KAZHDAN-LUSZTIG POLYNOMIALS FOR BOOLEAN ELEMENTS IN LINEAR COXETER SYSTEMS

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ABSTRACT. Kazhdan-Lusztig polynomials have been proven to play an important role in different fields. Despite this, there are still few explicit formulas for them. Here we give closed product formulas for the R-polynomials and for the Kazhdan-Lusztig polynomials when they are indexed by Boolean elements. Boolean elements are elements smaller than a reflection that admits a reduced expression of the form $s_1 \dots s_{n-1} s_n s_{n-1} \dots s_1$ ($s_i \in S$, $s_i \neq s_j$ if $i \neq j$). Then we provide several applications of this result concerning the combinatorial invariance of the Kazhdan-Lusztig polynomials, the classification of the pairs (u, v) with $u \prec v$, the Kazhdan-Lusztig elements and the intersection homology Poincaré polynomials of the Schubert varieties.

RÉSUMÉ. Les polynômes de Kazhdan-Lusztig jouent un rôle important dans differents domaines mathématiques. Cependant, il n'existe pour ceux-ci encore que peu de formules explicites. Nous donnons ici des formules de produit pour les R-polynômes et pour les polynômes de Kazhdan-Lusztig lorsqu'ils sont indexés par des éléments booléens. Les éléments booléens sont des éléments inférieurs à une réflection qui admet une expression réduite de la forme $s_1 \dots s_{n-1} s_n s_{n-1} \dots s_1$ ($s_i \in S$, $s_i \neq s_j$ if $i \neq j$). Ensuite nous donnons quelques applications de ce résultat concernant l'invariance combinatoire des polynômes de Kazhdan-Lusztig, la classification des couples (u, v) avec $u \prec v$, les éléments de Kazhdan-Lusztig et les polynômes de Poincaré de l'homologie d'intersection des variétés de Schubert.

1. INTRODUCTION

In their main work of 1979 [17] Kazhdan and Lusztig introduced a family of polynomials in one variable, indexed by pairs of elements in an arbitrary Coxeter group, which soon became well known thanks to their applications in different contexts such as in the geometry of Schubert varieties and in representation theory [13, 15, 5, 2, 18]. In order to prove the existence of these polynomials, now known as the Kazhdan-Lusztig polynomials, another family of polynomials was defined, the R-polynomials, which are related to the multiplicative structure of the Hecke algebra associated to the Coxeter system and are important because their knowledge allows the computation of the Kazhdan-Lusztig polynomials.

In the past 20 years, many efforts have been made to compute some classes of Kazhdan-Lusztig polynomials and to give explicit closed formulas for the R-polynomials (see [1, 3, 7, 8, 10, 19, 20, 22]). Here, for any Coxeter group W, we compute all the R-polynomials indexed by Boolean elements, which are elements smaller than a Boolean reflection, that is a reflection that admits a reduced expression of the type $s_1 \dots s_{n-1} s_n s_{n-1} \dots s_1$, with $s_i \in S$ for all i and $s_i \neq s_j$ if $i \neq j$. Furthermore, under the same hypotheses on the pair of elements, we find a closed product formula for the Kazhdan-Lusztig polynomials in the class of the linear Coxeter systems (for the definition see the end of Section 2), which includes some of the most important Coxeter systems, such as those of type A_n, B_n, F_n, H_n, $\tilde{A}_n$, $\tilde{C}_n$, $I_2(m)$. Thanks to this result, we can compute the Kazhdan-Lusztig polynomial $P_{u,v}$ very easily from certain reduced expressions of u and v and list explicitly all the pairs (u, v) of Boolean elements with $u \prec v$, namely with $\overline{\mu}(u, v) \neq 0$. This result can be useful also for the computation of other classes of Kazhdan-Lusztig polynomials since the function $\overline{\mu}$ is often the main obstacle in their recursive property (see, for example, [11, 12]). Moreover this formula turns out to have several other consequences. It allows us to compute and factorize the Kazhdan-Lusztig elements and the intersection homology Poincaré polynomials F_v indexed by Boolean elements, and to prove one of the most challenging conjectures of this theory, due to Dyer and Lusztig, that states that isomorphic intervals have the same Kazhdan-Lusztig polynomials; this, of course, under the restrictive hypothesis that the elements are Boolean. In all these results, (W, S) can be any linear Coxeter system except in the last one, where (W, S) is supposed to be strictly linear.

2. Preliminaries

This section reviews the background material on Coxeter systems that is needed in the rest of this work. We let $\mathbf{P} = \{1, 2, 3, \dots\}$, $\mathbf{N}=\mathbf{P} \cup \{0\}$, $\mathbf{Z}$ be the set of integers; for $a, b \in \mathbf{N}$, we let $[a, b] = \{a, a+1, a+2, \dots, b\}$ (where $[a, b] = \emptyset$ if $a \not\leq b$) and $[a] = [1, a]$.

We follow [16] for general Coxeter system notation and terminology. In particular, given a Coxeter system (W, S) and $u \in W$, we denote by $l(u)$ the length of u, with respect to S, and we let $D_L(u) = \{s \in S : \ l(su) < l(u)\}$, $D_R(u) = \{s \in S : \ l(us) < l(u)\}$ and $T(W) = \{usu^{-1} : s \in S, u \in W\}$ (the set of reflections of W). We denote by e the identity of W and by $m(s, s')$ the order of the product ss' (write ∞ if this is not finite). We will always assume that W is partially ordered by (strong) Bruhat order. Recall (see [16] §5.9) that $u \leq v$ means that there exist $t_1, \dots, t_r \in T(W)$ such that $t_r \dots t_1 u = v$ and $l(t_i \dots t_1 u) > l(t_{i-1} \dots t_1 u)$ for $i = 1, \dots, r$. It is well known that $u \leq v$ if and only if for any (equivalently every) reduced expression of v there exists a reduced expression of u which is a subword of it. We let $l(u, v) = l(v) - l(u)$ and $[u, v]_W = \{z \in W : u \leq z \leq v\}$, and we write $[u, v]$ when no confusion arises. Given a set G, we denote by $|G|$ its cardinality, and we let $S(G)$ be the set of all bijections $\pi : G \to G$ and $S_n = S([n])$. It is well known that (S_n, S), where $S = \{(1, 2), (2, 3), \dots, (n-1, n)\}$, is a Coxeter system, that $T(S_n) = \{(i, j) : 1 \leq i < j \leq n\}$ and that every transposition (i, j) admits $s_i s_{i+1} \dots s_{j-2} s_{j-1} s_{j-2} \dots s_{i+1} s_i$ as a reduced expression, where $s_k = (k, k+1)$. We abuse notation by referring to this Coxeter system simply by S_n.
The proof of the following fundamental result can be found in [16] §5.8.

Theorem 2.1 (Exchange Property). *Let $w \in W$, $s_1, s_2, \dots, s_r \in S$, $w = s_1 s_2 \dots s_r$ where this expression is reduced. Let $t \in T(W)$ be such that $l(wt) < l(w)$. Then there exists a unique $i \in [r]$ such that $wt = s_1 s_2 \dots \hat{s}_i \dots s_r$ (where $\hat{s}_i$ means that s_i has been omitted). In particular, if $t \in S$, this $i \in [r]$ is such that $s_{i+1}s_{i+2}\dots s_r s$ is reduced while $s_i s_{i+1} \dots s_r s$ is not.*

We now recall a result due to J. Tits [23] that later will be very useful. Given $s, s' \in S$ such that $m(s,s') < \infty$, let $\alpha_{s,s'} = \underbrace{ss'ss'\dots}_{m(s,s')}$, with exactly $m(s,s')$ letters. Two expressions are said to be linked by a braid move (respectively a nil move) if it is possible to obtain the first from the second by changing a factor $\alpha_{s,s'}$ to a factor $\alpha_{s',s}$ (respectively by deleting a factor ss).

Theorem 2.2 (Tits' Word Theorem). *Let $u \in W$. Then:*

- *any two reduced expressions of u are linked by a finite sequence of braid moves;*
- *any expression of u (not necessarily reduced) is linked to any reduced expression of u by a finite sequence of braid and nil moves.*

The Kazhdan-Lusztig polynomials were originally introduced in terms of the Hecke algebra ([17]). Given any Coxeter system (W,S), the Hecke Algebra $\mathcal{H}$ of W over $\mathbf{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$ is the free $\mathbf{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$-module with basis $\{T_w : w \in W\}$ and multiplication defined by:

$$T_s T_w = \begin{cases} T_{sw}, & \text{if } sw > w, \\ (q-1)T_w + qT_{sw}, & \text{if } sw < w. \end{cases}$$

Every T_w turns out to be invertible; as $l(w)$ increases, however, the expression of the inverse gets more and more complicated and this is the reason why the family $\{R_{u,v}(q)\}$ of R-polynomials was defined, essentially as its coordinates with respect to the canonical basis of $\mathcal{H}$. More precisely, $\{R_{u,v}(q)\}_{u,v\in W} \subseteq \mathbf{Z}[q]$ is the unique family of polynomials satisfying

$$(T_{w^{-1}})^{-1} = (-1)^{l(w)} q^{-l(w)} \sum_{u \leq w} (-1)^{l(u)} R_{u,w} T_u,$$

for all $w \in W$.
Define $i(q^{\frac{1}{2}}) = q^{-\frac{1}{2}}$ and $i(T_w) = (T_{w^{-1}})^{-1}$ and combine these assignments to obtain a ring automorphism $i : \mathcal{H} \to \mathcal{H}$, which is clearly an involution. We have the following:

Theorem 2.3. *There exists a unique basis $\mathcal{C} = \{C_w : w \in W\}$ of $\mathcal{H}$ such that:*

1. $i(C_w) = C_w$;
2. $C_w = q^{-\frac{l(w)}{2}} \sum_{u \leq w} P_{u,w}(q) T_u$;
3. $P_{u,w} \in \mathbf{Z}[q]$ *has degree at most* $\frac{1}{2}(l(u,w) - 1)$ *if* $u \neq w$, *and* $P_{w,w} = 1$.

The elements of the basis $\mathcal{C}$ are currently called Kazhdan-Lusztig elements while the polynomials $\{P_{u,v}(q)\}_{u,v\in W} \subseteq \mathbf{Z}[q]$ are the well known Kazhdan-Lusztig polynomials, or P-polynomials. As the coefficient of $q^{\frac{1}{2}(l(u,v)-1)}$ in $P_{u,v}(q)$ plays a very important role, we denote it, as customary, by $\overline{\mu}(u,v)$ and we write $u \prec v$ if $\overline{\mu}(u,v) \neq 0$. The following proposition deals with the multiplication of the Kazhdan-Lusztig elements and hence gives a recursive formula to compute them.

Proposition 2.4. *Let $s \in S$. Then*

$$C_s C_w = \begin{cases} C_{sw} + \sum_{s \in D_L(z)} \overline{\mu}(z,w) C_z, & \text{if } sw > w, \\ (q^{\frac{1}{2}} + q^{-\frac{1}{2}}) C_w, & \text{if } sw < w, \end{cases}$$

for all $w \in W$.
Hence, given $v \in W$ and $s \in D_L(v)$, we have that

$$C_v = C_s C_{sv} - \sum_{z : s \in D_L(z)} \overline{\mu}(z, sv) C_z.$$

Both R-polynomials and Kazhdan-Lusztig polynomials could be equivalently introduced in a purely combinatorial way through the following results.

Theorem 2.5. *Let (W,S) be a Coxeter system. Then there is a unique family of polynomials $\{R_{u,v}(q)\}_{u,v \in W} \subseteq \mathbf{Z}[q]$ satisfying the following conditions:*

1. $R_{u,v}(q) = 0$ *if* $u \not\leq v$;
2. $R_{u,u}(q) = 1$;
3. *if* $s \in D_L(v)$ *then*

$$R_{u,v}(q) = \begin{cases} R_{su,sv}(q), & \text{if } s \in D_L(u), \\ q R_{su,sv}(q) + (q-1) R_{u,sv}(q), & \text{if } s \notin D_L(u). \end{cases}$$

Theorem 2.6. *Let (W,S) be a Coxeter system. Then there is a unique family of polynomials $\{P_{u,v}(q)\}_{u,v \in W} \subseteq \mathbf{Z}[q]$ satisfying the following conditions:*

1. $P_{u,v}(q) = 0$ *if* $u \not\leq v$;
2. $P_{u,u}(q) = 1$;
3. $deg(P_{u,v}(q)) \leq \frac{1}{2}(l(u,v) - 1)$, *if* $u < v$;
4. *if* $u \leq v$

$$q^{l(u,v)} P_{u,v}\left(\frac{1}{q}\right) = \sum_{u \leq z \leq v} R_{u,z}(q) \, P_{z,v}(q) \, .$$

The following results will be useful because they will enable us to compute the Kazhdan-Lusztig polynomials by induction.

Theorem 2.7. *Let (W,S) be a Coxeter system, $u, v \in W$, $u \leq v$, and $s \in D_L(v)$. Then*

$$P_{u,v}(q) = q^{1-c} P_{su,sv}(q) + q^c P_{u,sv}(q) - \sum_{z : s \in D_L(z)} q^{\frac{l(z,v)}{2}} \overline{\mu}(z, sv) P_{u,z}(q),$$

where $c = \begin{cases} 1, & \text{if } s \in D_L(u), \\ 0, & \text{otherwise.} \end{cases}$

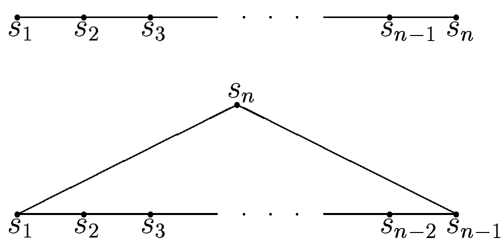
Corollary 2.8. *Let (W,S) be a Coxeter system, $u, v \in W$, $u < v$, and $s \in D_L(v)$. Then $P_{u,v}(q) = P_{su,v}(q)$.*

Proposition 2.4, Theorem 2.5, Theorem 2.7 and Theorem 2.8 can also be reformulated in right versions. We refer to [4, 16, 17] for proofs of these results and for more details concerning general Coxeter group theory.

In order to simplify notation, we introduce the following definitions. We call a reflection $t \in T(W)$ *Boolean* if it admits a reduced expression $s_1 \dots s_{n-1} s_n s_{n-1} \dots s_1$ with $s_h \in S$ for all $h \in [n]$ and $s_i \neq s_j$ if $i \neq j$. Call such expressions Boolean and note that a Boolean reflection can admit different Boolean expressions. Moreover, we call an element $w \in W$ *Boolean* if $w \leq t$, where t is a Boolean reflection. If $W = S_n$, by what we have already said, every reflection is Boolean and is smaller than the reflection $(1, n)$. So $w \in S_n$ is a Boolean element if and only if $w \leq (1, n)$.
Finally, we call an irreducible Coxeter system *linear* if it has Coxeter graph with no branch points, that is if it is isomorphic, for a certain n, to a Coxeter system $(W, S = \{s_1, \dots, s_n\})$ with:

$$\begin{cases} m(s_i, s_j) = 2, & \text{if } i \neq j, j \pm 1 \text{ and } (i,j) \neq (1,n), \\ m(s_i, s_j) \geq 3, & \text{if } i = j \pm 1. \end{cases}$$

(strictly linear if also $m(s_1, s_n) = 2$, non-strictly otherwise). These are the Coxeter graphs associated respectively to a strictly and to a non-strictly linear Coxeter system:



(all the edges may have any weight). This class includes many important Coxeter systems such as those of type A, B, F, H, $\tilde{C}$, $I(m)$ (which are strictly linear) and $\tilde{A}$. In particular S_n (type A) is linear.

3. Preliminary Lemmas

In this section we give some lemmas that are essential for working with Boolean elements.

Lemma 3.1. *Given a Coxeter system (W, S), let $s, t_1, \dots, t_n \in S$, $s \neq t_i$ for all $i \in [n]$, and $l(t_1 \dots t_n) = n$. Furthermore, let $t_{i_1} \dots t_{i_h}$ be a reduced subword of $t_1 \dots t_n$ such that $s t_{i_1} \dots t_{i_h} \leq t_1 \dots t_n s$. Then s commutes with each $t_{i_1}, \dots, t_{i_h}$.*

The following lemma essentially says what one gains in Tits' Word Theorem by adding the hypothesis that the element $u \in W$ is Boolean. A short braid move is, by definition, a braid move of the shortest type (namely $\alpha_{s,s'} = ss'$). Given any $s \in S$ and any word $\overline{v} \in S^*$ (where S^* denotes the free monoid on the set S), we denote by $\overline{v}(s)$ the number of occurrences of s in the word $\overline{v}$.

Lemma 3.2. *Given a Coxeter system (W, S), let $u \in W$ be a Boolean element and let $\overline{u}$ be a reduced expression of u which is subword of the Boolean expression $s_1 \dots s_n \dots s_1$. Then:*

- *any other reduced expression $\underline{u}$ of u which is a subword of $s_1 \dots s_n \dots s_1$ is linked to $\overline{u}$ by a sequence of short braid moves;*

- *any expression $\underline{u}$ of u (not necessarily reduced) which is a subword of $s_1 \dots s_n \dots s_1$ is linked to $\overline{u}$ by a sequence of short braid and nil moves.*

Corollary 3.3. *Given a Coxeter system (W,S), let $\overline{u}$, $\underline{u}$ be two reduced expressions of the same Boolean element $u \in W$ which are both subwords of a Boolean expression $s_1 \dots s_n \dots s_1$. Then $\overline{u}(s_i) = \underline{u}(s_i)$ for all $i \in [n]$.*

n-Boolean sequences. Let us specialize to the case $W = S_{n+1}$, where an element u is Boolean if and only if $u \leq (1, n+1)$. After Corollary 3.3, we denote by u_i the number of occurrences of s_i in any reduced expression of u which is a subword of the Boolean expression $s_1 \dots s_n \dots s_1$ of $(1, n+1)$. It is sometimes convenient to handle Boolean elements in terms of sequences. So we introduce a well-defined surjective map ϕ from the interval $[e,(1,n+1)]_{S_n}$ to the set of the n-Boolean sequences by sending u to $(u_1, \dots, u_n)$. A n-Boolean sequence is a sequence $(x_1, \dots, x_n)$ of n numbers chosen in $\{0,1,2\}$ that avoids the pattern $|2,0|$, where $|2,0|$-avoidance means that there does not exist an $i \in [n-1]$ such that $(x_i, x_{i+1}) = (2,0)$ and that $x_n \neq 2$. All the properties are easily checked.
Given a n-Boolean sequence $x = (x_1, \dots, x_n)$, we define:

$$\begin{aligned} l(x) &= \sum_{i \in [n]} x_i, \\ p(x) &= |\{i \in [n-1] : x_i = 1,\ x_{i+1} \neq 0\}|. \end{aligned}$$

Then the cardinality of the preimage of x is equal to $2^{p(x)}$ and $l(u) = l(\phi(u))$ for all $u \in [e,(1,n+1)]$.
If we endow the range with the component-wise partial order, ϕ is a morphism of posets.

Finally, we state the following technical propositions that are easy to prove and where we assume that the Coxeter systems have Coxeter graphs of the types drawn in Section 2.

Proposition 3.4. *Let (W,S) be a strictly linear Coxeter system and let $t \in W$ be a Boolean reflection. Then t admits a Boolean expression of one of the following types:*

(1) $s_a s_{a-1} \dots s_{i+1} s_b s_{b+1} \dots s_{i-1} s_i s_{i-1} \dots s_{b+1} s_b s_{i+1} \dots s_{a-1} s_a$,
(2) $s_b s_{b+1} \dots s_{i-1} s_a s_{a-1} \dots s_{i+1} s_i s_{i+1} \dots s_{a-1} s_a s_{i-1} \dots s_{b+1} s_b$,

for appropriate $0 < b \leq i \leq a \leq n$.

Proposition 3.5. *Let $(W, S = \{s_1, \dots, s_n\})$ be a non-strictly linear Coxeter system and let $t \in W$ be a Boolean reflection. Then, up to a "rotation" of the indices of the generators (that is up to adding a fixed $r \in [n-1]$ to their indices and taking the indices modulo n), t admits a Boolean expression of one of the following types:*

(1) $s_a s_{a-1} \dots s_{i+1} s_b s_{b+1} \dots s_{i-1} s_i s_{i-1} \dots s_{b+1} s_b s_{i+1} \dots s_{a-1} s_a$,
(2) $s_b s_{b+1} \dots s_{i-1} s_a s_{a-1} \dots s_{i+1} s_i s_{i+1} \dots s_{a-1} s_a s_{i-1} \dots s_{b+1} s_b$,

for appropriate $0 < b \leq i \leq a \leq n$. If $s_i \leq t$ for all $i \in [n]$, we can assume $a \neq (i+1)$ in (1), $b \neq (i-1)$ in (2).

4. Main Results

We are now prepared to tackle the main results of this paper. Their proofs, that we omit because rather technical, are based on the recursive properties of Theorems 2.5 and 2.7, and on the lemmas in Section 3.

Theorem 4.1. *Given a Coxeter system (W, S), let $u, v \in W$ be Boolean elements, $u \leq v$. Fix a reduced expression $\overline{v}$ of v which is a subword of a Boolean expression $s_1 \dots s_n \dots s_1$ and a reduced expression $\overline{u}$ of u which is a subword of $\overline{v}$. Then*

$$R_{u,v}(q) = (q-1)^{l(u,v)-2a}(q^2-q+1)^a,$$

where

$$a = |\{i \in [n] : \begin{array}{l} \overline{v}(s_i) = 2 \\ \overline{u}(s_i) = 0 \end{array} \text{ and } m(s_i, s_j) = 2,\ \forall j > i \text{ such that } \overline{u}(s_j) \neq 0\}|.$$

If $W = S_{n+1}$, this means that:

$$a = |\{i \in [n] : \begin{array}{ll} v_i = 2 & \\ u_i = 0 & u_{i+1} = 0 \end{array} \}|.$$

Theorem 4.2. *Let u and v be Boolean elements in S_{n+1}, $u \leq v$. Then*

$$P_{u,v}(q) = (1+q)^b,$$

where

$$b = |\{k \in [n] : \begin{array}{ll} v_k = 2 & v_{k+1} = 2 \\ & u_{k+1} = 0 \end{array} \}|.$$

We illustrate Theorems 4.1 and 4.2 with an example.

Example. Let $W = S_8$, $u = s_1 s_5 s_7$ and $v = s_1 s_2 s_3 s_4 s_5 s_6 s_7 s_6 s_5 s_3 s_2 s_1$. Then

$$\phi(v) = (2, 2, 2, 1, 2, 2, 1)$$

$$\phi(u) = (1, 0, 0, 0, 1, 0, 1).$$

There are exactly 2 sub-tableaux of the type

$$\begin{array}{|c|c} \hline 2 & \multicolumn{1}{|c}{} \\ \hline 0 & \multicolumn{1}{|c|}{0} \\ \hline \end{array}$$

and 3 sub-tableaux of the type

$$\begin{array}{|c|c|} \hline 2 & 2 \\ \hline \multicolumn{1}{c|}{} & 0 \\ \cline{2-2} \end{array}$$

in

$$\begin{array}{|c|c|c|c|c|c|c|} \hline 2 & 2 & 2 & 1 & 2 & 2 & 1 \\ \hline 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ \hline \end{array}.$$

Therefore $R_{u,v}(q) = (q-1)^5(q^2-q+1)^2$ and $P_{u,v}(q) = (1+q)^3$.

Now we extend the last result to other Coxeter systems. We show how Theorem 4.1 and Theorem 4.2, in conjunction with Lemma 3.2, imply the result for the strictly linear Coxeter systems. First we need the following lemma, where we use the same symbols $s_1, \dots, s_m$ for both the generators of W and the generators of S_{m+1}.

Lemma 4.3. *Let $(W, S = \{s_1, \dots, s_m\})$ be a strictly linear Coxeter system. Let $t \in W$ be a Boolean reflection with Boolean expression $\overline{t}$. Consider the map $\psi:[e,t]_W \longrightarrow S_{m+1}$ defined as follows: if $z \in [e,t]_W$ admits the reduced expression $\overline{z}$ which is a subword of $\overline{t}$, then $\psi(z)$ is the element of S_{m+1} represented by the same expression $\overline{z}$. Then ψ is an isomorphism of posets from $[e,t]_W$ to $[e,\psi(t)]_{S_{m+1}}$.*

Proof. The map ψ is well defined: in fact, by Lemma 3.2, any two such reduced expression of the same $z \in W$ are linked by short braid moves, and W and S_{m+1} share the same short braid moves. Moreover, the expression $\overline{t} = t_1 \dots t_{n-1} t_n t_{n-1} \dots t_1$ is reduced also in S_{m+1}. In fact, suppose, by contradiction, that there exists $k \in [n]$ such that $t_1 \dots t_{n-1} t_n t_{n-1} \dots t_{k-1}$ is reduced while $t_1 \dots t_{n-1} t_n t_{n-1} \dots t_k$ is not. Then, clearly, $t_k \dots t_{n-1} t_n t_{n-1} \dots t_k$ is not reduced (by hypothesis, $t_i \neq t_j$ if $i \neq j$). Hence, by Lemma 3.1, t_k commutes with t_j for all $j > k$ in S_{m+1}, and so also in W, and this is a contradiction because $\overline{t}$ is reduced in W. This means that $\overline{t}$ is a Boolean expression of the Boolean reflection $\psi(t)$ of S_{m+1}. Now Lemma 3.2 implies that $l(z) = l(\psi(z))$, for all $z \in [e,t]_W$, and that ψ is an isomorphism of posets from $[e,t]_W$ to $[e,\psi(t)]_{S_{m+1}}$ by the characterization of the Bruhat order in terms of reduced expressions. □

Theorem 4.4. *Let $(W, S = \{s_1, \dots, s_m\})$ be a strictly linear Coxeter system. Let $u, v \in W$ be such that $u \leq v \leq t$, where t is a Boolean reflection. Then*

$$P_{u,v}(q) = P_{\psi(u),\psi(v)}(q),$$

where ψ is as in Lemma 4.3, and $P_{\psi(u),\psi(v)}(q)$ can be computed as in Theorem 4.2.

Proof. First of all we fix a Boolean expression $\overline{t}$ of t, a reduced expression $\overline{v}$ of v which is a subword of $\overline{t}$ and a reduced expression $\overline{u}$ of u which is a subword of $\overline{v}$.
Recall that, if an element z has a reduced expression $\overline{z}$ which is a subword of $\overline{v}$, then the map ψ sends z to the element of S_{m+1} represented by the same expression $\overline{z}$. Theorem 4.1 shows that the R-polynomials depend only on the chosen reduced expression and on the commutation relations between the generators of the Coxeter system. So for all $x, y \in [u,v]_W$, $R_{x,y}(q) = R_{\psi(x),\psi(y)}(q)$. Finally property (4) of Theorem 2.6, in conjunction with Lemma 4.3, implies that the same equality holds also for the P-polynomials. □

Example. Let $(W, S = \{s_1, s_2, s_3, s_4\})$ be a strictly linear Coxeter system and let $v = s_4 s_1 s_2 s_3 s_2 s_1 s_4$, $u = s_4 s_1$. Then $\psi(v) = s_4 s_1 s_2 s_3 s_2 s_1 s_4 = s_1 s_2 s_3 s_4 s_3 s_2 s_1 \in S_5$, $\psi(u) = s_4 s_1 \in S_5$, and $P_{u,v}(q) = P_{\psi(u),\psi(v)}(q) = (1+q)^2$.

The following result deals with the non-strictly linear Coxeter systems.

Theorem 4.5. *Let $(W, S = \{s_1, \dots, s_m\})$ be a non-strictly linear Coxeter system. Let $u, v \in W$ be such that $u \leq v \leq t$ where t is a Boolean reflection that we can assume such that $s_i \leq t$ for all $i \in [m]$. Then there exists $b \in \mathbf{N}$ such that:*

$$P_{u,v}(q) = (1+q)^b.$$

Fix a Boolean expression $\overline{t} = t_1 \dots t_{n-1} t_n t_{n-1} \dots t_1$ for t of the type shown in Proposition 3.5, a reduced expression $\overline{v}$ of v which is a subword of $\overline{t}$ and a reduced expression $\overline{u}$

of u which is a subword of $\overline{v}$. Suppose that t_j is, together with t_2, the only other generator that does not commute with t_1. Then $P_{u,v}(q) = (1+q)^{b'} P_{u',v'}(q)$, where u' and v' are the elements represented by the expressions we obtain by erasing all the letters t_1 in $\overline{u}$ and $\overline{v}$, and where

$$b' = \begin{cases} 1, & \text{if } \overline{v}(t_1) = 2,\ \overline{u}(t_2) = 0 \text{ and } \overline{u}(t_j) = 0; \\ 0, & \text{otherwise.} \end{cases}$$

Then compute $P_{u',v'}(q)$ as in Theorem 4.4 (there are no longer occurrences of t_1).

Remarks.

(1) If the Coxeter system is not irreducible, (and $S = \bigcup S_i$ is the decomposition into irreducible components), the expression $t_1 \dots t_{n-1} t_n t_{n-1} \dots t_1$ is reduced only if all the generators t_j belong to the same S_i.

(2) If $W = S_n$, it is easy to see that a Boolean permutation v is always covexillary (3412 avoiding). Therefore, the polynomial $P_{u,v}$ can also be computed using the algorithm given in [19]. However, it seems to be difficult to derive the explicit formulas of Theorem 4.2 from this algorithm if $v < (1, n)$.

(3) The results in this section do not hold for general Coxeter systems. In fact, let (W, S) be a Coxeter system such that S contains s_1, s_2, s_3 and r with $m(s_i, s_j) = 2$ for all $i \neq j$, $m(s_i, r) \geq 3$ for all i. Then $P_{u,v}(q) = 1 + 2q$, where $v = s_1 s_2 r s_3 r s_2 s_1$, $u = s_3 s_2 s_1$.

5. Combinatorial Invariance

In this section we show that, given u and v Boolean elements in a strictly linear Coxeter system W, the polynomial $P_{u,v}$ can be easily computed from $l(u,v)$, $c_1(u,v)$ and $c_2(u,v)$, where

$$\begin{aligned} c_i(u,v) &= |C_i(u,v)|, \\ C_i(u,v) &= \{z \in [u,v] : l(z,v) = i\}, \end{aligned}$$

for $i = 1, 2$. The elements of $C_1(u,v)$ are the coatoms of $[u,v]$.
This result is in the spirit of the following long standing conjecture.

Conjecture 5.1 (Dyer,Lusztig). *Let (W,S) be a Coxeter system, u, $v \in W$. Then $P_{u,v}(q)$ (equivalently $R_{u,v}(q)$) depends only on the isomorphism type of the interval $[u,v]$ as a poset.*

Let $g_i(u,v) = |G_i(u,v)|$ and $h_i(u,v) = |H_i(u,v)|$, where

$$\begin{aligned} G_i(u,v) &= \{z \in [u,v] : z^{-1}v \in T(W), l(z,v) = (1+2i)\}, \\ H_i(u,v) &= \{z \in [u,v] : u^{-1}z \in T(W), l(u,z) = (1+2i)\}, \end{aligned}$$

for all possible $i \in \mathbf{N}$. Thanks to the following theorem due to Dyer [14], they are all combinatorial invariants of $[u,v]$ as a poset.

Theorem 5.2. *Let (W,S) be a Coxeter system, $u, v \in W$. The isomorphism type of the poset $[u,v]$ determines the isomorphism type of its Bruhat graph. This is the direct graph* *having* $[u, v]$ *as vertex set and where, for any* $x, y \in [u, v]$, $x \to y$ *if and only if* $l(x) < l(y)$ *and* $x^{-1}y \in T(W)$.

Conjecture 5.1 has till now been proven only in particular case. The most convincing result, due to Brenti [9], deals with the pairs (u, v) of elements in S_n in which $u = e$. If $l(u, v) \leq 4$, Dyer proved explicit formulae for $R_{u,v}(q)$ depending only on the $g_i(u, v)$ and the $h_i(u, v)$. Here we show that, in general, the $g_i(u, v)$ and the $h_i(u, v)$ do not determine the Kazhdan-Lusztig polynomial. The smallest S_n in which we can find a counterexample for Boolean elements is S_{10}.

Let us enunciate explicitely the results. We write, respectively, $a(u, v)$ and $b(u, v)$ for the exponents in Theorem 4.1 and in Theorem 4.2, and we always omit the dependence on (u, v) when no confusion arises.

Theorem 5.3. *Let* (W, S) *be a strictly linear Coxeter system,* u *and* v *be Boolean elements of* W. *Then* $R_{u,v}(q) = (q-1)^{l-2a}(q^2 - q + 1)^a$ *and* $P_{u,v}(q) = (1+q)^b$ *where*

$$\begin{aligned} a &= 2l + \frac{c_1}{2}(c_1 - 5) - c_2, \\ b &= l + \frac{c_1}{2}(c_1 - 3) - c_2. \end{aligned}$$

Proof. After Lemma 3.2, we can compute explicitely c_1 and c_2, and find their relation with a and b by Theorem 4.1 and Theorems 4.2 and 4.4. □

Example. Let $W = S_{10}$,

$$\begin{aligned} &v = s_1s_2s_3s_4s_5s_6s_7s_8s_9s_4s_3s_2s_1, \quad && v' = s_1s_2s_3s_4s_5s_6s_7s_8s_9s_8s_7s_5s_4s_2s_1, \\ &u = s_1s_4, && u' = s_1s_4s_7s_9. \end{aligned}$$

Then

$$\begin{aligned} \phi(v) &= (2,2,2,2,1,1,1,1,1) \quad & \phi(v') &= (2,2,1,2,2,1,2,2,1) \\ \phi(u) &= (1,0,0,1,0,0,0,0,0) \quad & \phi(u') &= (1,0,0,1,0,0,1,0,1) \end{aligned}$$

and $l = 11$, $c_1 = g_0 = 12$, $h_0 = 10$, $h_1 = 4$, and $g_1 = g_i = h_i = 0$, for $i > 1$, for both the intervals $[u, v]$ and $[u', v']$. However $P_{u,v}(q) = (1+q)^2$ while $P_{u',v'}(q) = (1+q)^3$. Of course, this agrees with the result in Theorem 5.3 since $c_2(u, v) = 63$ while $c_2(u', v') = 62$.

6. The Top Coefficient

In this section, as a consequence of our main result, we classify all those Kazhdan-Lusztig polynomials indexed by Boolean elements in a linear Coxeter system (W, S) which have the highest degree allowed. These particular polynomials play a fundamental role in the construction of the Kazhdan-Lusztig representations (see [17]). Moreover they appear in the recursive property of Theorem 2.7, and so Corollaries 6.1, 6.2 and 6.3 have applications in the computation of generic Kazhdan-Lusztig polynomials (see [11, 12]).

Let us treat first the case $W = S_{n+1}$, and let us handle the Boolean elements in S_{n+1} in terms of n-Boolean sequences (see Section 3).

Corollary 6.1. *Let $u, v \in S_{n+1}$ be Boolean elements such that $l(u,v) > 1$. Then $u \prec v$ if and only if there exist $1 \leq l_1 < l_2 < n$ such that*

$$\begin{array}{ll} v_k = u_k, & \text{if } 1 \leq k < l_1, \\ v_k = 2 \text{ and } u_k = 1, & \text{if } k = l_1, \\ v_k = 2 \text{ and } u_k = 0, & \text{if } l_1 < k \leq l_2, \\ v_k = u_k, & \text{if } k > l_2. \end{array}$$

Example. The Kazhdan-Lusztig polynomial $P_{u,v}(q)$ indexed by $u = s_1 s_3 s_7 s_4 s_3 s_2$ and $v = s_1 s_3 s_4 s_5 s_6 s_7 s_6 s_5 s_4 s_3 s_2$ in S_8 has the highest degree allowed. The Boolean sequences $(1,1,2,1,0,0,1)$ and $(1,1,2,2,2,2,1)$ associated to u and v satisfy the requirement of Corollary 6.1 with $l_1 = 4$ and $l_2 = 6$.

The case of (W,S) being generic linear Coxeter system is treated by the following theorems.

Corollary 6.2. *Under the hypotheses of Theorem 4.4, assume $l(u,v) > 1$. Then $u \prec v$ if and only if $\psi(u) \prec \psi(v)$ in S_{m+1}.*

Let W be a non-strictly linear Coxeter system, $w \in W$ be a Boolean element, and $\overline{w}$ be a reduced expression of w which is a subword of the Boolean expression $t_1 \ldots t_{n-1} t_n t_{n-1} \ldots t_1$. We denote by $i_{L,t_k}(w)$ (resp. $i_{R,t_k}(w)$) the element represented by the expression we obtain by inserting a factor t_k to the left (resp. to the right) in the appropriate position in $\overline{w}$. For instance, if $\overline{w} = t_1 t_3 t_4 t_2 t_1$, then $i_{L,t_2}(w) = t_1 t_2 t_3 t_4 t_2 t_1$ and $i_{R,t_3}(w) = t_1 t_3 t_4 t_3 t_2 t_1$.

Corollary 6.3. *Under the hypotheses of Theorem 4.5, assume $l(u,v) > 1$. Denote by u' and v' the elements represented by the expressions we obtain by deleting all the letters t_1 in $\overline{u}$ and $\overline{v}$. Then $u \prec v$ if and only if either*

$$\overline{v}(t_1) = \overline{u}(t_1) \text{ and } u' \prec v',$$

or

$$(\overline{v}(t_1), \overline{u}(t_1), \overline{u}(t_2), \overline{u}(t_j)) = (2,1,0,0) \text{ and}$$
$$\text{there exists } w \in \{i_{L,t_2}(u'), i_{R,t_2}(u'), i_{L,t_j}(u'), i_{R,t_j}(u')\} \text{ such that } w \prec v'.$$

7. Kazhdan-Lusztig elements

Consider the basis $\mathcal{C}$ of the Hecke algebra $\mathcal{H}$ associated to a Coxeter system (W,S) appearing in Theorems 2.3. In this section we compute those Kazhdan-Lusztig elements which are indexed by Boolean elements in any linear Coxeter system. For any expression $\overline{x} = s_{i_1} \ldots s_{i_r}$, we pose $C(\overline{x}) = C_{s_{i_1}} \ldots C_{s_{i_r}}$.
First we treat the case $W = S_{n+1}$. If $\overline{x}$ is a subword of $s_1 \ldots s_{n-1} s_n s_{n-1} \ldots s_1$ such that $\overline{x}(s_k) = 2$ and $\overline{x}(s_{k+1}) = 1$, we denote by $C^k(\overline{x})$ the element we obtain from $C(\overline{x})$ by deleting the factor $C_{s_{k+1}}$ and one of the two factors C_{s_k} (by Proposition 2.4, it is easy to see that it does not matter which one). We extend this notation to $C^K(\overline{x})$, for any $K \subseteq [n]$, making the same deletions for every $k \in K$.

Theorem 7.1. *Let $w \in S_{n+1}$ be a Boolean element. Fix a reduced expression $\overline{w}$ of w which is a subword of $s_1 \dots s_n \dots s_1$ and let $V = \{k \in [n] : w_k = 2, w_{k+1} = 1\}$. Then:*

$$C_w = \sum_{K \subseteq V} (-1)^{|K|} C^K(\overline{w}).$$

As a corollary, we have the following nice factorization.

Corollary 7.2. *Let $w \in S_{n+1}$ be a Boolean element. Fix a reduced expression $\overline{w}$ of w which is a subword of $s_1 \dots s_n \dots s_1$ and let $V' = V + 1 = \{k \in [n] : w_{k-1} = 2, w_k = 1\}$. Then C_w is obtained from $C(\overline{w})$ by changing the factor C_{s_k} to $[C_{s_k} - (q^{\frac{1}{2}} + q^{-\frac{1}{2}})^{-1} C_e]$ for all $k \in V'$.*

Example. Let $w = s_1 s_2 s_3 s_5 s_4 s_3 s_1 \in S_6$. Then $V = \{1, 3\}$ and

$$C_w = C_{s_1} C_{s_2} C_{s_3} C_{s_5} C_{s_4} C_{s_3} C_{s_1} - C_{s_3} C_{s_5} C_{s_4} C_{s_3} C_{s_1} - C_{s_1} C_{s_2} C_{s_3} C_{s_5} C_{s_1} + C_{s_3} C_{s_5} C_{s_1},$$

while $V' = \{2, 4\}$ and we obtain the factorization:

$$C_w = C_{s_1} [C_{s_2} - (q^{\frac{1}{2}} + q^{-\frac{1}{2}})^{-1} C_e] C_{s_3} C_{s_5} [C_{s_4} - (q^{\frac{1}{2}} + q^{-\frac{1}{2}})^{-1} C_e] C_{s_3} C_{s_1}.$$

Now we treat the case of a strictly linear Coxeter system (W, S). Let $t \in T(W)$ be a Boolean reflection with Boolean expression $\overline{t} = t_1 \dots t_{n-1} t_n t_{n-1} \dots t_1$ that we can assume equal to

$$s_a s_{a-1} \dots s_{i+1} s_b s_{b+1} \dots s_{i-1} s_i s_{i-1} \dots s_{b+1} s_b s_{i+1} \dots s_{a-1} s_a$$

by Proposition 3.4. Suppose that t_j is s_{i+1}. If $\overline{x}$ is a subword of $t_1 \dots t_{n-1} t_n t_{n-1} \dots t_1$ such that $\overline{x}(t_k) = 2$ and $\overline{x}$ has only one factor $t_{k'}$, $k' > k$, that does not commute with t_k (i.e. $t_{k'} = t_{k+1}$, if $k \neq j$, $t_{k'} = t_n$, if $k = j$), we denote by $C^k(\overline{x})$ the element we obtain from $C(\overline{x})$ by deleting the factor $C_{t_{k'}}$ and one of the two factors C_{t_k}. We extend this notation to $C^K(\overline{x})$, for any $K \subseteq [n]$, making the same deletions for every $k \in K$. Keeping these notations, we have the following.

Theorem 7.3. *Let $(W, S = \{s_1, \dots, s_m\})$ be a strictly linear Coxeter system, $w \in W$, $w \leq t$. Fix a reduced expression $\overline{w}$ of w which is a subword of $t_1 \dots t_n \dots t_1$, and let $V' = \{k \in [n] \setminus \{j\} : \overline{w}(t_k) = 2, \overline{w}(t_{k+1}) = 1\}$ and*

$$V = \begin{cases} V' \cup \{j\} & \text{if } \overline{w}(t_j) = 2,\ \overline{w}(t_{n-1}) \neq 2\ , \\ V' & \text{otherwise.} \end{cases}$$

Then

$$C_w = \sum_{K \subseteq V} (-1)^{|K|} C^K(\overline{w}).$$

Theorem 7.4. *Let $(W, S = \{s_1, \dots, s_m\})$ be a non-strictly linear Coxeter system, $t \in T(W)$ be a Boolean reflection. Let $w \in W$, $w \leq t$ be such that $s_i \leq w$ for all $i \in [m]$. Fix a Boolean expression $\overline{t} = t_1 \dots t_m \dots t_1$ of the type of Proposition 3.5 and a reduced expression $\overline{w}$ of w which is a subword of $\overline{t}$. Then*

$$C_w = \begin{cases} C_{t_1} C_{w'} C_{t_1}, & \text{if } \overline{w}(t_1) = 2, \\ C_{t_1} C_{w'}, & \text{if } \overline{w} \text{ has only a factor } t_1 \text{ at the leftmost place,} \\ C_{w'} C_{t_1}, & \text{if } \overline{w} \text{ has only a factor } t_1 \text{ at the rightmost place,} \end{cases}$$

where w' is the element represented by the expression we obtain from $\overline{w}$ by erasing all the factors t_1. Hence $C_{w'}$ can be computed as in Theorem 7.3.

8. Poincaré Polynomials

Given $v \in W$, define $F_v(q) = \sum_{u \leq v} q^{l(u)} P_{u,v}(q)$. It is known that, if W is any Weyl or affine Weyl group, $F_v(q)$ is the intersection homology Poincaré polynomial of the Schubert variety indexed by v (see [18]). In this section, we want to compute these polynomials when W is a linear Coxeter system and $v \in W$ is a Boolean element.

First let us do this computation for $W = S_{n+1}$ where we treat the Boolean elements in terms of n-Boolean sequences as in Section 3. Let us restrict the domain of ϕ to the interval $[e, v]$. Given any Boolean sequence $u = (u_i, \ldots, u_n) \leq \phi(v)$ in the component-wise partial order, we define

$$\begin{aligned} n(u, \phi(v)) &= |\{i \in [n-1] \,:\, v_i = 2,\ u_i = 1,\ u_{i+1} \neq 0\}|; \\ b(u, \phi(v)) &= |\{i \in [n-1] \,:\, v_i = 2,\ v_{i+1} = 2,\ u_{i+1} = 0\}|. \end{aligned}$$

With these notations,

$$\phi^{-1}_{|_{[e,v]}}(u) = 2^{n(u,\phi(v))},$$

and, by Theorem 4.2,

$$F_v(q) = \sum_{u \leq \phi(v)} q^{l(u)} (1+q)^{b(u,\phi(v))} 2^{n(u,\phi(v))}.$$

We have the following theorem.

Theorem 8.1. *Let $v \in S_{n+1}$ be a Boolean element. Then*

$$F_v(q) = (1+q)^{l(v)-2f(v)} (1+q+q^2)^{f(v)},$$

where $f(v)$ is the number of occurrences of the pattern $|2,1|$ in the sequence $\phi(v)$.

Example. Let $v \in S_8$, $v = s_1 s_2 s_4 s_5 s_6 s_7 s_5 s_4 s_3 s_2$. Then the Boolean sequence associated to v is $(1, 2, 1, 2, 2, 2, 1, 1)$, $f(v) = 2$ and $F_v(q) = (1+q)^{l(v)-4}(1+q+q^2)^2$.

The following two theorems treat respectively the case of a strictly and of a non-strictly linear Coxeter system.

Theorem 8.2. *Let $(W, S = \{s_1, \ldots, s_m\})$ be a strictly linear Coxeter system, $t \in W$ a Boolean reflection and $v \in W$, $v \leq t$. Then $F_v(q) = F_{\psi(v)}(q)$, where ψ is as in Lemma 4.3 and $F_{\psi(v)}(q)$ can be computed as in Theorem 8.1.*

Theorem 8.3. *Let $(W, S = \{s_1, \ldots, s_m\})$ be a non-strictly linear Coxeter system, $t \in T(W)$ a Boolean reflection that we can assume such that $s_i \leq t$ for all $i \in [m]$, and $v \in W$, $v \leq t$. Fix a Boolean expression $\overline{t} = t_1 \ldots t_{n-1} t_n t_{n-1} \ldots t_1$ of t of the type shown in Proposition 3.5 and a reduced expression $\overline{v}$ of v which is a subword of $\overline{t}$. Then $F_v(q) = (1+q)^{\overline{v}(t_1)} F_{v'}(q)$, where v' is the element of W represented by the expression we obtain from $\overline{v}$ by deleting all the letters t_1 and $F_{v'}(q)$ can be computed as in Theorem 8.2.*

Remark. The polynomials $F_v(q)$ computed in this section are all symmetric and unimodal. For Weyl or affine Weyl groups W, this is a consequence of the fact that (middle perversity) intersection cohomology satisfies Poincaré duality and the "Hard Lefschetz Theorem". So this result is consistent with the idea that there may be geometric objects associated to any Coxeter group analogous to Schubert varieties.

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References

[1] S. Billey, G. Warrington, *Kazhdan-Lusztig polynomials for 321-hexagon avoiding permutations*, J. Algebraic Combin., **13** (2001), 111-136.

[2] S. Billey, G. Warrington, *Maximal singular loci of Schubert varieties in $SL(n)/B$*, Trans. Amer. Math. Soc., to appear.

[3] B. Boe, *Kazhdan-Lusztig polynomials for hermitian spaces*, Trans. Amer. Math. Soc., **309** (1988), 279-294.

[4] N. Bourbaki, *Groupes et algèbres de Lie*, Chap. 4-6, Hermann, Paris, 1968.

[5] T. Braden, R. MacPherson, *From moment graph to intersection cohomology*, Math. Ann., **321** (2001), 533-551.

[6] F. Brenti, *A combinatorial formula for Kazhdan-Lusztig polynomials*, Invent. Math., **118** (1994), 371-394.

[7] F. Brenti, *Combinatorial properties of the Kazhdan-Lusztig R-polynomials for S_n*, Adv. Math., **126** (1997), 21-51.

[8] F. Brenti, *Lattice paths and Kazhdan-Lusztig polynomials*, J. Amer. Math. Soc., **11** (1998), 229-259.

[9] F. Brenti, *The intersection cohomology of Schubert varieties is a combinatorial invariant*, preprint.

[10] F. Brenti, R. Simion, *Explicit formulae for some Kazhdan-Lusztig polynomials*, J. Algebraic Combin., **11** (2000), 187-196.

[11] F. Caselli, *Proof of two conjectures of Brenti and Simion on Kazhdan-Lusztig polynomials*, J. Algebraic Combin., to appear.

[12] F. Caselli, M. Marietti, *Computing the Kazhdan-Lusztig polynomials for S_n*, preprint.

[13] V. Deodhar, *A brief survey of Kazhdan-Lusztig theory and related topics*, Proc. Sympos. Pure Math., **56**, (1994), 105-124.

[14] M. Dyer, *On the Bruhat graph of a Coxeter system*, Compositio Math., **78**, (1991), 185-191.

[15] A. Garsia, T. McLarnan, *Relations between Young's natural and Kazhdan-Lusztig representations of S_n*, Adv. Math., **69** (1988), 32-92.

[16] J. E. Humphreys, *Reflection Groups and Coxeter Groups*, Cambridge Studies in Advanced Mathematics, no.29, Cambridge Univ. Press, Cambridge, 1990.

[17] D. Kazhdan, G. Lusztig, *Representations of Coxeter groups and Hecke algebras*, Invent. Math., **53** (1979), 165-184.

[18] D. Kazhdan, G. Lusztig, *Schubert varieties and Poincaré duality*, Proc. Sympos. Pure Math., **36**, (1980), 185-203.

[19] A. Lascoux, *Polynômes de Kazhdan et Lusztig pour les variétés de Schubert vexillaires*, C. R. Acad. Sci. Paris Sér. I Math., **321** (1995), 667-670.

[20] A. Lascoux, M. Schützenberger, *Polynômes de Kazhdan et Lusztig pour les grassmanniennes*, Astérisque, **87-88** (1981), 249-266.

[21] M. Marietti, *Closed product formulas for certain R-polynomials*, European J. Combin., **23** (2002), 57-62.

[22] B. Shapiro, M. Shapiro, A. Vainshtein, *Kazhdan-Lusztig polynomials for certain varieties of incomplete flags*, Discrete Math., **180** (1998), 345-355.

[23] J. Tits, *Le problème des mots dans les groupes de Coxeter*, Symposia Mathematica, vol 1, INDAM, Roma, 1969, 175-185.

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