Two linear transformations each tridiagonal with respect to an eigenbasis of the other*

Paul Terwilliger

Abstract

In this paper we give an elementary introduction to the theory of Leonard pairs. A Leonard pair is defined as follows. Let $\mathbb{K}$ denote a field and let V denote a vector space over $\mathbb{K}$ with finite positive dimension. By a *Leonard pair* on V we mean an ordered pair of linear transformations $A : V \to V$ and $B : V \to V$ that satisfy conditions (i), (ii) below.

(i) There exists a basis for V with respect to which the matrix representing A is irreducible tridiagonal and the matrix representing B is diagonal.

(ii) There exists a basis for V with respect to which the matrix representing A is diagonal and the matrix representing B is irreducible tridiagonal.

We give several examples of Leonard pairs. Using these we illustrate how Leonard pairs arise in representation theory, combinatorics, and the theory of orthogonal polynomials.

1 Leonard pairs

In this paper we give an elementary introduction to the theory of Leonard pairs [24], [25], [35], [37], [39], [40], [41], [42], [43], [44], [45]. We define a Leonard pair and give some examples. Using these examples we illustrate how Leonard pairs arise in representation theory, combinatorics, and the theory of orthogonal polynomials.

We define the notion of a Leonard pair. To do this, we first recall what it means for a square matrix to be *tridiagonal*.

The following matrices are tridiagonal.

$$\begin{pmatrix} 2 & 3 & 0 & 0 \\ 1 & 4 & 2 & 0 \\ 0 & 5 & 3 & 3 \\ 0 & 0 & 3 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 2 & 3 & 0 & 0 \\ 0 & 4 & 2 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 5 \end{pmatrix}.$$

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Tridiagonal means each nonzero entry lies on either the diagonal, the subdiagonal, or the superdiagonal.

The tridiagonal matrix on the left is *irreducible*. This means each entry on the subdiagonal is nonzero and each entry on the superdiagonal is nonzero.

We now define a Leonard pair. For the rest of this paper $\mathbb{K}$ will denote a field.

Definition 1.1 Let V denote a vector space over $\mathbb{K}$ with finite positive dimension. By a *Leonard pair* on V we mean an ordered pair of linear transformations $A : V \to V$ and $B : V \to V$ that satisfy conditions (i), (ii) below.

(i) There exists a basis for V with respect to which the matrix representing A is irreducible tridiagonal and the matrix representing B is diagonal.

(ii) There exists a basis for V with respect to which the matrix representing A is diagonal and the matrix representing B is irreducible tridiagonal.

Our use of the name "Leonard pair" is motivated by a connection to a theorem of D. Leonard [3, p. 260], [33] involving the q-Racah and related polynomials of the Askey Scheme. See [39, App. A] and [37] for more information on this.

Here is an example of a Leonard pair. Set $V = \mathbb{K}^4$ (column vectors), set

$$A = \begin{pmatrix} 0 & 3 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 3 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix},$$

and view A and B as linear transformations on V. We assume the characteristic of $\mathbb{K}$ is not 2 or 3 to ensure A is irreducible. Then the pair A, B is a Leonard pair on V. Indeed condition (i) in Definition 1.1 is satisfied by the basis for V consisting of the columns of the 4 by 4 identity matrix. To verify condition (ii), we display an invertible matrix P such that $P^{-1}AP$ is diagonal and $P^{-1}BP$ is irreducible tridiagonal. Set

$$P = \begin{pmatrix} 1 & 3 & 3 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -3 & 3 & -1 \end{pmatrix}.$$

By matrix multiplication $P^2 = 8I$, where I denotes the identity, so P^{-1} exists. Also by matrix multiplication,

$$AP = PB. \tag{1}$$

Apparently $P^{-1}AP$ is equal to B and is therefore diagonal. By (1) and since P^{-1} is a scalar multiple of P, we find $P^{-1}BP$ is equal to A and is therefore irreducible tridiagonal. Now condition (ii) of Definition 1.1 is satisfied by the basis for V consisting of the columns of P.

The above example is a member of the following infinite family of Leonard pairs. For any nonnegative integer d, the pair

$$A = \begin{pmatrix} 0 & d & & & & \mathbf{0} \\ 1 & 0 & d-1 & & & \\ & 2 & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & 1 \\ \mathbf{0} & & & & d & 0 \end{pmatrix}, \qquad B = \mathrm{diag}(d, d-2, d-4, \ldots, -d) \tag{2}$$

is a Leonard pair on the vector space $\mathbb{K}^{d+1}$, provided the characteristic of $\mathbb{K}$ is zero or an odd prime greater than d. This can be proved by modifying the proof for $d = 3$ given above. One shows $P^2 = 2^d I$ and $AP = PB$, where P denotes the matrix with ij entry

$$P_{ij} = \binom{d}{j} \sum_{n=0}^{d} \frac{(-i)_n (-j)_n 2^n}{(-d)_n n!} \qquad (0 \leq i, j \leq d), \tag{3}$$

where

$$(a)_n = a(a+1)(a+2)\cdots(a+n-1) \qquad n = 0, 1, 2 \ldots$$

The details of the above calculation can be found in [37, Sect. 16].

In line (3) the sum on the right is the hypergeometric series

$${}_2F_1\left(\begin{matrix} -i, -j \\ -d \end{matrix} \middle| 2 \right).$$

For nonnegative integers r, s there exist analogs of the above ${}_2F_1$ hypergeometric series called ${}_rF_s$ hypergeometric series [2]. Moreover there exist q-analogs of these called ${}_r\phi_s$ basic hypergeometric series [2], [11]. As we will see, there exist Leonard pairs similar to the one above in which the series of type ${}_2F_1$ is replaced by a series of one of the following types: ${}_2F_1, {}_3F_2, {}_4F_3, {}_2\phi_1, {}_3\phi_2, {}_4\phi_3$.

2 Leonard pairs and orthogonal polynomials

There is a natural correspondence between Leonard pairs and a family of orthogonal polynomials consisting of the q-Racah polynomials and related polynomials in the Askey scheme [39, App. A]. Applying this correspondence to the Leonard pairs in Section 1 we get a class of *Krawtchouk* polynomials. The details are as follows.

To keep things simple, we assume throughout this section that our field $\mathbb{K}$ has characteristic zero. We fix a nonnegative integer d and consider the subset of $\mathbb{K}$ consisting of

$$d, \quad d-2, \quad d-4, \quad \ldots, \quad 2-d, \quad -d.$$

We call this subset Ω. Let V denote the vector space over $\mathbb{K}$ consisting of all functions from Ω to $\mathbb{K}$. The cardinality of Ω is $d+1$ so the dimension of V is $d+1$. We define two linear transformations $A : V \to V$ and $B : V \to V$.

We begin with A. For all $f \in V$, let Af denote the element in V satisfying

$$(Af)(\theta) = \theta f(\theta) \qquad\qquad (\forall \theta \in \Omega).$$

We observe A is a linear transformation on V.

We now define B. For all $f \in V$, let Bf denote the element in V satisfying

$$(Bf)(\theta) = \frac{d+\theta}{2} f(\theta-2) + \frac{d-\theta}{2} f(\theta+2) \qquad\qquad (\forall \theta \in \Omega).$$

We observe B is a linear transformation on V. One might call B a "linear difference operator".

We claim the pair A, B is a Leonard pair on V. To show this, we first display a basis for V with respect to which the matrix representing A is irreducible tridiagonal and the matrix representing B is diagonal. We use the following notation.

$$\theta_i = d - 2i \qquad\qquad (0 \leq i \leq d).$$

For $0 \leq j \leq d$ let K_j denote the element in V satisfying

$$K_j(\theta_i) = \binom{d}{j} {}_2F_1\left(\begin{matrix} -i, -j \\ -d \end{matrix} \,\middle|\, 2 \right) \qquad\qquad (0 \leq i \leq d).$$

We observe $K_j(\theta)$ is a polynomial of degree j in θ. For instance

$$K_0(\theta) = 1, \qquad K_1(\theta) = \theta, \qquad K_2(\theta) = \frac{\theta^2 - d}{2} \qquad\qquad (\forall \theta \in \Omega).$$

The polynomials $K_0, K_1, \ldots, K_d$ are *Krawtchouk* polynomials but not the most general ones [2, p. 347], [27]. The sequence $K_0, K_1, \ldots, K_d$ forms a basis for V. With respect to this basis the matrices representing A and B are

$$A: \begin{pmatrix} 0 & d & & & & \mathbf{0} \\ 1 & 0 & d-1 & & & \\ & 2 & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & 1 \\ \mathbf{0} & & & & d & 0 \end{pmatrix}, \qquad B: \ \mathrm{diag}(d, d-2, d-4, \ldots, -d).$$

Apparently A, B form a Leonard pair on V which is the example from Section 1 in disguise.

Since the pair A, B is a Leonard pair on V, there exists a basis for V with respect to which the matrix representing A is diagonal and the matrix representing B is irreducible tridiagonal. We now display this basis. For $0 \leq j \leq d$ let K_j^* denote the element in V which satisfies

$$K_j^*(\theta_i) = \delta_{ij} \qquad\qquad (0 \leq i \leq d),$$

where δ_{ij} denotes the Kronecker delta. The sequence $K_0^*, K_1^*, \ldots, K_d^*$ forms a basis for V. With respect to this basis the matrices representing A and B are

$$A: \ \mathrm{diag}(d, d-2, d-4, \ldots, -d), \qquad B: \left(\begin{array}{cccccc} 0 & d & & & & \mathbf{0} \\ 1 & 0 & d-1 & & & \\ & 2 & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & 1 \\ \mathbf{0} & & & & d & 0 \end{array} \right).$$

We have now shown how the above Krawtchouk polynomials correspond to the Leonard pairs in Section 1. The polynomials in the following table are related to Leonard pairs in a similar fashion.

type	polynomial
${}_4F_3$	Racah
${}_3F_2$	Hahn, dual Hahn
${}_2F_1$	Krawtchouk
${}_4\phi_3$	q-Racah
${}_3\phi_2$	q-Hahn, dual q-Hahn
${}_2\phi_1$	q-Krawtchouk (classical, affine, quantum, dual)

Definitions of the above polynomials can be found in [27].

We have a classification of Leonard pairs [39, Theorem 1.9], [43, Theorem 5.16]. By that classification the above examples exhaust essentially all Leonard pairs in the following sense. Associated with any Leonard pair is a certain scalar q [37, Cor. 12.8]. The above examples exhaust all Leonard pairs for which $q \neq -1$. For a discussion of the polynomials corresponding to Leonard pairs with $q = -1$ see [3, p. 260].

3 Leonard pairs and the Lie algebra sl_2

Leonard pairs appear naturally in representation theory. To illustrate this we show how the Leonard pairs from Section 1 correspond to irreducible finite dimensional modules for the Lie algebra sl_2.

In this section we assume the field $\mathbb{K}$ is algebraically closed with characteristic zero.

We recall the Lie algebra $sl_2 = sl_2(\mathbb{K})$. This algebra has a basis e, f, h satisfying

$$[h, e] = 2e, \qquad [h, f] = -2f, \qquad [e, f] = h,$$

where $[\, , \,]$ denotes the Lie bracket.

We recall the irreducible finite dimensional modules for sl_2.

Lemma 3.1 *[26, p. 102] There exists a family*

$$V_d \qquad\qquad d = 0, 1, 2 \ldots \tag{4}$$

of irreducible finite dimensional sl_2-modules with the following properties. The module V_d has a basis $v_0, v_1, \ldots, v_d$ satisfying $hv_i = (d-2i)v_i$ for $0 \le i \le d$, $fv_i = (i+1)v_{i+1}$ for $0 \le i \le d-1$, $fv_d = 0$, $ev_i = (d-i+1)v_{i-1}$ for $1 \le i \le d$, $ev_0 = 0$. Every irreducible finite dimensional sl_2-module is isomorphic to exactly one of the modules in line (4).

Example 3.2 *Let A and B denote the following elements of sl_2.*

$$A = e + f, \qquad\qquad B = h.$$

Let d denote a nonnegative integer and consider the action of A, B on the module V_d. With respect to the basis $v_0, v_1, \ldots, v_d$ from Lemma 3.1, the matrices representing A and B are

$$A: \begin{pmatrix} 0 & d & & & & \mathbf{0} \\ 1 & 0 & d-1 & & & \\ & 2 & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & 1 \\ \mathbf{0} & & & & d & 0 \end{pmatrix}, \qquad B: \; diag(d, d-2, d-4, \ldots, -d).$$

The pair A, B acts on V_d as a Leonard pair. Once again, the resulting Leonard pair is the one from Section 1 in disguise.

The Leonard pairs in Example 3.2 are not the only ones associated with sl_2. To get more Leonard pairs we replace A and B by more general elements in sl_2. Our result is the following.

Example 3.3 *[24, Ex. 1.5] Let A and B denote semi-simple elements in sl_2 and assume sl_2 is generated by these elements. Let V denote an irreducible finite dimensional module for sl_2. Then the pair A, B acts on V as a Leonard pair.*

We remark the Leonard pairs in Example 3.3 correspond to the general Krawtchouk polynomials [27].

A bit later in this paper we will obtain Leonard pairs from each irreducible finite dimensional module for the quantum algebra $U_q(sl_2)$. To prepare for this we recall the split representation of a Leonard pair.

4 The split representation

Given a Leonard pair A, B it is natural to represent one of A, B by an irreducible tridiagonal matrix and the other by a diagonal matrix. There is another representation of interest; we call this the *split representation*. In order to distinguish the two representations we make a definition.

Definition 4.1 Let V denote a vector space over $\mathbb{K}$ with finite positive dimension and let A, B denote a Leonard pair on V. By a *standard basis* for this pair we mean a basis for V with respect to which the matrix representing A is irreducible tridiagonal and the matrix representing B is diagonal. (We remark that if $v_0, v_1, \ldots, v_d$ is a standard basis for the pair A, B then so is $v_d, v_{d-1}, \ldots, v_0$.)

Before proceeding we recall what it means for a square matrix to be *lower bidiagonal* or *upper bidiagonal*.

The following matrices are lower bidiagonal.

$$\begin{pmatrix} 2 & 0 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 1 & 5 \end{pmatrix}, \qquad \begin{pmatrix} 2 & 0 & 0 & 0 \\ 1 & 4 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 2 & 6 \end{pmatrix}.$$

Lower bidiagonal means every nonzero entry lies on either the diagonal or the subdiagonal. The lower bidiagonal matrix on the left is *irreducible*. This means each entry on the subdiagonal is nonzero.

A matrix is *upper bidiagonal* (resp. irreducible upper bidiagonal) whenever its transpose is lower bidiagonal (resp. irreducible lower bidiagonal).

Definition 4.2 Let V denote a vector space over $\mathbb{K}$ with finite positive dimension. Let A, B denote a Leonard pair on V. By a *split basis* for this pair we mean a basis for V with respect to which the matrix representing A is irreducible lower bidiagonal and the matrix representing B is irreducible upper bidiagonal. (We remark that if $u_0, u_1, \ldots, u_d$ is a split basis for the pair A, B then $u_d, u_{d-1}, \ldots, u_0$ is a split basis for the Leonard pair B, A.)

Each Leonard pair has many split bases and these are obtained as follows. Let V denote a vector space over $\mathbb{K}$ with finite positive dimension and let A, B denote a Leonard pair on V. Let $v_0, v_1, \ldots, v_d$ denote a standard basis for A, B as in Definition 4.1. Let $v_0^*, v_1^*, \ldots, v_d^*$ denote a standard basis for the Leonard pair B, A. For $0 \leq i \leq d$ the space

$$\mathrm{Span}(v_0, v_1, \ldots, v_i) \cap \mathrm{Span}(v_i^*, v_{i+1}^*, \ldots, v_d^*)$$

has dimension 1 [39, Lem. 3.8]; let u_i denote a nonzero vector in this space. Then the sequence $u_0, u_1, \ldots, u_d$ is a split basis for A, B [39, Lem. 3.8]. Every split basis for A, B is obtained in this fashion. This follows from [39, Thm. 3.2] and [39, Lem. 4.10].

We illustrate with the Leonard pairs from Section 1. Let A, B denote the Leonard pair given in (2). With respect to an appropriate split basis for this pair the matrices representing A and B are as follows.

$$A: \begin{pmatrix} d & & & & & \mathbf{0} \\ -1 & d-2 & & & & \\ & -2 & \cdot & & & \\ & & \cdot & \cdot & & \\ & & & \cdot & \cdot & \\ \mathbf{0} & & & & -d & -d \end{pmatrix}, \qquad B: \begin{pmatrix} d & 2d & & & & \mathbf{0} \\ & d-2 & 2d-2 & & & \\ & & \cdot & \cdot & & \\ & & & \cdot & \cdot & \\ & & & & \cdot & 2 \\ \mathbf{0} & & & & & -d \end{pmatrix}.$$

There is some information on split bases in [24] and [39]. For a more detailed study see [37].

5 Recognizing a Leonard pair

The following theorem provides a way to recognize a Leonard pair.

Theorem 5.1 *[39, Cor. 14.2] Let V denote a vector space over $\mathbb{K}$ with finite positive dimension. Let $A : V \to V$ and $B : V \to V$ denote linear transformations. Let us assume there exists a basis for V with respect to which the matrices representing A and B have the following form.*

$$A : \begin{pmatrix} \theta_0 & & & & & \mathbf{0} \\ 1 & \theta_1 & & & & \\ & 1 & \theta_2 & & & \\ & & \cdot & \cdot & & \\ & & & \cdot & \cdot & \\ \mathbf{0} & & & & 1 & \theta_d \end{pmatrix}, \qquad B : \begin{pmatrix} \theta^*_0 & \varphi_1 & & & & \mathbf{0} \\ & \theta^*_1 & \varphi_2 & & & \\ & & \theta^*_2 & \cdot & & \\ & & & \cdot & \cdot & \\ & & & & \cdot & \varphi_d \\ \mathbf{0} & & & & & \theta^*_d \end{pmatrix}. \tag{5}$$

Then the pair A, B is a Leonard pair on V if and only if there exist scalars ϕ_i $(1 \leq i \leq d)$ in $\mathbb{K}$ such that conditions (i)–(v) hold below.

(i) $\varphi_i \neq 0, \quad \phi_i \neq 0 \qquad (1 \leq i \leq d).$

(ii) $\theta_i \neq \theta_j, \quad \theta^*_i \neq \theta^*_j \quad \text{if} \quad i \neq j, \qquad (0 \leq i, j \leq d).$

(iii) $$\varphi_i = \phi_1 \sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} + (\theta^*_i - \theta^*_0)(\theta_{i-1} - \theta_d) \qquad (1 \leq i \leq d).$$

(iv) $$\phi_i = \varphi_1 \sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} + (\theta^*_i - \theta^*_0)(\theta_{d-i+1} - \theta_0) \qquad (1 \leq i \leq d).$$

(v) The expressions

$$\frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i}, \qquad \frac{\theta^*_{i-2} - \theta^*_{i+1}}{\theta^*_{i-1} - \theta^*_i} \tag{6}$$

are equal and independent of i for $2 \leq i \leq d - 1$.

We will demonstrate the utility of the above theorem in the next section.

6 Leonard pairs and $U_q(sl_2)$

Given a Leonard pair it is often more natural to work with a split basis rather than a standard basis. We illustrate this with an example based on the quantum algebra $U_q(sl_2)$.

In this section we assume $\mathbb{K}$ is algebraically closed. We fix a nonzero scalar $q \in \mathbb{K}$ which is not a root of unity.

Definition 6.1 [26, p.122] Let $U_q(sl_2)$ denote the associative $\mathbb{K}$-algebra with 1 generated by symbols e, f, k, k^{-1} subject to the relations

$$kk^{-1} = k^{-1}k = 1,$$

$$ke = q^2 ek, \qquad\qquad kf = q^{-2} fk,$$

$$ef - fe = \frac{k - k^{-1}}{q - q^{-1}}.$$

We recall the irreducible finite dimensional modules for $U_q(sl_2)$. We use the following notation.

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}} \qquad\qquad n \in \mathbb{Z}.$$

Lemma 6.2 *[26, p. 128] With reference to Definition 6.1, there exists a family*

$$V_{\varepsilon,d} \qquad\qquad \varepsilon \in \{1, -1\}, \qquad\qquad d = 0, 1, 2 \ldots \tag{7}$$

of irreducible finite dimensional $U_q(sl_2)$-modules with the following properties. The module $V_{\varepsilon,d}$ has a basis $u_0, u_1, \ldots, u_d$ satisfying $ku_i = \varepsilon q^{d-2i} u_i$ for $0 \leq i \leq d$, $fu_i = [i+1]_q u_{i+1}$ for $0 \leq i \leq d-1$, $fu_d = 0$, $eu_i = \varepsilon [d-i+1]_q u_{i-1}$ for $1 \leq i \leq d$, $eu_0 = 0$. Every irreducible finite dimensional $U_q(sl_2)$-module is isomorphic to exactly one of the modules $V_{\varepsilon,d}$. (Referring to line (7), if $\mathbb{K}$ has characteristic 2 we interpret the set $\{1, -1\}$ as having a single element.)

We use the above $U_q(sl_2)$-modules to get Leonard pairs. The following result was proved by the present author in [35] and is implicit in the results of Koelink and Van der Jeugt [30], [31]. For related results see [29], [32], [34], [40].

Example 6.3 *[30], [31], [35] Referring to Definition 6.1 and Lemma 6.2, let α, β denote nonzero scalars in $\mathbb{K}$ and define A, B as follows.*

$$A = \alpha f + \frac{k}{q - q^{-1}}, \qquad\qquad B = \beta e + \frac{k^{-1}}{q - q^{-1}}. \tag{8}$$

Let d denote a nonnegative integer and choose $\varepsilon \in \{1, -1\}$. Then the pair A, B acts on $V_{\varepsilon,d}$ as a Leonard pair provided $\varepsilon\alpha\beta$ is not among $q^{d-1}, q^{d-3}, \ldots, q^{1-d}$. For this Leonard pair the basis $u_0, u_1, \ldots, u_d$ from Lemma 6.2 is a split basis.

Proof. We apply Theorem 5.1. We begin by displaying a basis for $V_{\varepsilon,d}$ with respect to which the matrices representing A, B have the form (5). We obtain this basis by modifying the basis $u_0, u_1, \ldots, u_d$ given in Lemma 6.2. For $0 \leq i \leq d$ we define $u'_i = \alpha^i [1]_q [2]_q \cdots [i]_q u_i$. We observe $u'_0, u'_1, \ldots, u'_d$ is a basis for $V_{\varepsilon,d}$. The elements k, f, e act on this basis as follows. We have $ku'_i = \varepsilon q^{d-2i} u'_i$ for $0 \leq i \leq d$, $fu'_i = \alpha^{-1} u'_{i+1}$ for $0 \leq i \leq d-1$, $fu'_d = 0$, $eu'_i = \varepsilon\alpha [i]_q [d-i+1]_q u'_{i-1}$ for $1 \leq i \leq d$, $eu'_0 = 0$. Using these comments and (8) we find that with respect to the basis $u'_0, u'_1, \ldots, u'_d$ the matrices representing A, B are given in (5), where

$$\theta_i = \frac{\varepsilon q^{d-2i}}{q - q^{-1}}, \qquad \theta^*_i = \frac{\varepsilon q^{2i-d}}{q - q^{-1}} \qquad (0 \le i \le d),$$
$$\varphi_i = \varepsilon\alpha\beta[i]_q[d-i+1]_q \qquad (1 \le i \le d).$$

Define

$$\phi_i = [i]_q[d-i+1]_q(\varepsilon\alpha\beta - q^{2i-d-1}) \qquad (1 \le i \le d).$$

Let us assume $\varepsilon\alpha\beta$ is not among $q^{d-1}, q^{d-3}, \ldots, q^{1-d}$. Then the above scalars $\theta_i, \theta^*_i, \varphi_i, \phi_i$ satisfy conditions (i)–(v) of Theorem 5.1. The verification is routine and left to the reader. To aid in this verification we mention the above scalars θ_i satisfy

$$\sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} = \frac{[i]_q[d-i+1]_q}{[d]_q} \qquad (1 \le i \le d).$$

Applying Theorem 5.1 we find the pair A, B acts on $V_{\varepsilon,d}$ as a Leonard pair. With respect to the basis $u_0, u_1, \ldots, u_d$ the matrix representing A (resp. B) is irreducible lower bidiagonal (resp. irreducible upper bidiagonal). Therefore this basis is a split basis for A, B in view of Definition 4.2. $\square$

We remark the Leonard pairs in Example 6.3 are related to the quantum q-Krawtchouk polynomials [27], [29].

7 Leonard pairs in combinatorics

Leonard pairs arise in many branches of combinatorics. For instance they arise in the theory of partially ordered sets (posets). We illustrate this with a poset called the subspace lattice $L_n(q)$.

In this section we assume our field $\mathbb{K}$ is the field $\mathbb{C}$ of complex numbers.

To define the subspace lattice we introduce a second field. Let $GF(q)$ denote a finite field of order q. Let n denote a positive integer and let W denote an n-dimensional vector space over $GF(q)$. Let P denote the set consisting of all subspaces of W. The set P, together with the containment relation, is a poset called $L_n(q)$.

Using $L_n(q)$ we obtain a family of Leonard pairs as follows. Let $\mathbb{C}P$ denote the vector space over $\mathbb{C}$ consisting of all formal $\mathbb{C}$-linear combinations of elements of P. We observe P is a basis for $\mathbb{C}P$ so the dimension of $\mathbb{C}P$ is equal to the cardinality of P.

We define three linear transformations on $\mathbb{C}P$. We call these K, R (for "raising"), L (for "lowering").

We begin with K. For all $x \in P$,

$$Kx = q^{n/2 - dim\, x} x.$$

Apparently each element of P is an eigenvector for K.

To define R and L we use the following notation. For $x, y \in P$ we say y *covers* x whenever (i) $x \subseteq y$ and (ii) $\dim y = 1 + \dim x$.

The maps R and L are defined as follows. For all $x \in P$,

$$Rx = \sum_{y \text{ covers } x} y.$$

Similarly

$$Lx = q^{(1-n)/2} \sum_{x \text{ covers } y} y.$$

(The scalar $q^{(1-n)/2}$ is included for aesthetic reasons.)

We consider the properties of K, R, L. From the construction we find K^{-1} exists. By combinatorial counting we verify

$$KL = qLK, \qquad KR = q^{-1}RK,$$
$$LR - RL = \frac{K - K^{-1}}{q^{1/2} - q^{-1/2}}.$$

We recognize these equations. They are the defining relations for $U_{q^{1/2}}(sl_2)$. Apparently K, R, L turn $\mathbb{C}P$ into a module for $U_{q^{1/2}}(sl_2)$.

We now see how to get Leonard pairs from $L_n(q)$. Let α, β denote nonzero complex scalars and define A, B as follows.

$$A = \alpha R + \frac{K}{q^{1/2} - q^{-1/2}}, \qquad B = \beta L + \frac{K^{-1}}{q^{1/2} - q^{-1/2}}.$$

To avoid degenerate situations we assume $\alpha\beta$ is not among $q^{(n-1)/2}, q^{(n-3)/2}, \ldots, q^{(1-n)/2}$.

The $U_{q^{1/2}}(sl_2)$-module $\mathbb{C}P$ is completely reducible [26, p. 144]. In other words $\mathbb{C}P$ is a direct sum of irreducible $U_{q^{1/2}}(sl_2)$-modules. On each irreducible module in this sum the pair A, B acts as a Leonard pair. This follows from Example 6.3.

We just saw how the subspace lattice gives Leonard pairs. It is implicit in [36] that the following posets give Leonard pairs in a similar fashion: the subset lattice, the Hamming semi-lattice, the attenuated spaces, and the classical polar spaces. Definitions of these posets can be found in [36].

8 Further reading

We mention some additional topics which are related to Leonard pairs.

Earlier in this paper we obtained Leonard pairs from the irreducible finite dimensional modules for the Lie algebra sl_2 and the quantum algebra $U_q(sl_2)$. We cite some other algebras whose modules are related to Leonard pairs. These are the Askey-Wilson algebra [12], [13], [14], [15], [16], [46], [47], [48], the Onsager algebra [1], [8], [9], [10], and the Tridiagonal algebra [24], [39], [40].

We discussed how certain classical posets give Leonard pairs. Another combinatorial object which gives Leonard pairs is a P- and Q-polynomial association scheme [3], [4], [38]. Leonard pairs have been used to describe certain irreducible modules for the subconstituent algebra of these schemes [5], [6], [7], [24], [38].

The topic of Leonard pairs is closely related to the work of Grünbaum and Haine on the "bispectral problem" [19], [20]. See [17], [18], [21], [22], [23] for related work.

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Paul Terwilliger, Department of Mathematics, University of Wisconsin, 480 Lincoln Drive, Madison, Wisconsin, 53706, USA
Email: terwilli@math.wisc.edu