The importance of 2 in algebraic combinatorics

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Abstract

I give different examples from classical algebraic combinatorics where the operation of cutting in two (sets, alphabets, kernels or generating functions) constitutes the main tool.

Résumé. Je donne différents exemples en combinatoire algébrique où l'outil essentiel est la division en deux ou par deux: ensembles d'interpolation, alphabets, noyau de Cauchy, fonctions génératrices.

1 Lagrange Interpolation

Given two sets of indeterminates $\mathbb{A}$, $\mathbb{B}$, called *alphabets*, let $R(\mathbb{A},\mathbb{B}) := \prod_{a\in\mathbb{A},b\in\mathbb{B}}(a-b)$ denote the *resultant* of the two polynomials $\prod_{a\in\mathbb{A}}(x-a)$ and $\prod_{b\in\mathbb{B}}(x-b)$.

The Lagrange interpolation can be interpreted as an operator on functions of one variable :

$$f \to \sum_{a\in\mathbb{A}} f(a)/R(a,\mathbb{A}\setminus a) \ , \tag{1}$$

where $\mathbb{A}\setminus a$ denotes the set difference $\mathbb{A}\setminus\{a\}$.

It is a fundamental too for the reduction of rational fractions in one variable, or for computation of residues. Since the start, mathematicians looked for generalizations. For example, the main theorem in Jacobi's thesis [7] deals with the operator

$$\mathfrak{Sym}(n')\otimes\mathfrak{Sym}(n'') \ni f\otimes g \longrightarrow \sum_{\mathbb{A}',\mathbb{A}'':\mathbb{A}'\cup\mathbb{A}''=\mathbb{A}} f(\mathbb{A}')g(\mathbb{A}'')/R(\mathbb{A}',\mathbb{A}'') \ , \tag{2}$$

sum over all decompositions of $\mathbb{A}$ into two disjoint subsets of cardinalities n', n'', f and g being symmetrical functions.

In modern language, the operator considered by Jacobi is the *Gysin morphism* associated to a relative Grassmannian. Its main property is expressed by Bott's theorem, which states that if f and g are two Schur functions, then the image of $f \otimes g$ is equal to 0 or $\pm$ a Schur function. Indexing Schur functions S_I by increasing partitions, if $f = S_I$, $g = S_J$, $I = [i_1, \dots, i_{n'}]$, $J = [j_1, \dots, j_{n''}]$, then the precise statement is that the image is $S_K(\mathbb{A})$, with

$$K = [i_1, \dots, i_{n'}, j_1 - n', \dots, j_{n''} - n'] \ .$$

In other words, Jacobi's operator essentially consists in concatenating partitions indexing Schur functions. However, such an easy operator is sufficient to recover much of the theory of symmetric functions. Jacobi did find Schur functions, not using the preceding operator but a simpler related one :

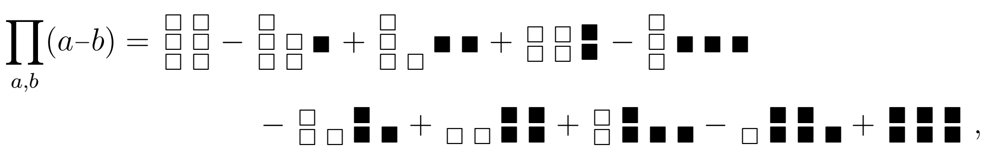
$$f(a_1, \dots, a_n) \longrightarrow \sum\nolimits_{\sigma \in \mathfrak{S}_n} \pm f^\sigma / \Delta(\mathbb{A}) \ , \qquad (3)$$

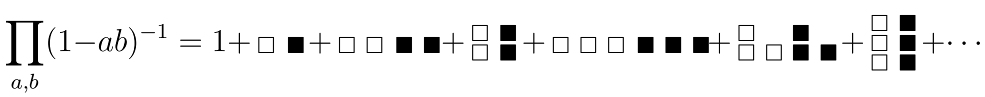
sum over all permutations of $a_1, \dots, a_n$, $\Delta(\mathbb{A}) = \prod_{i<j}(a_i - a_j)$ being the Vandermonde in $\mathbb{A}$.

2 The Resultant

The *resultant* $R(\mathbb{A}, \mathbb{B}) = \prod_{a,b}(a-b)$ is a fundamental topic of algebra since the 18[e] century. It also leads to Schur functions, but it moreover can be used to define a fundamental scalar product on the space of symmetric polynomials, relative to which the set of Schur function is a fundamental orthonormal basis.

Coding Schur functions by diagrams of partitions, one has the two following equivalent decompositions attributed to Cauchy:





white diagrams coding Schur functions in $\mathbb{A}$, black ones in $\mathbb{B}$ (we have taken $\mathbb{A}, \mathbb{B}$ of cardinalities $3, 2$ in the first expression; the enumeration involves all pairs of partitions of complementary shape in the rectangle 3×2; the second summation runs over all pairs of partitions of the same shape, with a limit on the number of parts equal to the minimum of $|\mathbb{A}|, |\mathbb{B}|$).

Both formulas are related to the Cauchy matrix $\left[(1-ab)^{-1}\right]$ which can be factorized in two infinite Vandermonde matrices, one in $\mathbb{A}$, the other in $\mathbb{B}$. Binet-Cauchy formula for the minors of a product of two matrices gives the two expansions (cf. [14]).

3 Non-commutative symmetric functions

To access to non-commutativity is easy, representing the resultant planarly:

$$R(\mathbb{A},\mathbb{B}) = \begin{matrix} (b_1+a_3) & (b_2+a_3) & (b_3+a_3) \\ (b_1+a_2) & (b_2+a_2) & (b_3+a_2) \\ (b_1+a_1) & (b_2+a_1) & (b_3+a_1) \end{matrix} \tag{4}$$

This time, alphabets are totally ordered and letters do not commute.

Expanding $R(\mathbb{A},\mathbb{B})$ means choosing in each box either a letter a or a letter b.

Reading letters a's by columns (downwards, then from left to right), and reading in the same way letters b's after reversing the axes of coordinates, one gets pairs of words.

For example the term $\begin{matrix} a_3 & b_2 & a_3 \\ b_1 & a_2 & b_3 \\ a_1 & b_2 & b_3 \end{matrix}$ gives the pair of words

$$a_3a_1 \cdot a_2 \cdot a_3 \qquad b_3b_2 \cdot b_3b_1 \cdot b_2$$

At this stage, $R(\mathbb{A},\mathbb{B})$ can be interpreted as an element in the tensor product of two free algebra $\mathbb{Z} < \mathbb{A}^* > \otimes \mathbb{Z} < \mathbb{B}^* >$. Quotienting each free algebra by the Robinson-Schensted relations (which transform words into tableaux), one gets a sum in the tensor product $\mathfrak{Plac}(\mathbb{A}) \otimes \mathfrak{Plac}(\mathbb{B})$ of two *plactic algebras* :

$$R(\mathbb{A},\mathbb{B}) = \sum \pm S_I(\mathbb{A}) S_J(\mathbb{B}) \ , \tag{5}$$

where a Schur function S_I is now to be interpreted as the sum of all tableaux of shape I (or of the corresponding words, reading tableaux by columns), and where I,J are complementary partitions, as in the commutative case.

For example, when $|\mathbb{A}| = 3 = |\mathbb{B}|$, a term $S_{122}(\mathbb{A})S_{13}(\mathbb{B})$ means

$$\sum \begin{array}{|c|c} \hline a_3 & \\ \hline a_2 & \multicolumn{1}{c|}{a_5} \\ \hline a_1 & \multicolumn{1}{c|}{a_4} \\ \hline \end{array} \sum \begin{array}{|c|c|c|} \hline b_2 & \multicolumn{2}{c}{} \\ \hline b_1 & b_3 & b_4 \\ \hline \end{array} = \sum a_3a_2a_1 \cdot a_5a_4 \sum b_2b_1 \cdot b_3 \cdot b_4 \ ,$$

sum over all letters such that $a_3 > a_2 > a_1$; $a_2 \leq a_5$; $a_1 \leq a_4$; $b_2 > b_1$; $b_1 \leq b_3 \leq b_4$.

This non-commutative version of Cauchy formula (in fact, rather the one for the expansion of $\prod(1-ab)^{-1}$) is due to Bender and Knuth [1].

Let us mention that there are more sophisticated theories of non-commutative symmetric functions (cf. [6]).

4 Schubert polynomials

The resultant $R(\mathbb{A},\mathbb{B})$ is symmetrical in all the a's , and all the b's, and also symmetrical in $\mathbb{A}$ and $\mathbb{B}$.

One keeps a symmetrical role for $\mathbb{A}$ and $\mathbb{B}$, but disymmetrize each alphabet by taking *half of the resultant*:

$$K_n(\mathbb{A},\mathbb{B}) := \prod_{i,j:i+j\leq n} (a_i - b_j) \ .$$

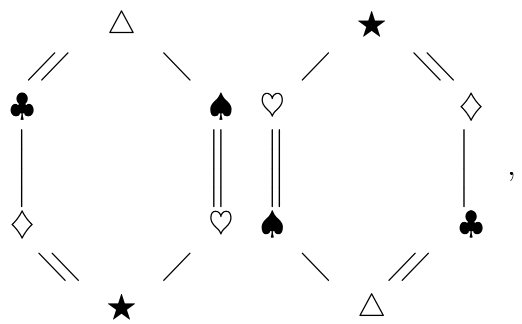
There is still a "canonical" decomposition :

$$K_n(\mathbb{A},\mathbb{B}) = \sum_{\sigma',\,\sigma''} \pm X_{\sigma'}(\mathbb{A})\, X_{\sigma''}(\mathbb{B}) \ , \tag{6}$$

but this time, the indices of the *Schubert polynomials* X_σ are permutations, and one passes from $\sigma' = [i_1,\ldots,i_n]$ to σ'' by reversion: $\sigma'' = [i_n,\ldots,i_1]$, instead of taking complementary partitions..

For example $K_3(\mathbb{A},\mathbb{B})$ decomposes as

$$\begin{matrix} (a_2-b_1) & \\ (a_1-b_1) & (a_1-b_2) \end{matrix} =$$



using symbols for the 6 permutations indexing the pairs of Schubert polynomials; more explicitely the above display stands for the decomposition :

$$\begin{aligned}(a_2-b_1)(a_1-b_1)(a_1-b_2) &= X_{321}(\mathbb{A})X_{123}(\mathbb{B}) - X_{312}(\mathbb{A})X_{213}(\mathbb{B}) \\ &\quad - X_{231}(\mathbb{A})X_{132}(\mathbb{B}) + X_{132}(\mathbb{A})X_{231}(\mathbb{B}) + X_{213}(\mathbb{A})X_{312}(\mathbb{B}) \\ &\quad - X_{123}(\mathbb{A})X_{321}(\mathbb{B}) \ .\end{aligned}$$

Letting n tends towards ∞, one gets a linear basis $X_\sigma(\mathbb{A})$ of polynomials in $a_1, a_2, \ldots$, indexed by infinite permutations permuting only a finite number of integers. Schur functions $S_J(a_1, \ldots, a_n)$, any $n \in \mathbb{N}$, any partition J, occur as a sub-family.

Many algebraic computations of symmetric can be visualized at the level of diagrams of partitions. The same is true for what concerns Schubert polynomials, but this time one needs *diagrams of permutations*, which were defined by Rothe in 1800 to represent the inversions of a permutation and settle the problem of signs in the expansion of a determinant. In his thesis, Kohnert [8] used deformations of Rothe diagrams to generate Schubert polynomials. His conjectural rule gave rise to much work still in progress.

Let us mention another way of using the Rothe diagram, extending the notion to *alternating sign matrices* [2], instead of permutation matrices only. Each box of coordinate $[i, j]$ of a diagram gives a contribution $a_i + b_j - a_i b_j$, but apart from usual boxes, there are also entries –1, which give a contribution $(1 - a_i)(1 - b_j)$. The *weight* of a diagram is the product of such elementary weights, and a Schubert polynomial (in $\mathbb{A}, \mathbb{B}$) is the sum of the weights of a certain set of alternating sign matrices. Thus, Schubert or Grothendieck polynomials are obtained by just recording coordinates of boxes of diagrams [11], as in Kohnert's rule.

5 Free Schubert polynomials

The planar representation of the kernel $K_n(\mathbb{A}, \mathbb{B})$ instantly furnishes a free version, using the same rules for expansion as in (4).

Each term in the expansion of $K_n(\mathbb{A}, \mathbb{B})$ corresponds to a choice of letters a or b. Deciding to read the a's by columns, then reversing the axes of coordinates and reading in the same way the b's, one gets from $K_n(\mathbb{A}, \mathbb{B})$ a sum of pairs of words, then, by projection, an element in $\mathfrak{Plac}(\mathbb{A}) \otimes \mathfrak{Plac}(\mathbb{B})$:

$$K_n(\mathbb{A}, \mathbb{B}) = \sum \pm X_{\sigma'}(\mathbb{A})\, X_{\sigma''}(\mathbb{B}) \ , \qquad (7)$$

this time Schubert polynomials being sums of tableaux.

For example, in the expansion of $K_4(\mathbb{A}, \mathbb{B})$, there will be a term $X_{1342}(\mathbb{A}) X_{2431}(\mathbb{B})$ which will appear as

$$\left(\begin{array}{|c|}\hline a_3 \\ \hline a_2 \\ \hline \end{array} + \begin{array}{|c|}\hline a_3 \\ \hline a_1 \\ \hline \end{array} + \begin{array}{|c|}\hline a_2 \\ \hline a_1 \\ \hline \end{array} \right) \left(\begin{array}{|c|c}\cline{1-1} b_3 & \\ \cline{1-1} b_2 & \\ \hline b_1 & \multicolumn{1}{c|}{b_2} \\ \hline \end{array} + \begin{array}{|c|c}\cline{1-1} b_3 & \\ \cline{1-1} b_2 & \\ \hline b_1 & \multicolumn{1}{c|}{b_1} \\ \hline \end{array} \right) .$$

However, the tableaux in the expansion of a Schubert polynomial do not necessarily have the same shape, and the possible pairs of tableaux are not as easily characterized as in the case of the resultant.

6 Quadratic relations between minors

Plücker relations between minors of any matrix have already been obtained in the 18^e century (cf. the first volume of Muir [15]).

A typical one looks like

$$\begin{array}{|c|c|}\hline 2&4\\ 1&3\\ \hline\end{array} - \begin{array}{|c|c|}\hline 3&4\\ 1&2\\ \hline\end{array} + \begin{array}{|c|c|}\hline 4&3\\ 1&2\\ \hline\end{array} = 0\ .$$

Bigger relations appear in the literature, such as Garnir's relations

$$\begin{array}{|c|c|}\hline 4&8\\ 3&7\\ 2&6\\ 1&5\\ \hline\end{array} = \begin{array}{|c|c|}\hline 6&8\\ 5&7\\ 2&4\\ 1&3\\ \hline\end{array} + \begin{array}{|c|c|}\hline 6&8\\ 3&7\\ 5&4\\ 1&2\\ \hline\end{array} + \begin{array}{|c|c|}\hline 6&8\\ 3&7\\ 2&4\\ 5&1\\ \hline\end{array} + \begin{array}{|c|c|}\hline 4&8\\ 6&7\\ 5&3\\ 1&2\\ \hline\end{array} + \begin{array}{|c|c|}\hline 4&8\\ 6&7\\ 2&3\\ 5&1\\ \hline\end{array} + \begin{array}{|c|c|}\hline 4&8\\ 3&7\\ 6&2\\ 5&1\\ \hline\end{array}\ .$$

A combinatorialist will note that the right side is obtained by placing $6, 5$ in all possible manners in the first column. He will then inquire about the meaning of the symbols. Here one has a $4 \times \infty$ matrix, a column denotes a minor of order 4, a pair of columns, a product of two minors, and the above relation is a quadratic relation between minors of a matrix (the only restriction is that the entries of the matrix belong to a commutative algebra).

In fact, in the case considered by Garnir, the matrix was the Vandermonde matrix of powers of $x_1, x_2, \ldots$. Thus the column $[4, 3, 2, 1]$ denotes the Vandermonde $\prod_{1\le i<j\le 4}(x_i - x_j)$. We shall rather write $\Delta(1234)$. Now, Garnir's relation becomes

$$\begin{aligned}\Delta(1234)\Delta(5678) = \Delta(1256)\Delta(3478) + \Delta(1536)\Delta(2478) + \Delta(5236)\Delta(1478)\\ + \Delta(1564)\Delta(2378) + \Delta(5264)\Delta(13787) + \Delta(5634)\Delta(1278)\ . \quad (8)\end{aligned}$$

Dividing by $\Delta(1234)\Delta(56)\Delta(78)$, and using the notation $R(56, 78)$ for the resultant $R(\{x_5, x_6\}, \{x_7, x_8\}$ one gets

$$R(56, 78) = \frac{R(12, 56)R(34, 78)}{R(12, 34} + \frac{R(13, 56)R(24, 78)}{R(13, 24)} + \cdots + \frac{R(34, 56)R(12, 78)}{R(34, 12)}\ , \quad (9)$$

and we are now back to the case of Jacobi, with a summation over all subsets of $\{1, 2, 3, 4\}$ of order 2. One will find in the work of Sylvester [16] many computations of this type.

The restriction of having used minors of a Vandermonde matrix is lifted by a multi-linearity argument.

7 Ordering the symmetric group

Following Dedekind, the best way to define real numbers is to cut the field of rational numbers into two complementary intervals.

There is an order on the symmetric group, which is imposed by geometry and has been defined by Ehresmann [3]. A cut will be a decomposition of $\mathfrak{S}_n$ into two disjoint intervals, i.e. will be defined by a pair of permutations σ, ν such that $\mathfrak{S}_n$ is the disjoint union of the two intervals $[1, \sigma]$ and $[\nu, \omega]$, with $\omega = [n, \ldots, 1]$. It is not difficult [12] to find that such a decomposition is possible iff ν is a permutation of the type

$$[1, \ldots, \alpha{-}1,\ \beta, \ldots, \beta{+}\gamma,\ \alpha, \ldots, \beta{-}1,\ \beta{+}\gamma{+}1, \ldots n]\ .$$

However, the interpretation of such a construction is that it defines an embedding of the symmetric group into a lattice (which happens to be distributive), and the elements of this lattice can be identified with alternating sign matrices. In fact, any finite ordered set admits a *MacNeille completion*, and the preceding construction is just an explicit description of it (and is valid for any finite Coxeter group, cf. [5]).

This construction allows to recover sup and inf operations on permutations; sup and inf were indeed occurring in the theory of symmetric functions, at the level of partitions (the supremum of two partitions being obtained by superposing diagrams). Whenever the Ehresmann-Bruhat order is involved, it seems likely that the lattice of alternating sign matrices has a role to play (I used it to compute some Kazhdan-Lusztig polynomials).

8 Division by 2

There is another way of understanding what it means to cut an alphabet into two pieces.

First, "doubling an alphabet" $\mathbb{A}$ means taking for each letter $a \in \mathbb{A}$ a copy a', thus obtaining a new set $\mathbb{A} \cup \mathbb{A}'$ with twice as much letters. Symmetric functions in $\mathbb{A} \cup \mathbb{A}'$, after erasing the diacritics, can be considered as symmetric functions in $2\mathbb{A}$. At the level of power sums, it simply means passing from $p_k := \sum a^k$ to $\sum a^k + (a')^k \to 2\sum a^k = 2p_k$. It just consists into doubling each power sum.

The converse operation, dividing an alphabet by 2, is therefore defined as $p_k(\mathbb{A}) \to p_k(\mathbb{A}/2) := p_k(\mathbb{A})/2$. I have shown in [10] some relations between this "splitting of alphabets" and the work of Gian-Carlo Rota.

Let us give a different example, this time in relation with elliptic integrals and the work of Abel and Jacobi. Using generating series of complete or symmetric functions, the operation $\mathbb{A} \to \mathbb{A}/2$ consists now in taking square roots. For example, the polynomial $\prod_{a\in\mathbb{A}}(1-ax)$, whose coefficients are, up to sign, the elementary symmetric functions of $\mathbb{A}$, is transformed into its square root $\phi(x) := \sqrt{\prod(1-ax)}$.

Let us take $\mathbb{A}$ of cardinality 3. The basic problem solved by Abel and Jacobi was to integrate $1/\phi(x)$, and to give addition or multiplication formulas for the corresponding elliptic functions. Part of the story is purely algebraic. For example, multiplication of the argument of an elliptic function by an integer involves computing determinants of the type $\left|P_{i+j+c}\right|_{0\le i,j\le n-1}$, where $P_k := \frac{1}{k!}D^k\phi(x)$, $c \in \mathbb{N}$, D being the derivative with respect to x.

Iterating Leibnitz' formula applied to $\phi(x)^2$, one has

$$\frac{1}{n!}D^n(\phi^2) = \sum_{i,j:i+j=n} \frac{1}{i!}D^i(\phi)\,\frac{1}{j!}D^j(\phi)\ . \tag{10}$$

Identifying $D^i(\phi)/i!$ with the i-th elementary symmetric function $\Lambda^i(\mathcal{D})$ of some alphabet $\mathcal{D}$, then $D^n(\phi^2)/n!$ is the n-th elementary symmetric function of $2\mathcal{D}$ (Λ^0 should be equal to 1, and not to ϕ, but this discrepancy is settled by homogeneity). However, ϕ^2 is a polynomial of degree 3 in x, and therefore $D^i(\phi^2) = 0$ for all $i > 3$. In other words, $2\mathcal{D}$ is of rank 3, i.e.

$$\left(\sum_{i=0}^{\infty} z^i \frac{1}{i!} D^i(\phi)\right)^2$$

is a polynomial of degree 3 in z.

Going back to the determinants $\left|P_{i+j+c}\right|$, one recognizes in them Schur functions ($= S_{n^{n-1+c}}((2\mathcal{D})/2)$) of half an alphabet of cardinality 3. The problem of multiplication of elliptic functions has been transformed into a pure problem of symmetric functions: finding the proper relations between the Schur functions of $\mathbb{B}/2$, with $|\mathbb{B}| = 3$.

9 Multiplication by $1-q$

We have just seen than "doubling" an alphabet can be realized by multiplying each power sum by 2. The reader will therefore accept than multiplying an alphabet by $1-q$ amounts to transforming each power sum p_i into $(1-q^i)\,p_i$. In the commutative case, the transformation $\mathbb{A} \to (1-q)\mathbb{A}$, and its inverse $\mathbb{A} \to (1-q)^{-1}\mathbb{A}$ are of fundamental importance in the theory of Hall-Littlewood polynomials. They are also used in the computation of characters of finite linear groups, or of Hecke algebras for the type A.

In the non-commutative case, this transformation is related to the q-bracket and has far-reaching consequences. It gives, for example, families of Lie idempotents interpolating between the classical idempotents due to Dynkin, Solomon and Klyachko. We refer to the series of papers "Non-commutative symmetric functions $II, III, \ldots$ for details [9]. Computing with the quantum rule $[a, b]_q = ab - qba$ reaches a degree of sophistication much higher than the topics I tackle in this text.

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