for overpartitions of a theorem of Uchimura on partitions into distinct parts [12, 22]

Theorem 5.3. *The sum of all overpartitions of n weighted by $(-1)^{k-1}m$, where m is the smallest part and k is the number of parts, is equal to twice the number of odd divisors of n.*

Proof. Set $b = -1$ in the q-Gauss summation,

$$\sum_{n=0}^{\infty} \frac{(a,b;q)_n (c/ab)^n}{(c,q;q)_n} = \frac{(c/a, c/b;q)_\infty}{(c, c/ab;q)_\infty}.$$

Then take $\frac{d}{da}$ of both sides and set $a = 1$. This yields

$$\sum_{n=1}^{\infty} \frac{(-1;q)_n (-1)^{n-1} q^n}{(q;q)_{n-1}(1-q^n)^2} = \sum_{n=1}^{\infty} \frac{2q^n}{(1-q^{2n})}.$$

The right side generates odd divisors, while expanding $q^n/(1-q^n)^2 = (q^n + 2q^{2n} + \cdots)$ and appealing to Proposition 2.1 shows that the left side generates the weighted count of overpartitions. □

We shall return to the relationship between overpartitions and divisor functions when more intricate examples are treated in §5.3 in the context of Bailey chains and again in §5.4 in the context of partial theta functions.

5.2. **Overpartitions and Theta Series.** Here we use (1.8) to reveal that certain generating functions for overpartitions are given by theta-type series. The example below is related to the *perimeter* of an overpartition, which is defined to be the largest part plus the number of parts.

Theorem 5.4. *Let $D_2(m,n)$ denote the number of overpartitions of n with perimeter m having largest part even minus the number having largest part odd. Then we have*

$$D_2(m,n) = \begin{cases} 2(-1)^n, & m = 2k \text{ and } n = k^2 \\ 0, & \text{otherwise.} \end{cases} \tag{5.4}$$

Proof. In the Rogers-Fine identity (1.8) let $t = -z, a = b = zq$, multiply both sides by $-2qz^2/(1-zq)$, and shift the summation to get

$$\sum_{n=1}^{\infty} \frac{2(-zq)_{n-1}(-q)^n z^{n+1}}{(zq;q)_n} = 2\sum_{n=1}^{\infty}(-1)^n z^{2n} q^{n^2}. \tag{5.5}$$

Now the left side generates the nonempty overpartitions with an even number of parts minus the nonempty overpartitions with an odd number of parts, where the exponent on z is equal to 1 plus the number of parts plus the rank plus the co-rank. This is easily seen to be equal to the perimeter. □

A combinatorial proof of this result is essentially contained in [13].

5.3. **Overpartitions and the Bailey chain.** Next we give two samples of how overpartitions fit nicely into the theory of Bailey chains by applying Andrews' multiple series generalization of Watson's transformation [4]. As usual, the combinatorics is in terms of the Durfee dissection of a partition [6]. We recall that the Ferrers diagram of a partition λ has a largest upper-left justified square called the Durfee square. Since there is a partition to the right of this square, we identify its Durfee square as the second Durfee square of the partition λ. Continuing in this way, we obtain a sequence of successive squares. For example for the partition $(17, 15, 13, 9, 9, 7, 4, 2, 2)$ we get the sequence $(6, 3, 3, 2, 1, 1, 1)$. We also recall that the associated partition of a partition into disinct non-negative parts is obtained by writing the parts in increasing order and then removing $j-1$ from the jth part.

Proof of Theorem 1.3. We apply Andrews' multiple series transformation [4] with all variables besides c_k, a, and q tending to infinity to find that for any natural number k we have

$$\sum_{n_{k-1}\geq\cdots\geq n_1\geq 0}\frac{(c_k;q)_{n_{k-1}}q^{n_{k-1}(n_{k-1}+1)/2+n_{k-2}^2+\cdots+n_1^2}a^{n_{k-1}+\cdots+n_1}}{(q;q)_{n_{k-1}-n_{k-2}}\cdots(q;q)_{n_2-n_1}(q;q)_{n_1}(-c_k)^{n_{k-1}}} \tag{5.6}$$

$$= \frac{(aq/c_k;q)_\infty}{(q;q)_\infty}\sum_{n=0}^{\infty}\frac{(1-aq^{2n})(a,c_k;q)_n a^{kn}q^{kn^2}}{(1-a)(q,aq/c_k;q)_n c_k^n}.$$

If we set $a=1$ and $c_k=-1$ then the right side becomes a theta series which sums (by the triple product identity [18, p. 239, Eq. II.28]) to

$$\frac{(-q;q)_\infty(q^k;q^k)_\infty}{(-q^k;q^k)_\infty(q;q)_\infty}.$$

This is the generating function for overpartitions with parts not divisble by k. The $(k-1)$-fold summation on the left is interpreted as a generating function for Frobenius partitions counted by $p_{\mathcal{O},Q}(n)$. First, n_{k-1} is the number of columns. Next, by Proposition 2.1, the quotient $(-1;q)_{n_{k-1}}/(q;q)_{n_{k-1}}$ generates an overpartition into exactly n_{k-1} non-negative parts for the top row. Finally, the top row is generated by the rest of the summand, which, as detailed in [5, p.54-55], is the generating function for partitions into n_{k-1} distinct positive parts whose associated partition has at most $k-2$ Durfee squares. □

Now we consider a different application of (5.6) which links a weighted count of the overpartitions in Theorem 1.3 to a generalization of a divisor function which arose from a study of certain types of identities appearing in Ramanujan's lost notebook [11]. Namely, let $m_k(n)$ denote the number of k-middle divisors of n, that is, the number of divisors of n which occur in the interval $[\sqrt{n/k},\sqrt{kn})$. For the proof, see [16].

Theorem 5.5. *Let $D_4^{\pm}(k,n)$ denote the number of overpartitions whose Frobenius representations have a top row with at most $k-2$ Durfee squares in the associated partition and a bottom row with an even (odd) number of non-overlined parts less than the largest. Then*

$$D_4^{+}(k,n)-D_4^{-}(k,n)=2m_k(n). \tag{5.7}$$

5.4. **Frobenius overpartitions and partial theta functions.** Identities involving partial theta functions appeared in Ramanujan's lost notebook and have been extensively studied [7, 23]. Consider, for example, the following infinite product representations for two sums involving partial theta products $(aq,q/a;q)_n$ [20]:

$$\frac{a}{(1-a)^2}+\sum_{n=1}^{\infty}\frac{(q;q)_{n-1}^2q^n}{(aq,q/a;q)_n}=\frac{a(q;q)_\infty^2}{(1-a)^2(aq,q/a;q)_\infty}. \tag{5.8}$$

By now these are easily recognizable as statments about *Frobenius overpartitions*, i.e., Frobenius partitions counted by $p_{\mathcal{O},\mathcal{O}}(n)$. We highlight just one special case, where we again encounter divisor functions.

Theorem 5.6. *Let $D_5(n)$ denote the number of Frobenius overpartitions of n in which the sum of the largest parts in the top and bottom row is odd minus those for which it is even.*

Then

$$D_5(n) = 4(-1)^n(d_1(n) - d_3(n)), \tag{5.9}$$

where $d_i(n)$ *denotes the number of divisors of* n *congruent to* i *modulo* 4.

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