TEST SETS IN INTEGER PROGRAMMING AND THE COMPLEX OF MAXIMAL LATTICE FREE CONVEX BODIES

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Abstract. For a certain class of integer programming problems, test sets can be used to decide the optimality of a feasible solution. The vectors in the test set correspond to lattice point free convex bodies. There is a simplicial complex, the complex of maximal lattice free convex bodies, associated with this class of integer programs. The complex has nice properties that will be explained. For instance, the group Z^n acts on it, and it is homeomorphic to a Euclidean space of suitable dimension.

Résumé: Pour une certaine classe de problèmes en programmation entière, on peut utiliser des ensembles tests pour décider si une solution réalisable est optimale ou non. Les vecteurs de l'ensemble test correspondent à des corps convexes ne contenant aucun point à coordonnées entières. Il existe un complexe simplicial, le complexe de ces corps convexes maximaux, associé à cette classe de programmes. Ce complexe a des propriétés intéressantes que nous décrirons. Par exemple, il y a une action du groupe Z^n sur ce complexe qui est lui même homéomorphe à un espace euclidien de dimension appropriée.

Test sets for integer programming were introduced by Graver [Gr] and Scarf [Sc]. They provide a way of telling if a feasible solution $z \in Z^n$ is optimal or not by checking, for each h in the test set, whether $z + h$ is feasible and yields an improved value of the objective function.

The test set of Scarf, the set of neighbors of the origin, is associated with a matrix A of size m by n, and is applied to the class of problems of the form

$$\begin{aligned} &\text{minimize } \ a_1 z \\ &\text{subject to } a_i z \ \le b_i \ (i = 2, ..., m), \ z \in Z^n \end{aligned}$$

in which the first row of A becomes the objective, and the remaining rows are used, with arbitrary b_i, to form the constraints.

For each lattice point $h \in Z^n$, the smallest body of the form

$$K_b = \{x \in R^n \,:\, Ax \le b\}$$

containing 0 and h is given by $b_i = \max\{0,\ a_i h\}$, for $i = 1,\ 2,\ ...,\ m$. We designate this body by $\langle 0, h\rangle$. (We assume throughout that the rows of A positively span the space since otherwise K_b is not bounded.) The lattice point $h \in Z^n$ ($h \neq 0$) is defined to be a *neighbor of the origin* if $\langle 0, h\rangle$ contains no lattice points in its *interior*. The collection of such neighbors is denoted by $N(A)$. Note that in this definition the special role of a_1 as the objective function has disappeared.

There are various conditions, of non-degeneracy type, on A to ensure that $N(A)$ is a test set for the integer programs (IP), or that $N(A)$ is nonempty and finite. Finiteness of $N(A)$ is proved in quantitative form. We can characterize matrices with identical sets of neighbors. It turns out that this collection $C(A)$ is a polyhedral set depending on the cones

$$C_i = \mathrm{pos}\{h \in N(A)\ :\ a_i h < 0\}$$

where A is a generic matrix. $C(A)$ has a product structure since the rows of the matrices in it vary in the interior of C_i^*, the polar of C_i, independently of each other.

There is a simplicial complex $K(A)$, associated with the matrix A, in which the neighbors of the origin play an important role. Assume, for simplicity's sake, that A is an $(n+1)\times n$ generic matrix. Then all bodies K_b are homothetic simplices. K_b is called *a maximal lattice point free convex body*, MLFC for short, if it contains no lattice point in its interior, but any other convex body which properly contain K_b does have a lattice point in its interior. It follows that the ith facet of a MLFC contains a lattice point, z_i, which is unique if A is generic, and z_i and z_j are distinct if $i \neq j$. The abstract simplicial complex $K(A)$ consists of all of the formal simplices $\{z_1, \ldots, z_{n+1}\}$, together with all of their subsimplices. $K(A)$ is an infinite simplicial complex since an arbitrary lattice translate of a MLFC is another MLFC. It can be shown, however, that $K(A)$ is locally finite: every lattice point is a vertex of finitely many simplices. In this setting h is a neighbor of the origin if and only if $\{0, h\}$ is a simplex of $K(A)$.

It is shown in [BHS] that the body of $K(A)$ is homeomorphic to R^n. A key step in the proof is the use of the so-called exponential map which is a simplicial map from $K(A)$ to the boundary of an $(n+1)$-dimensional unbounded convex body. The definition of $K(A)$ and the result on the body of $K(A)$ can be extended to the case when A is an $m \times n$ generic matrix [BSS]. Further, the group Z^n acts on $K(A)$ since the lattice translate of a simplex is also a simplex in $K(A)$. Thus one can factor out Z^n to obtain a CW complex $KZ(A)$, whose body turns out to be homeomorphic to the n-torus.

In a joint paper with Lovász and Scarf [BLS] we give a natural isomorphism $H_{n-1}(KZ(A)) \to Z^n$. A consequence is a minimax theorem relating the test set $N(A)$ to this isomorphism.

The above results have found applications and extensions elsewhere. For instance, Anders Björner [Bj] gives linear relations for the f-vector of $KZ(A)$. Peeva and Sturmfels use the complex produce resolutions of generic lattice ideals [PS].

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