Lifting the toric g-vector inequalities*

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Abstract

We present a method of lifting linear inequalities for the flag f-vector of polytopes to higher dimensions. Known inequalities that can be lifted using this technique are the non-negativity of the toric g-vector and that the simplex minimizes the **cd**-index. We obtain new inequalities for 6-dimensional polytopes. In the last section we present the currently best known inequalities for dimensions 5 through 8.

Résumé

Nous présentons un methode d'augmenter des inégalités linéaires portant sur le vecteur f drapeau d'un polytope, aux dimensions plus élevées. Parmis les inégalités connues qui peuvent être augmentées de cette façon on trouve la positivité du vecteur g, et le fait que le simplexe minimalise l'index **cd**. Nous obtenons des nouvelles inégalités pour les polytopes de dimension 6. Dans le dernier paragraphe nous présentons les meilleurs inégalités connues, de dimension 5 jusqu'à dimension 8.

1 Introduction

A great open problem in discrete geometry is to classify f-vectors of convex polytopes. The f-vector enumerates faces of different dimensions of the polytope. For an n-dimensional polytope P and $0 \leq i \leq n-1$, let f_i denote the number of faces of dimension i. For three dimensional polytopes the classification problem was solved by Steinitz [18] almost a century ago. By Euler's relation the number of edges f_1 is determined by the number vertices f_0 and the number of faces f_2. Steinitz proved that f_0 and f_2 satisfies the two inequalities

$$f_2 \leq 2 \cdot f_0 - 4 \quad \text{and} \quad f_0 \leq 2 \cdot f_2 - 4. \tag{1.1}$$

Interestingly, the reverse is also true. Given two integers f_0 and f_2 that satisfies the two inequalities in (1.1), there is a three dimensional polytope with f_0 vertices and f_2 faces.

*The complete version of this paper can be found at the webpage: `http://www.ms.uky.edu/~jrge`.

†The author was supported by NSF grant 0200624.

When studying polytopes in higher dimensions it is natural to consider the more general notion of the flag f-vector. The entries in the flag f-vector enumerate incidences between faces of different dimensions of the polytope. However, already for four dimensions the problem of classifying flag f-vectors (and f-vectors) is open. In three dimensions the classification is a cone in the (f_0, f_2)-plane where the simplex is at the apex of this cone. Thus one might guess that the classification would yield a cone with the n-dimensional simplex at its apex. Unfortunately, this is not true. There are holes in the set of flag f-vectors arising from polytopes and hence it is a more complicated structure; see the discussion in [1]. Thus one has to turn one's attention to the smallest polyhedral set that contains the set of flag f-vectors of polytopes. This polyhedral set is not known in dimension 4 and higher. In order to find this set one has to find linear inequalities satisfied by the flag f-vector of polytopes.

The currently known linear inequalities are as follows. They are expressed in terms of the toric g-vector, $(g_i^n(P))_{0 \leq i \leq n/2}$, and the **cd**-index, $\Psi(P)$, two linear invariants of the flag f-vector of an n-dimensional polytope P.

(0) Two straightforward facts: The number of vertices (and the number of facets) is at least $n + 1$ for an n-dimensional polytope. Also if L is a non-negative linear functional on flag vectors of n-dimensional polytopes, then so is the dual functional L^* defined by $L^*(P) = L(P^*)$, where P^* denotes the dual polytope.

(i) The toric g-vector measures the intersection homology Betti numbers of the toric variety associated with a rational polytope (a polytope which can be embedded in space such that each vertex has rational coordinates). Using the hard Lefschetz theorem Stanley [15] proved that the toric g-vector of a rational polytope P is non-negative. That is, $g_i^n(P) \geq 0$ for $0 \leq i \leq n/2$. Observe that the cases $i = 0, 1$ do not give any new information since $g_0^n(P) = 1$ and $g_1^n(P) = f_0 - (n + 1)$.

(ii) Using rigidity theory Kalai [8] proved that the second entry of the toric g-vector of any polytope P is non-negative, that is, $g_2^n(P) \geq 0$.

(iii) Recently Karu [10] proved the hard Lefschetz theorem for combinatorial intersection cohomology, and as consequence the toric g-vector is non-negative for all polytopes.

(iv) Kalai [9] devised a convolution product so that given two inequalities for m-, respectively n-, dimensional polytopes yields an inequality for $(m + n + 1)$-dimensional polytopes. In Section 2 we describe this convolution.

(v) Stanley [16] proved that the **cd**-index of a polytope P, a non-commutative polynomial in the variables **c** and **d**, has non-negative coefficients. This result was superseded by the next result on this list. Stanley's work played a major role in obtaining this improvement.

(vi) Billera and Ehrenborg [5] proved that the **cd**-index of a polytope is coefficientwise minimized on the simplex, that is, if Δ_n denotes the n-dimensional simplex then we have the coefficientwise inequality $\Psi(P) \geq \Psi(\Delta_n)$ for all n-dimensional polytopes P.

The purpose of this paper is to describe a new lifting technique for polytopal inequalities; see Theorem 3.1. Given a linear inequality on k-dimensional polytopes, we can produce inequalities in dimensions larger than k. For instance, when applying the lifting technique to the minimization inequalities of Billera-Ehrenborg, we obtain a large class of inequalities; see Theorem 3.4. One consequence is that the coefficients of the **cd**-index are increasing when replacing $\mathbf{c}^2$ with $\mathbf{d}$. Hence the **cd**-monomial with the largest coefficient in the **cd**-index of a polytope has no consecutive **c**'s. Another inequality that will generate more inequalities when lifted is the non-negativity of the toric g-vector; see Theorem 4.4.

Using our lifting technique we can now explicitly state the currently best known inequalities for polytopes of low dimensions. Dimension 4 has been described by Bayer [1]. We describe the inequalities for 5-dimensional polytopes in Section 5. Since one can deduce many inequalities by applying the convolution, we only present the irreducible inequalities for polytopes in dimensions 6 through 8. In the last section we discuss open problems and further research.

2 Preliminaries

Let P be an n-dimensional polytope. For $S = \{s_1, \ldots, s_k\}$ a subset of $\{0, 1, \ldots, n-1\}$ define f_S to be the number of flags (chains) of faces $F_1 \subseteq F_2 \subseteq \cdots \subseteq F_k$ such that $\dim(F_i) = s_i$. The 2^n values f_S constitute the flag f-vector of the polytope P. Let **a** and **b** be two non-commutative variables. For S a subset of $\{0, \ldots, n-1\}$ define a polynomial v_S of degree n by letting $v_S = v_0 v_1 \cdots v_{n-1}$ where $v_i = \mathbf{a} - \mathbf{b}$ if $i \notin S$ and $v_i = \mathbf{b}$ otherwise. The **ab**-index $\Psi(P)$ of a polytope P is defined by

$$\Psi(P) = \sum_S f_S \cdot v_S,$$

where S ranges over all subsets of $\{0, \ldots, n-1\}$. The **ab**-index encodes the flag f-vector of a polytope P. Its use is demonstrated by the following theorem, due to Bayer and Klapper [4].

Theorem 2.1 *Let P be polytope. Then the* **ab***-index of P, $\Psi(P)$, can be written in terms of* $\mathbf{c} = \mathbf{a} + \mathbf{b}$ *and* $\mathbf{d} = \mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{a}$.

When $\Psi(P)$ is expressed in terms of **c** and **d**, it is called the **cd**-index. Observe that **c** has degree 1 and **d** has degree 2. Hence there are F_n **cd**-monomials of degree of n, where F_n is the nth Fibonacci number. The flag f-vector information is encoded as the coefficients of these monomials. Also knowing the **cd**-index of a polytope is equivalent to knowing the flag f-vector.

The existence of the **cd**-index is equivalent to the generalized Dehn-Somerville relations due to Bayer and Billera [2]. These relations are all the linear relations that hold among the entries of flag f-vector. The **cd**-monomials offers an explicit linear basis for the subspace cut out by the generalized Dehn-Somerville relations.

In order to discuss inequalities for polytopes, define a bilinear from $\langle \cdot | \cdot \rangle : \mathbb{R}\langle \mathbf{c}, \mathbf{d} \rangle \times \mathbb{R}\langle \mathbf{c}, \mathbf{d} \rangle \rightarrow \mathbb{R}\langle \mathbf{c}, \mathbf{d} \rangle$ by $\langle u|v \rangle = \delta_{u,v}$ for all **cd**-monomials u and v. A linear functional L on the flag f-vectors of n-dimensional polytopes can now be written in terms of the bilinear form as $L(P) = \langle z|\Psi(P)\rangle$, where z is a **cd**-polynomial homogeneous of degree n.

Kalai's convolution is defined as follows; see [9]. Let M and L be two linear functionals on flag f-vectors of m- and n-dimensional polytopes, respectively. Define the linear functional $M * L$ on $(m + n + 1)$-dimensional polytopes P by

$$(M * L)(P) = \sum_{F} M(F) \cdot L(P/F),$$

where F ranges over all m-dimensional faces of P and P/F denotes the face figure of F. It is straightforward to see that if M and L are non-negative on all polytopes, so is their convolution $M * L$.

Kalai's convolution defines a product on $\mathbb{R}\langle \mathbf{c}, \mathbf{d} \rangle$ by

$$\langle z * w|\Psi(P)\rangle = \sum_{F} \langle z|\Psi(F)\rangle \cdot \langle w|\Psi(P/F)\rangle .$$

This product has an explicit expression in terms of **cd**-polynomials. The following result is independently due to Mahajan [11], Reading [13], and Stenson [19].

Proposition 2.2 *For two* **cd***-monomials u and v we have*

$$\begin{aligned} u\mathbf{c} * \mathbf{c}v &= 2 \cdot u\mathbf{c}^3 v + u\mathbf{dc}v + u\mathbf{cd}v, \\ u\mathbf{d} * \mathbf{c}v &= 2 \cdot u\mathbf{dc}^2 v + u\mathbf{d}^2 v, \\ u\mathbf{c} * \mathbf{d}v &= 2 \cdot u\mathbf{c}^2\mathbf{d}v + u\mathbf{d}^2 v, \\ u\mathbf{d} * \mathbf{d}v &= 2 \cdot u\mathbf{dcd}v. \end{aligned}$$

Also we have $1 * 1 = 2 \cdot \mathbf{c}$, $1 * \mathbf{c}v = 2 \cdot \mathbf{c}^2 v + \mathbf{d}v$, $1 * \mathbf{d}v = 2 \cdot \mathbf{cd}v$, $u\mathbf{c} * 1 = 2 \cdot u\mathbf{c}^2 + u\mathbf{d}$ *and* $u\mathbf{d} * 1 = 2 \cdot u\mathbf{dc}$.

We end this section with a direct consequence of Proposition 2.2.

Lemma 2.3 *Let u, q, r and v be four* **cd***-monomials such that u does not end in* **c** *and v does not begin with* **c***. Then the following associative law holds between the product and the convolution:*

$$u \cdot (q * r) \cdot v = (u \cdot q) * (r \cdot v).$$

3 The lifting theorem

We now present our lifting theorem, which allow us to obtain more inequalities on the flag f-vectors of polytopes.

Theorem 3.1 *Assume that the inequality $\langle H|\Psi(P)\rangle \geq 0$ holds for all (rational) polytopes P. Then for all (rational) polytopes P we have the inequality*

$$\langle u \cdot H \cdot v|\Psi(P)\rangle \geq 0,$$

where u and v are **cd***-monomials such that u does not end in* **c** *and v does not begin with* **c***.*

We present two examples of Theorem 3.1.

Example 3.2 We have that $\left\langle \mathbf{c}^k|\Psi(P)\right\rangle = \delta_{k,\dim(P)} \geq 0$. Since every **cd**-monomial w factors into the form $w = \mathbf{c}^k \cdot v$, where v does not begin **c**, we have that $\langle w|\Psi(P)\rangle \geq 0$. This is Stanley's result that the **cd**-index of a polytope has non-negative coefficients; see [16].

The next example shows that it is not necessary to lift inequalities obtained by convolutions. Instead, it is better to first lift each term and then convolute the lifted inequalities.

Example 3.3 Assume that for $i = 1, 2$ we have the inequalities $\langle H_i|\Psi(P)\rangle \geq 0$. Then the lifting of the convoluted inequality gives

$$\langle (u \cdot H_1) * (H_2 \cdot v)|\Psi(P)\rangle = \langle u \cdot (H_1 * H_2) \cdot v|\Psi(P)\rangle \geq 0, \tag{3.1}$$

by Lemma 2.3. Now lift each of the inequalities and then convolute gives

$$\langle (u_1 \cdot H_1 \cdot v_1) * (u_2 \cdot H_2 \cdot v_2)|\Psi(P)\rangle \geq 0. \tag{3.2}$$

Observe that the inequality in (3.1) is a special case of inequality in (3.2).

Billera and Ehrenborg [5] proved that the **cd**-index over all k-dimensional polytopes is coefficientwise minimized on the k-dimensional simplex Δ_k. Letting Δ_q denote the coefficient of q in $\Psi(\Delta_k)$, we can write $[q]\Psi(P) \geq [q]\Psi(\Delta_k) = \Delta_q = \Delta_q \cdot [\mathbf{c}^k]\Psi(P)$. Thus we have $\left\langle q - \Delta_q \cdot \mathbf{c}^k|\Psi(P)\right\rangle \geq 0$. Lifting this inequality we obtain the following result:

Theorem 3.4 *Let P be a polytope of dimension n and let u, q and v be three* **cd***-monomials such that the sum of the degrees of u, q and v is n, the degree of q is k, u does not end with a* **c** *and v does not begin with a* **c***. Then we have*

$$\langle u \cdot q \cdot v|\Psi(P)\rangle \geq \Delta_q \cdot \left\langle u \cdot \mathbf{c}^k \cdot v|\Psi(P)\right\rangle. \tag{3.3}$$

Inequality (3.3) can also be extended to hold for all **cd**-monomials u and v. However this does not give any sharper inequalities.

4 Lifting the toric g-vector

Let us now turn our attention to the toric g-vector. It is defined by a recursion; see for instance Stanley [14, Chapter 3.14]. However we build on the work of Bayer-Ehrenborg who described the toric g-polynomial in terms of the **cd**-index. Recall that the toric g-vector is the coefficients of the g-polynomial, that is,

$$g(P, x) = \sum_{i=0}^{\lfloor n/2 \rfloor} g_i^n(P) \cdot x^i.$$

Before we begin, a few definitions are necessary. Define $p(k, j)$ to denote the difference $\binom{k}{j} - \binom{k-1}{j}$. Also we need two polynomial sequences. First define $Q_k(x)$ by $Q_k(x) = \sum_{j=0}^{\lfloor (k-1)/2 \rfloor} (-1)^j \cdot p(k-1, j) \cdot x^j$. Now define $T_k(x)$ for k odd as $T_k(x) = (-1)^{(k-1)/2} \cdot C_{(k-1)/2} \cdot x^{(k-1)/2}$, where $C_n = p(2n, n)$ denotes the nth Catalan number. For even k, let $T_k(x) = 0$. We are now able to state the result of Bayer and Ehrenborg [3, Theorem 4.2].

Theorem 4.1 *Let g be the linear map from $\mathbb{R}\langle \mathbf{c}, \mathbf{d} \rangle$ to $\mathbb{R}[x]$ such that*

$$g(\mathbf{c}^{k_1}\mathbf{d}\mathbf{c}^{k_2}\mathbf{d} \cdots \mathbf{d}\mathbf{c}^{k_r}\mathbf{d}\mathbf{c}^{k}) = x^r \cdot Q_{k+1}(x) \cdot \prod_{j=1}^{r} T_{k_j+1}(x). \tag{4.1}$$

Then the toric g-polynomial of a polytope P is described by $g(\Psi(P)) = g(P, x)$.

Observe that the entry g_i^n in the toric g-vector is a linear functional on **cd**-polynomials of degree n. Hence we view g_i^n as a homogeneous **cd**-polynomial of degree n such that

$$\langle g_i^n | \Psi(P) \rangle = g_i^n(P),$$

for all n-dimensional polytopes P.

For v a **cd**-monomial of degree $2i$ we define a polynomial $b(v, n)$ in the variable n. If v cannot be written in terms of $\mathbf{c}^2$ and $\mathbf{d}$ then $b(v, n) = 0$. Otherwise let

$$b(v, n) = (-1)^{i-r} \cdot \prod_{j=1}^{r} C_{\ell_j} \cdot p(n - 2i + 2\ell_{r+1}, \ell_{r+1})$$

where $v = \mathbf{c}^{2\ell_1}\mathbf{d}\mathbf{c}^{2\ell_2}\mathbf{d} \cdots \mathbf{d}\mathbf{c}^{2\ell_{r+1}}$.

Theorem 4.2 *The toric* **cd**-*polynomial g_i^n is described by*

$$g_i^n = \left(\sum_{v} b(v, n) \cdot v \right) \cdot \mathbf{c}^{n-2i},$$

where the sum ranges over all **cd**-*monomials v of degree $2i$.*

The three first examples of Theorem 4.2 are $g_0^n = \mathbf{c}^n$, $g_1^n = \mathbf{dc}^n - (n-1)\cdot\mathbf{c}^n$ and

$$g_2^n = \mathbf{d}^2\mathbf{c}^{n-4} - \mathbf{c}^2\mathbf{dc}^{n-4} - (n-3)\cdot\mathbf{dc}^{n-2} + \left(\binom{n-1}{2} - 1\right)\cdot\mathbf{c}^n.$$

Observe that $b(v, 2i) = b(v^*, 2i)$ for v of degree $2i$. From this the classical duality $g_i^{2i} = g_i^{2i*}$ follows.

Proposition 4.3 *The toric* $\mathbf{cd}$*-polynomial* g_i^k *satisfies the following identity*

$$g_i^k \cdot \mathbf{c}^j = \sum_{m=0}^{i} \binom{j+i-m-1}{i-m} \cdot g_m^{k+j}.$$

Applying our main result Theorem 3.1 to $H = g_i^k \cdot \mathbf{c}^j$ we have the following result.

Theorem 4.4 *Let P be a polytope of dimension n, u and v two* $\mathbf{cd}$*-monomials such that u does not end in* $\mathbf{c}$*, the sum of the degrees of u and v is $n-k$ and $2 \le i \le n/2$. Then*

$$\left\langle u \cdot g_i^k \cdot v \middle| \Psi(P) \right\rangle \ge 0.$$

Note that in the theorem we dropped the condition that v does not begin with a $\mathbf{c}$. However, by Proposition 4.3 allowing the monomial v to begin with a $\mathbf{c}$ does not give any sharper inequalities than restricting v to not beginning with a $\mathbf{c}$.

5 Inequalities for 5 through 8-dimensional polytopes

The purpose of this section is to present the currently best-known linear inequalities for polytopes of dimensions 5 through 8. We introduce two notations to simplify the presentation. First we will write $w \ge 0$ instead of the longer $\langle w|\Psi(P)\rangle \ge 0$. Second for a $\mathbf{cd}$-monomial q of degree k let $\{q\}$ denote the polynomial $q - \Delta_q \cdot \mathbf{c}^k$. For instance, inequality (3.3) in Theorem 3.4 can be written as $u \cdot \{q\} \cdot v \ge 0$. Also note that the two inequalities $\{\mathbf{dc}^{n-2}\} \ge 0$ and $\{\mathbf{c}^{n-2}\mathbf{d}\} \ge 0$ are just the classical statements that an n-dimensional polytope has at least $n+1$ vertices, respectively $n+1$ facets.

Before we consider 5 through 8-dimensional polytopes, let us briefly review the lower dimensional cases. (Also observe that we omit Theorem 5.1 in order to keep the numbering consistent with the dimensions.)

Theorem 5.2 *The* **cd**-*index (equivalently the* f*-vector) of a polygon* P *satisfies the inequality*

$$\{\mathbf{d}\} \geq 0 \tag{5.2.1}$$

Theorem 5.3 *The* **cd**-*index (equivalently the* f*-vector) of a 3-dimensional polytope* P *satisfies the following two inequalities*

$$1 * \{\mathbf{d}\} \geq 0 \qquad \{\mathbf{d}\} * 1 \geq 0 \qquad (5.3.1) \quad (5.3.1^*)$$

Theorem 5.3 is due to Steinitz [18]. As mentioned in the introduction, the converse of this theorem is the more interesting part. The best known result for 4-dimensional polytopes is due to Bayer [1]:

Theorem 5.4 *The* **cd**-*index (equivalently the flag* f*-vector) of a 4-dimensional polytope* P *satisfies the following list of six inequalities.*

$$\begin{array}{rcl rcl l l}
\{\mathbf{dc}^2\} & \geq & 0 & \{\mathbf{c}^2\mathbf{d}\} & \geq & 0 & (5.4.1) & (5.4.1^*) \\
g_2^4 & \geq & 0 & & & & (5.4.2) & \\
1 * \{\mathbf{d}\} * 1 & \geq & 0 & & & & (5.4.3) & \\
\mathbf{c} * \{\mathbf{d}\} & \geq & 0 & \{\mathbf{d}\} * \mathbf{c} & \geq & 0 & (5.4.4) & (5.4.4^*)
\end{array}$$

We now list the currently best inequalities for 5-dimensional polytopes.

Theorem 5.5 *The* **cd**-*index of a 5-dimensional polytope* P *satisfies the following list of* 13 *inequalities.*

$$\begin{array}{rcl rcl l l}
\{\mathbf{dc}^3\} & \geq & 0 & \{\mathbf{c}^3\mathbf{d}\} & \geq & 0 & (5.5.1) & (5.5.1^*) \\
1 * \{\mathbf{dc}^2\} & \geq & 0 & \{\mathbf{c}^2\mathbf{d}\} * 1 & \geq & 0 & (5.5.2) & (5.5.2^*) \\
1 * \{\mathbf{c}^2\mathbf{d}\} & \geq & 0 & \{\mathbf{dc}^2\} * 1 & \geq & 0 & (5.5.3) & (5.5.3^*) \\
1 * g_2^4 & \geq & 0 & g_2^4 * 1 & \geq & 0 & (5.5.4) & (5.5.4^*) \\
\mathbf{c} * \{\mathbf{d}\} * 1 & \geq & 0 & 1 * \{\mathbf{d}\} * \mathbf{c} & \geq & 0 & (5.5.5) & (5.5.5^*) \\
\mathbf{c}^2 * \{\mathbf{d}\} & \geq & 0 & \{\mathbf{d}\} * \mathbf{c}^2 & \geq & 0 & (5.5.6) & (5.5.6^*) \\
\{\mathbf{d}\} * \{\mathbf{d}\} & \geq & 0 & & & & (5.5.7) &
\end{array}$$

Before continuing with dimension 6 two observations are needed. First, so far the inequalities have described a cone. From now on, the inequalities we present determines a polyhedron. Second, the number of facets of the polyhedron grows rapidly. Hence we will only list the irreducible inequalities in dimensions 6 through 8, that is, inequalities that cannot be factored using the convolution.

Theorem 5.6 *The* **cd***-index of a* 6*-dimensional polytope* P *satisfies the following list of irreducible inequalities.*

$$\begin{array}{rclrcll}
\{\mathbf{dc}^4\} & \geq & 0 & \{\mathbf{c}^4\mathbf{d}\} & \geq & 0 & (5.6.1) \quad (5.6.1^*) \\
\{\mathbf{c}^2\mathbf{dc}^2\} & \geq & 0 & & & & (5.6.2) \\
\{\mathbf{dc}^2\}\cdot\mathbf{d} & \geq & 0 & \mathbf{d}\cdot\{\mathbf{c}^2\mathbf{d}\} & \geq & 0 & (5.6.3) \quad (5.6.3^*) \\
g_2^6 & \geq & 0 & g_2^{6^*} & \geq & 0 & (5.6.4) \quad (5.6.4^*) \\
g_3^6 & \geq & 0 & & & & (5.6.5)
\end{array}$$

Theorem 5.7 *The* **cd***-index of a* 7*-dimensional polytope* P *satisfies the following list of eight irreducible inequalities.*

$$\begin{array}{rclrcll}
\{\mathbf{dc}^5\} & \geq & 0 & \{\mathbf{c}^5\mathbf{d}\} & \geq & 0 & (5.7.1) \quad (5.7.1^*) \\
\{\mathbf{c}^2\mathbf{dc}^3\} & \geq & 0 & \{\mathbf{c}^3\mathbf{dc}^2\} & \geq & 0 & (5.7.2) \quad (5.7.2^*) \\
\{\mathbf{dc}^3\}\cdot\mathbf{d} & \geq & 0 & \mathbf{d}\cdot\{\mathbf{c}^3\mathbf{d}\} & \geq & 0 & (5.7.3) \quad (5.7.3^*) \\
g_2^7 & \geq & 0 & g_2^{7^*} & \geq & 0 & (5.7.4) \quad (5.7.4^*)
\end{array}$$

Theorem 5.8 *The* **cd***-index of an* 8*-dimensional polytope* P *satisfies the following list of irreducible inequalities.*

$$\begin{array}{rclrcll}
\{\mathbf{dc}^6\} & \geq & 0 & \{\mathbf{c}^6\mathbf{d}\} & \geq & 0 & (5.8.1) \quad (5.8.1^*) \\
\{\mathbf{c}^2\mathbf{dc}^4\} & \geq & 0 & \{\mathbf{c}^4\mathbf{dc}^2\} & \geq & 0 & (5.8.2) \quad (5.8.2^*) \\
\{\mathbf{c}^3\mathbf{dc}^3\} & \geq & 0 & & & & (5.8.3) \\
\{\mathbf{dc}^2\mathbf{dc}^2\} & \geq & 0 & \{\mathbf{c}^2\mathbf{dc}^2\mathbf{d}\} & \geq & 0 & (5.8.4) \quad (5.8.4^*) \\
\{\mathbf{dc}^4\}\cdot\mathbf{d} & \geq & 0 & \mathbf{d}\cdot\{\mathbf{c}^4\mathbf{d}\} & \geq & 0 & (5.8.5) \quad (5.8.5^*) \\
\{\mathbf{dc}^2\}\cdot\mathbf{dc}^2 & \geq & 0 & \mathbf{c}^2\mathbf{d}\cdot\{\mathbf{c}^2\mathbf{d}\} & \geq & 0 & (5.8.6) \quad (5.8.6^*) \\
\{\mathbf{dc}^2\}\cdot\mathbf{d}^2 & \geq & 0 & \mathbf{d}^2\cdot\{\mathbf{c}^2\mathbf{d}\} & \geq & 0 & (5.8.7) \quad (5.8.7^*) \\
\{\mathbf{c}^2\mathbf{d}\}\cdot\mathbf{dc}^2 & \geq & 0 & \mathbf{c}^2\mathbf{d}\cdot\{\mathbf{dc}^2\} & \geq & 0 & (5.8.8) \quad (5.8.8^*) \\
g_2^8 & \geq & 0 & g_2^{8^*} & \geq & 0 & (5.8.9) \quad (5.8.9^*) \\
g_2^6\cdot\mathbf{d} & \geq & 0 & \mathbf{d}\cdot g_2^{6^*} & \geq & 0 & (5.8.10) \quad (5.8.10^*) \\
g_3^8 & \geq & 0 & g_3^{8^*} & \geq & 0 & (5.8.11) \quad (5.8.11^*) \\
g_4^8 & \geq & 0 & & & & (5.8.12)
\end{array}$$

The calculations in Theorems 5.5 through 5.8 were carried out in Maple. We end this section by summarizing some data on these polyhedra.

n	2	3	4	5	6	7	8
$F_n - 1$	1	2	4	7	12	20	33
# facets of the polyhedron	1	2	6	13	29	60	119
# irreducible facets of the polyhedron	1	0	3	2	8	8	22

6 Concluding remarks

Theorem 3.1 produces many new inequalities for us to consider. However, these lifted inequalities does not give an equality when applied to the simplex. Thus it is natural to consider the following generalization of Theorem 3.1.

Conjecture 6.1 *Let H be a* $\mathbf{cd}$*-polynomial such that the inequality $\langle H|\Psi(L)\rangle \geq 0$ holds for all Gorenstein* lattices L. Moreover, let u and v be two* $\mathbf{cd}$*-monomials such that u does not end in* $\mathbf{c}$*, v does not begin with* $\mathbf{c}$ *and they are not both equal to 1. Then the following inequality holds for all Gorenstein* lattices L of rank $n+1$:*

$$\langle u \cdot H \cdot v|\Psi(L) - \Psi(\Delta_n)\rangle \geq 0.$$

This conjecture extends Conjecture 2.7 of Stanley [17].

Two questions deserve a deeper study. First, when is a new inequality new? That is, it is not implied by non-negative linear combinations of known inequalities. Second, when do we stop trying to find linear inequalities. In other words, how do we recognize that we have the smallest polyhedral set containing all flag f-vectors of polytopes?

Recall the two inequalities mentioned in item (0) in the introduction: that an n-dimensional polytope has at least $n+1$ vertices and at least $n+1$ facets. In terms of the $\mathbf{cd}$-monomial basis they are expressed as $\{\mathbf{dc}^{n-2}\} \geq 0$ and $\{\mathbf{c}^{n-2}\mathbf{d}\} \geq 0$. Observe that in dimensions 4 through 8 these two inequalities appear as facets of the polyhedral sets. However, there is only one polytope appearing on these facets, namely the simplex. Hence it is a challenging problem to determine if these inequalities are sharp, or if it is possible to sharpen them.

Also when studying the irreducible facet inequalities in Theorems 5.5 and 5.7 one might suspect that the two inequalities $\langle g_2^5|\Psi(P)\rangle \geq 0$ and $\langle g_3^7|\Psi(P)\rangle \geq 0$ are missing. These inequalities are not facet inequalities. This fact follows from an identity due to Stenson [19], namely

$$(k+2) \cdot g_k^{2k+1} = \sum_{i=0}^{k} (i+1) \cdot g_i^{2i} * g_{k-i}^{2(k-i)}.$$

Moreover, Stenson proved that the inequalities $\langle \mathbf{c}^i\mathbf{dc}^j|\Psi(P) - \Psi(\Delta_n)\rangle \geq 0$, where $i, j \geq 2$ and $i+j+2 = n$, are not implied by convolutions of the non-negativity of the toric g-vector. These inequalities are expressed as $\{\mathbf{c}^i\mathbf{dc}^j\} \geq 0$ in Theorems 5.6 through 5.8.

Meisinger, Kleinschmidt and Kalai proved that a 9-dimensional rational polytope has a three-dimensional face that has less than 78 vertices or less than 78 faces [12]. However, with the recent proof that the entries in the toric g-vector are non-negative [10], their result now extends to all polytopes. Their proof uses the following observation. Assume that P is a 9-dimensional polytope with every three-dimensional face having at least m vertices and at least m faces. If the inequality $\langle L|\Psi(Q)\rangle \geq 0$ holds for all 5-dimensional polytopes then the two inequalities

$$\big\langle (\mathbf{dc} - (m-2)\cdot \mathbf{c}^3) * L|\Psi(P)\big\rangle \geq 0 \quad \text{and} \quad \big\langle (\mathbf{cd} - (m-2)\cdot \mathbf{c}^3) * L|\Psi(P)\big\rangle \geq 0$$

also hold. Hence consider the system of linear inequalities

$$\begin{cases} \langle (\mathbf{dc} - 76\mathbf{c}^3) * L|z\rangle & \geq & 0, \\ \langle (\mathbf{cd} - 76\mathbf{c}^3) * L|z\rangle & \geq & 0, \\ \langle K|z\rangle & \geq & 0, \end{cases},$$

where L ranges over linear inequalities for 5-dimensional polytopes and K ranges over linear inequalities for 9-dimensional polytopes. They showed that this system is infeasible which implies that there is no 9-dimensional polytope with all its three-dimensional faces having at least 78 vertices and at least 78 faces. Using this technique and the inequalities derived from Theorem 3.1, we were able to improve upon the constant 78.

Theorem 6.2 *A 9-dimensional polytope has a three-dimensional face that has less than 72 vertices or less than 72 faces.*

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