On edge tricolorations of triangulations of simply connected surfaces

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Abstract

This paper studies the tricolorations of edges of triangulations of simply connected orientable surfaces such that the degree of each interior vertex is even. Using previous results on lozenge tilings, we give a linear algorithm of coloration for triangulations of the sphere, or of planar regions with the constraint that the boundary is monochromatic.

We define a flip as a shift of colors on a cycle of edges using only two colors. We prove flip connectivity of the set of solutions for the cases seen above, and prove that there is no flip accessibility in the general case where the boundary is not assumed to be monochromatic. Nevertheless, using flips, we obtain a tiling invariant, even in the general case.

We finish relaxing the condition, allowing monochromatic triangles. With this hypothesis, there exists some local flips. We give a linear algorithm of coloration, and strong structural results on the set of solutions.

Résumé

Cet article étudie les tricolorations des arêtes des triangulations des surfaces simplement connexes orientables, telles que le degré de chaque sommet interieur soit pair. A partir de résultats précédents sur les pavages par des losanges, nous donnons un algorithme linéaire de colorations pour les triangulations de la sphère, ou pour des regions du plan sous la contrainte que le bord est monochromatique.

Nous définissons un flip comme étant une inversion des couleurs sur un cycle d'aretes n'utilisant que deux couleurs. Nous prouvons la connexite par flips de l'ensemble des solutions dans les cas vus ci-dessus, et montrons que la connexité n'est pas toujours obtenue quand le bord n'est pas monochromatique. Néanmoins, grace aux flips, nous avons un invariant de pavage, valable dans le cas général.

Nous terminons en relachant les conditions par l'introduction de tuiles monochromatiques. Dans ce cas, il existe des flips locaux. Nous donnons un algorithme linéaire de colorations et des résultats structurels sur l'espace des solutions.

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1   Introduction

In 1990, W. P. Thurston [14] gave an algebraic study of tilings problems, based on ideas from J. H. Conway and J. C. Lagarias [4]. Especially, in his paper, W. P. Thurston studies the tilings simply connected regions of the triangular lattice, with lozenges formed from two triangles of the lattice. Using a height function and local transformations (called flips), an algorithm of tiling (i. e. an algorithm to produce a tiling, or decide that there is none) is exhibited. Remark that such a tiling can be seen as a coloration of edges in two colors (blue and red) such that each triangle has two blue edges and a red edge and the boundary is blue.

In the present paper, we add a color for edges, and study tricolorations. We are focused in the most natural problem, where the constrains are that each triangle must have an edge of each color and the boundary has to be monochromatic.

Our framework is larger than simply connected bounded regions of the triangular lattice studied by W. P. Thurston : we work in finite triangulations of orientable compact surfaces, such that the degree of each interior vertex is even. Our arguments hold in this framework, which points out the importance of topologic properties for this type of problem.

Using results about lozenge tilings, we first give an algorithm of coloration, and an algorithm of exhaustive generation of colorations. Afterwards we introduce *flips* (which are shifts of colors on a cycle of edges using only two colors). At the opposite as for the classical cases as lozenge tilings, these flips are not local, which creates a strong difficulty. We prove that one can transform any coloration into any other one by a sequence of our flips. The flip connectivity is an essential property for problems of uniform random generation [9].

If we assume that the coloration of the boundary is fixed, but not monochromatic, then the flip connectivity is lost. Nevertheless, even in this general framework, flips can be used to prove that the number of direct triangles (i. e. those whose colors yellow, blue, red are seen in this order turning clockwise around the triangle) is a invariant which does not depend on the coloration.

We finish allowing monochromatic triangles: in this case, local flips appear, which allow to get a linear algorithm of coloration and strong structural properties of the set of solutions.

2   Definitions and previous results

2.1   Triangulations

A *triangulation* (see [10] for details) of a compact surface S is a finite family $\{Tr_1, Tr_2, \ldots, Tr_p\}$ of closed subsets of S that cover S, and a family of homeomorphisms $\varphi_i : Tr_i' \to Tr_i$, where each Tr_i' is a a triangle of the plane $\mathbb{R}^2$. The subsets Tr_i are called "triangles", the subsets of Tr_i which are the images of the vertices and edges of the triangle Tr_i' under φ_i are also called *vertices* and *edges*. It is required that any two distinct triangles, Tr_i and Tr_j, either be disjoint, have a single vertex in common, or have an entire edge in common. Two vertices of a same triangle, or two triangles with an common edge, are said neighbors.

Let Φ be a triangulation. We say that Φ is *even* if Φ is a triangulation of a simply connected (i. e. each cycle can be contracted) orientable surface (i. e. homeomorphic to a sphere or a compact surface of the plane $\mathbb{R}^2$) such that each interior vertex has an even number of neighbors. This notion comes from the fundamental example of bounded subsets of the classical triangular lattice of the plane, without hole.

If Φ is even, then that there exists a bicoloration of triangles with white or black colors, in such a way that two neighbors triangles have not the same color. Now, we fix such a coloration. Edges of triangles of Φ can be directed in such a way that the three edges of any black triangle form a clockwise circuit, and the three edges of any white triangle form a counterclockwise circuit (see figure 1). For the sequel, we denote G_Φ as the directed graph defined by this way. For each pair (v, v') of neighbor vertices, we define $orient(v, v')$ by : $orient(v, v') = 1$ if (v, v') is a directed edge of G_Φ, and $orient(v, v') = -1$ otherwise. The corresponding indirected edge is denoted by $[v, v']$.

The main goal of this paper is the study of *tiling by trichromatic-edges triangles* (or trichromatic tiling for short), i. e. colorations of the edges of Φ with three colors (yellow, blue or red) in such a way that each triangle has exactly one edge of each color. To do it, we need some results on lozenge tilings.

2.2 Lozenge tilings

A *lozenge* is a pair of neighbor triangles of Φ. The common (undirected) edge is called the *central axis* of the lozenge. A *lozenge tiling* of Φ is a set of lozenges which cover the whole surface with neither gap nor overlap. In other words, it is a perfect matching on the triangles of Φ.

There exists a very powerful tool to study lozenge tilings on even triangulations : it is the notion of *height function*, introduced by W. P. Thurston [14] and independently in the statistical physics literature (see [2] for a review) for simply connected regions of the triangular lattice, and precisely studied and generalized by several authors ([3], [12], [13], [6]). The main results of the study are summarized below (see especially [13], [6] for details). The extension to even triangulations is straightforwards. Notice that the notions below can be applied in a more general framework ([1], [3], [12]).

2.2.1 Height functions

A lozenge tiling T of an even triangulation can be encoded by a height function h_T, defined as follows (see figure 1) : fix an origin vertex O of G_Φ (in the boundary of the surface, when it is not empty), for which $h_T(O) = 0$, and the following rule : if (v, v') is a directed edge of G_Φ such that $[v, v']$ is the central axis of a lozenge of T, then $h_T(v') = h_T(v) - 2$; otherwise $h_T(v') = h_T(v) + 1$. This definition is coherent, since it is coherent on each triangle and we have the simple connectivity.

2.2.2 Order

Let (T, T') be a pair of tilings of Φ. We say that $T \leq T'$ if for each vertex v of G_Φ, $h_T(v) \leq h_{T'}(v)$. The functions $h_{inf(T,T')} = min(h_T, h_{T'})$ and $h_{sup(T,T')} = max(h_T, h_{T'})$ are height functions of tilings, (which can be interpreted in order theory that the set
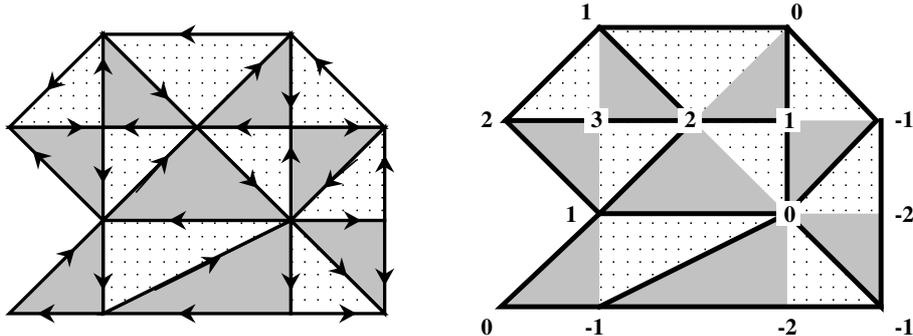
Figure 1: Left : An even triangulation and the induced orientation of edges. Right : A lozenge tiling and its height function

of tilings has a structure of distributive lattice (see for example [5] for basis of lattice theory).

2.2.3 Flips

Let v be an interior vertex such that all the directed edges of G_Φ ending in v correspond to the central axes of lozenges of a tiling T. A flip is the replacement of all these lozenges by lozenges whose central axis correspond to an edge starting in v (see figure 2). A new tiling T_{flip} is obtained by this way, and T and T_{flip} are comparable for the order defined above on tilings. Moreover $T \leq T'$ if and only if there exists an increasing sequence $(T = T_0, T_1, \ldots, T_p = T')$ of tilings such that for each integer i such that $0 \leq i < p$, T_{i+1} is deduced from T_i by a flip. As a corollary we get the *flip connectivity* : given any pair (T, T') of tilings of Φ, on can pass from T to T' by a sequence of flips and, more precisely, the minimal number of flips to pass from T to T' is $\sum_v |h_T(v) - h_{T'}(v')|/3$.
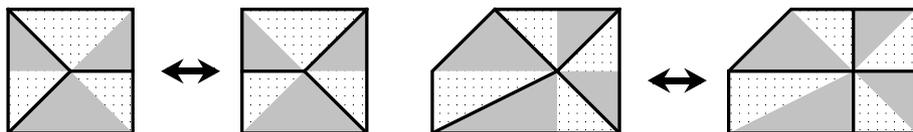

Figure 2: Examples of flips

2.2.4 Construction

There exists a minimal tiling which has a convexity property that no local maximum can exist, except on the boundary or in the origin vertex, since, otherwise, a flip can be done around the local maximum.

From this property, there exists a linear (in the number of vertices) algorithm of tiling, which constructs the minimal tiling when there exists a tiling, or, otherwise, claims that there is no tiling [14].

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2.2.5 Exhaustive generation

Recently, using the lattice structure, efficient algorithms to generate all the lozenge tilings of a polygon have been exhibited [6], [7]. Especially, in the algorithm of [7], the waiting time between two tilings is linear in the area of the polygon and the memory space is polynomial.

3 Construction and generation of trichromatic tilings

3.1 Algorithm of tiling

In this section, we first apply the previous results to exhibit an algorithm, which given a triangulation whose edges of the boundary are colored in yellow, either colors the other edges (in either yellow, blue or red) in such a way that a trichromatic tiling is obtained, or claims that the triangulation has no trichromatic tiling.

Theorem 3.1 *Let Φ be a triangulation and Φ_{inter} denote the new triangulation obtained removing triangles with an edge on the boundary of the surface.*

The triangulation Φ has a trichromatic tiling with yellow boundary if and only if Φ_{inter} has a lozenge tiling.

Moreover, we have a polynomial (linear for even triangulations) time algorithm to build a trichromatic tiling when there exists one (and claim that there is no tiling, otherwise).

Proof. We suppose that we have a tiling by trichromatic-edges triangles. Then, the red (resp. yellow or blue) edges are clearly the central axes of lozenges of a tiling of Φ_{inter}.

Conversely, assume that we have a lozenge tiling T of Φ_{inter}. Color the central axes of the lozenges of T in yellow. Now, we denote by $H_{T,y}$ the symmetric graph on the cells of Φ and such that two cells are joined by an edge if and only if this two cells are adjacent by a non-yellow edge. It is clear that this graph is a disjoint union of elementary even cycles. Moreover, its edges are in one to one correspondence with the non-yellow edge of triangulation. It suffices to alternatively color on each cycle the edges in red and blue to obtain a tiling by trichromatic-edges triangles.

Moreover, given a tiling T of Φ_{inter}, the method above of alternatively coloring cycles of $H_{T,y}$ gives a trichromatic tiling in linear time. This gives the algorithmic part of the theorem, since such a lozenge tiling of Φ_{inter} can be obtained in polynomial time (and linear time for even triangulations, using height functions). $\qquad\square$

3.2 Exhaustive generation

If we have an algorithm to generate all the tilings of Φ_{inter} with yellow boundary, then we easily deduce an algorithm which generates all the trichromatic tilings of Φ: it suffices, for each tiling T of Φ_{inter}, to generate all the corresponding trichromatic tilings of Φ.

Precisely, this can be done as follows: let $(C_1, C_2, \ldots, C_{p_T})$ be a fixed sequence consisting in all the anti-yellow cycles of ϕ induced by T. There exists 2^{p_T} trichromatic tilings which may be constructed from T and each of them can be encoded by a sequence $(b_1, b_2, \ldots, b_{p_T})$ of bits, each bit encoding one of the two possible colorations of the edges

of the corresponding cycle. Thus, all these trichromatic tilings can be successively generated, starting from the tiling corresponding to the sequence of bits $(0, 0, \ldots, 0)$, and finishing by the tiling corresponding the sequence $(1, 1, \ldots, 1)$, using the lexicographic order.

Thus, for the even triangulations, using the algorithm of [7] to generate lozenge tilings, we obtain an algorithm of exhaustive generation of trichromatic tilings with yellow boundary. For this algorithm, the waiting time between two tilings is linear in the number of vertices of the triangulation and the memory space is polynomial.

4 Accessibility by cyclic flips

Let T be a tiling by trichromatic-edges triangles. We recall that we denote by $H_{T,y}$ (resp. $H_{T,b}$, $H_{T,r}$) the symmetric graph on the cells of Φ and such that two triangles are joined by an edge if and only if this two triangles are linked by a non-yellow (resp. non-blue, non-red) edge. This graphs are disjoint unions of elementary even cycles. We call an *anti-yellow cycle* (resp. anti-blue cycle, anti-red cycle), a cycle in $H_{T,y}$ (resp. $H_{T,b}$, $H_{T,r}$). An anti-yellow (resp. anti-red, anti-blue) *cyclic flip* is the inversion of the red and blue (resp. blue and yellow, yellow and red) edges in an anti-yellow (resp. anti-red, anti-blue) cycle. We obtain by this transformation a new tiling by trichromatic-edges triangles. A natural question is to know if we can obtain all the other tilings of Φ from T by a sequence of cyclic flips.

Theorem 4.1 *Let Φ be an even triangulation. All the tilings by trichromatic-edges triangles of Φ with yellow boundary are mutually flip accessible.*

Moreover, the flip accessibility is preserved if the only anti-red and anti-blue cyclic flips allowed are local (i. e. on cycles formed by all triangles sharing a common vertex).

Proof. We have seen below that we can associate to a tiling by trichromatic-edges triangles T a lozenge tiling T_y in considering the yellow edges of T as the central axes of lozenges of T_y.

Now, suppose that we have two tilings by trichromatic-edges triangles T_1 and T_2, then we know by the results of section 2 that there exists a sequence of lozenge flips to transform $T_{1,y}$ into $T_{2,y}$. We will prove by induction on the length $\Delta(T_1, T_2)$ of the minimum sequence of flips which transforms $T_{1,y}$ into $T_{2,y}$ that T_1 and T_2 are accessible by cyclic flips.

If $\Delta(T_1, T_2) = 0$, then $T_{1,y} = T_{2,y}$, and so, $T_{1,y}$ and $T_{2,y}$ have exactly the same anti-yellow cycles. It is easy to see that we can obtain T_2 from T_1 by a sequence of (anti-yellow) cycle flips.

Suppose that we have the property for any pair (T_0, T_0') such that $\Delta(T_0, T_0') = n$ and take a pair (T_1, T_2) such that $\Delta(T_1, T_2) = n + 1$. We will prove that there exists a trichromatic tiling T' such that T' can be deduced from T_1 by a sequence of cyclic flips and $\Delta(T', T_2) = n$. If we prove this point it is clear that we have achieved the proof by induction.

We denote by x the vertex where we do the first flip f_1. Consider the set ϕ_x of triangles a vertex of which is x. Each of these triangles has a unique edge which does not contain

x, and ϕ_x is an even triangulation of a surface S_x, homeomorphic to a closed disk. We distinguish two cases according to the colors on the boundary of S_x.

The simple case is when this boundary is monochrome, for example red, in T_1. In this case, we have an anti-red cyclic flip around x, (which also is a lozenge flip) which transforms T_1 into T', and $\Delta(T', T_2) = n$, which gives the result.

The tricky case is when the boundary of S_x is not monochrome in T. Consider the lozenge tiling L_x of ϕ_x such that non-yellow edges issued from x are central axes of lozenges of L_x. Two neighbors triangles ϕ_x of are in the same lozenge of L_x if and only if they are in the same cycle. Notice that if triangles are ordered clockwise, all the first triangles of lozenges of L_x have the same color, which can be assumed to be white.

Then, take an anti-yellow cycle C of T which contains at least one of the lozenges of L_x. Let $(l_1, l_2, \ldots, l_p)$ be the sequence of lozenges of L_x in C, in a clockwise order around x (see figure 3). We direct C in such a way that C comes into l_1 by its white triangle and leaves l_1 by a black triangle. Notice that, following C, all the edges which are crossed when one passes from a white triangle to a black triangle have the same color.
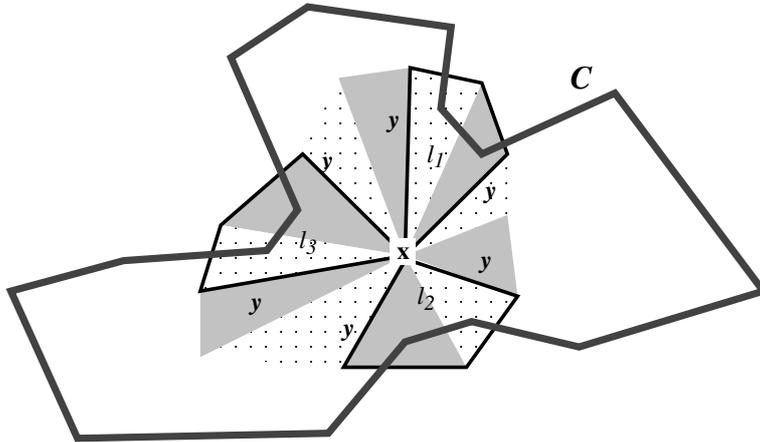

Figure 3: The intersection of the anti-yellow cycle C with S_x

What happens, following C, after having left l_1 ? Because of planarity (especially Jordan's theorem on loops of the plane), the next time that C comes back into S_x, then C necessarily comes into l_2, (since otherwise, l_2 cannot be visited, later in the cycle) by its white triangle (since otherwise C cannot visit other lozenges without cutting itself). Thus, repeating the argument, C comes into all the lozenges of the sequence $(l_1, l_2, \ldots, l_p)$ by the white triangles.

In conclusion of this study, all the edges of the boundary of ϕ_x which are edges of triangles of C have the same color. So, we can make cycle flips around some of the anti-yellow cycles which meet S_x to obtain a tiling T'_1 which has a monochrome boundary of S_x. Moreover, observe that $T_{1,y} = T'_{1,y}$. So, we have gone back to the first simple case. This achieves the proof. □

4.1 Application to random sampling

The above theorem proves that the following Markov chain is connected.

Markov Process: Choose at random an interior vertex or an interior edge, and, if it is possible, make a (local, anti-blue or anti-red) flip around the chosen vertex, or an anti-yellow flip on the cycle containing the chosen edge.

Notice that this chain is also symmetric, and has some loops. Thus, it converges to the uniform distribution. An open question is to know if it is rapidly mixing or not.

4.2   A counter-example in the planar general case

In the figure below, we present a example which shows that there is no general flip accessibility, when the boundary is not monochromatic.
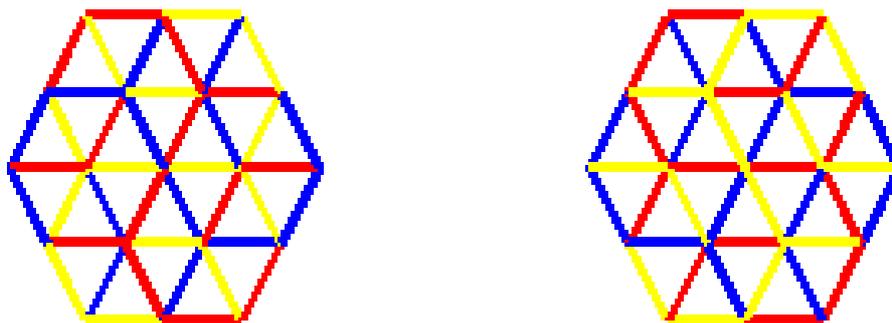

Figure 4: Two tricolorations, equal on the boundary, with no accessibility from one to the other one, since each of these colorations has no cycle.

5   Invariant of orientations of triangles

The goal of this section is to describe an invariant for the tilings. In other words, we look for some properties of the tilings which depend only on the triangulation.

We limit ourselves to even (consequently orientable) triangulations. In this case, remark that there exists to type of colored triangles : the *direct* colored ones and the *indirect*. If we turn counterclockwise around a direct triangle, we see consecutively a red, a blue and a yellow edge. Conversely, If we turn around a indirect triangle, we see a red, a yellow and a blue edge.

For each trichromatic tiling T, $direct(T)$ is the number of direct triangles of T and $indirect(T)$ is the number of indirect triangles of T. In this section, we will prove that these values are of tiling invariants of tilings.

5.1   The sphere

We start with the simple case of the sphere, from which the result will be extended later.

Theorem 5.1 (invariant for the sphere)   *All the tilings by trichromatic-edges triangles of an even triangulation of the sphere have the same number of direct (and indirect) triangles.*

Proof. As all tilings are accessible by cyclic flips, it suffices to show that the number of direct triangles in a cycle C in $H_{T,y}$ or $H_{T,b}$ or $H_{T,r}$ does not change when we make a flip on C. Indeed, when we make a flip on C, we transform all its direct triangles into indirect ones, and conversely.

So, we have to prove that, for any cycle, there is the same number of direct triangles than indirected ones.

5.1.1 Coding function

To do it, we introduce the *coding functions* of trichromatic tilings. Let T such a tiling, the function f_T is defined from the set of vertices of G_Φ to the set $\mathbb{C}$ of complex numbers, by : $f_T(O) = 0$ (where O denotes the origin vertex), and for each directed edge (v, v') of G_Φ $f_T(v') - f_T(v)$ is equal to 1 (resp. j, j^2) if the edge $[v, v']$ is yellow (resp. blue, red) in T (where j denotes the unique complex number such that $j^3 = 1$ and the (purely) complex component of j is positive). This definition is coherent, since it is coherent for any triangle, and the sphere is simply connected.

5.1.2 Combinatorial boundaries of cycles

Now, let C be an anti-yellow cycle of T, assumed to be directed clockwise. let Tr be a triangle of C, the *successor* of Tr is the first triangle Tr' of C, obtained after Tr in the cycle, such that the yellow edges of Tr and Tr' are not disjoint. By this definition, we define some (two in fact) disjoint circuits of triangles, whose union cover C. Each of these circuits induces a circuit $(v_0, v_1, \ldots, v_p = v_0)$ of vertices (called a combinatorial boundary of C) such that, for each integer i such that $0 \le i \le p$, $[v_i, v_{i+1}]$ is the yellow edge of the successor Tr' of the triangle Tr whose yellow edge is $[v_{i-1}, v_i]$.
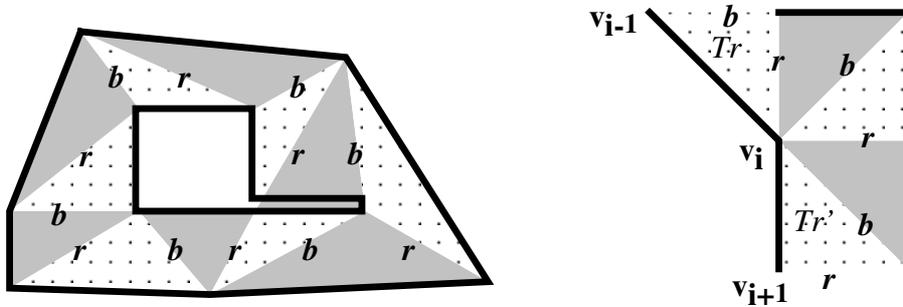

Figure 5: Left : the combinatorial boundaries of a cycle. Right : a triangle and its successor have the same orientation if and only if they have the same color.

Notice that Tr and Tr' have the same orientation if and only if Tr and Tr' have the same color, that is if and only if $orient(v_{i-1}, v_i) = orient(v_i, v_{i+1})$ (since indirected edges issued from v_i crossing the cycle C alternatively have red and blue colors) (see figure 5). Moreover, for each combinatorial boundary, $\sum_{i=1}^{p} orient(v_{i-1}, v_i) = \sum_{i=1}^{p} (f_T(v_i) - f_T(v_{i-1})) = f_T(v_p) - f_T(v_0) = 0$, which means that, the number of edges of positive orientation is equal to the number of edges of negative orientation. Thus, for each circuit of triangles, the number of direct triangles is equal to the number of indirect triangles.

The same argument can be done (up to the existence of a factor j or j^2 in the sequence of equalities) for blue or red cycles, which achieves the proof. $\square$

5.2 General case

We now prove the analog theorem for the triangulations of regions region of the plane. Notice that we do not necessarily assume that the boundary is monochromatic.

Theorem 5.2 (invariant for planar surfaces) *All the tilings by trichromatic-edges triangles of an even triangulation of a simply connected compact surface of the plane with the same fixed coloration of the boundary have the same number of direct (and indirect) triangles.*

Proof. We use the theorem 5.1 to prove that. Let T_1 and T_2 be two tilings of an even triangulation Φ of a simply connected compact surface of the plane with the same fixed coloration of the boundary. We denote by $[T_1, T_2]$ the tiling of the triangulation[1] of the sphere obtained gluing (by identification) the boundary of T_1 with the boundary of T_2. Notice that $[T_1, T_2]$ tiles an even triangulation.

Now, consider both tilings $[T_1, T_1]$ and $[T_1, T_2]$, by the theorem 5.1 they have the same number of direct (and indirect) triangles. Obviously, this involves that T_1 and T_2 have the same number of direct (and indirect) triangles. $\square$

6 Related problems

In this section, we allow monochromatic triangles on even triangulations. We see that the problem becomes much easier since there exists local flips. In each case below, we use a height function and the ideas previously used in [11] for Wang tiles.

6.1 Tiling by trichromatic-edges and monochromatic-edges triangles

Firstly, we relax our condition allowing all types of monochromatic triangles. Such a tiling T can be encoded using a function g_T from the set of vertices of G_Φ to the set $\mathbb{Z}/3\mathbb{Z}$ of integers modulo 3, such that $g_T(O) = 0$ (where O denotes the origin vertex) and for each directed edge (v, v') of G_Φ, $g_T(v') - g_T(v)$ is equal to 0 (resp. 1, 2) if the edge $[v, v']$ is yellow (resp. blue, red) in T. This definition is coherent, since it is coherent for any triangle, and we have the simple connectivity.

Notice that, conversely, each function g from vertices of G_Φ to $\mathbb{Z}/3\mathbb{Z}$ such that $g(O) = 0$ induces a coloration of Φ with trichromatic and monochromatic triangles. Thus

- there exists such a coloration if and only if the function g_T can be defined on the boundary with no contradiction (this condition is, of course satisfied in the sphere, which has an empty boundary; in the other cases, we just have to follow the boundary to construct g_T on the boundary),

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[1]Formally, we do not exactly have a triangulation when a triangle of Φ has two edges on the boundary (since two glued triangles share two edges), nevertheless our arguments can be obviously adapted to this case

- given a coloration of the boundary, there exists an easy linear algorithm which constructs a tiling, if there exists one, or claims that there is no tiling otherwise,

- when a tiling exists, there exists 3^N tilings satisfying the same boundary condition, where N denotes the number of free vertices (i. e. vertices which are not the origin for the spheres, vertices which are not on the boundary in the other cases),

- we define a flip as the change the value $g_T(v)$ of a free vertex v (which changes all the colors of the edges with v as endpoint). We have the flip connectivity for each set of tilings satisfying the same boundary condition, and moreover the minimal number of necessary flips to pass from a tiling T to a tiling T' is $\sum_v (1 - \delta(g_T(v), g_{T'}(v)))$ (where $\delta(i, j) = 1$ if $i = j$ and $\delta(i, j) = 0$ otherwise),

- the problems of exhaustive generation and uniform random sampling are (nearly) trivial.

6.2 Tiling by trichromatic-edges and one fixed monochromatic-edges triangle

We now only allow monochromatic yellow triangles (and trichromatic triangles). This study is nearly the same as the one for lozenge tilings, so we sketch it.

Such a tiling T can be encoded using a height function g'_T from the set of vertices of G_Φ to the set $\mathbb{Z}$ of integers, such that $g_T(O) = 0$ (where O denotes the origin vertex) and for each directed edge (v, v') of G_Φ, $g_T(v') - g_T(v)$ is equal to 0 (resp. 1, −1) if the edge $[v, v']$ is yellow (resp. blue, red) in T. Conversely, for each function g' from vertices of G_Φ to $\mathbb{Z}$ such that $g'(O) = 0$ and for each edge (v, v') of G_Φ, $|g'(v) - g'(v')| \leq 1$, there exists a tiling T such that $g' = g'_T$. If g' and g'' are functions satisfying the above conditions, then $min(g', g'')$ and $max(g', g'')$ also satisfy the same conditions. This implies that each set of tilings with the same boundary has a structure of distributive lattice for the order defined by : $T \leq T'$ if, for each vertex v of Φ, $g_T(v) \leq g_{T'}(v)$.

We define a flip as, when it is possible, the change of the value $g'_T(v)$ of a free vertex v *of one unit* (which changes all the colors of the edges with v as endpoint). Let T and T' be tilings such that $T < T'$ and v_0 be a vertex such that $g_{T'}(v_0) - g_T(v_0)$ is maximal, and $g_T(v_0)$ is minimal with the previous condition. One easily sees that a flip can be done in v_0, which increases the height function. This yields, repeating the argument that there exists an increasing sequence $(T = T_0, T_1, \ldots, T_p = T')$ of tilings such that for each integer i such that $0 \leq i < p$, T_{i+1} is deduced from T_i by a flip. As a corollary, we get the flip formula : the minimal number of flips to pass from T to T' is $\sum_v |g'_T(v) - g'_{T'}(v')|$.

Finally, remark that the minimal tiling T_{min} of has no local maximum for free vertices (since, otherwise, a height decreasing flip can be done), which yields to a linear algorithm of tiling of the same kind as the algorithm of W. P. Thurston [14] for lozenges and dominoes. We also have an algorithm of generation, which is similar to the algorithm for lozenge tiling of [7].

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