FACE NUMBERS OF SIMPLICIAL MANIFOLDS AND PSEUDOMANIFOLDS

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Abstract. The f-vector of a simplicial complex K is a vector whose i-th coordinate is the number of faces of dimension i in K (for $i = 0, 1, \ldots$). One of the important problems in combinatorics is to characterize the collection of f-vectors for certain families of simplicial complexes, or, at least, to find substantial necessary conditions. This problem was addressed in works of Klee, McMullen, Stanley, and others. In the talk I will present results on the subject for several classes of simplicial complexes, among them simplicial manifolds and pseudomanifolds with isolated singularities.

Résumé: Le f-vecteur d'un complexe simplicial K est un vecteur dont l' i-ième coordonnée est le nombre de faces de dimension i dans K ($i = 0, 1, 2, \ldots$). Un des problèmes les plus importants de combinatorique est de caractériser la collection des f-vecteurs de certaines familles de complexes simplicials, ou, au moins, de trouver des conditions nécessaires importantes. Ce problème a été traité dans les travaux de Klee, McMullen, Stanley et autres. Je présenterai des résultats sur le sujet pour plusieurs classes de complexes simplicials, parmi lesquelles, les variétés topologiques simpliciales et les pseudo-variétés topologiques avec points singuliers isolés.

The f-vector of a simplicial complex K is a vector whose i-th coordinate is the number of faces of dimension i in K (for $i = 0, 1, \ldots$). One of the central problems in geometric combinatorics is to characterize the collection of f-vectors for certain classes of simplicial complexes. This problem is solved in many important cases. In particular, the complete characterization of f-vectors is known for the family of all simplicial complexes (Kruskal-Katona theorem), all multicomplexes (Macaulay's theorem), all Cohen-Macaulay complexes (due to Stanley), all simplicial complexes with prescribed Betti numbers (due to Björner – Kalai), and all simplicial polytopes. The last result is the celebrated g-theorem conjectured by McMullen and proved by Stanley (necessity) and Billera–Lee (sufficiency). (Recently a more elementary proof of the necessity of conditions was given by McMullen.)

Although a complete characterization of f-vectors is lacking even for the family of all simplicial spheres (that is, simplicial complexes whose geometric realization is homeomorphic to a sphere), several substantial necessary conditions on the face numbers of simplicial spheres, simplicial manifolds and certain classes of odd-dimensional pseudomanifolds with isolated singularities are known. Among them is the Upper Bound Theorem (abbreviated UBT) originally conjectured by Motzkin for polytopes and later extended by Klee to all Eulerian complexes. It asserts that in the class of all Eulerian complexes of dimension $(d-1)$ on n vertices, the boundary of the cyclic d-polytope has the largest number of i-dimensional faces for all i. The UBT is known to hold for all Eulerian simplicial complexes with a sufficiently large number of vertices (Klee), all polytopes (McMullen), all simplicial spheres (Stanley), and several classes of simplicial manifolds (Novik) and odd-dimensional pseudomanifolds with isolated singularities (Hersh–Novik).

While all odd-dimensional manifolds have Euler characteristic zero, the Euler characteristic of even-dimensional manifolds can be arbitrary. Thus another interesting question is to find tight upper bounds on the Euler characteristic of simplicial manifolds of a fixed dimension and with a fixed number of vertices. Such a bound was conjectured by Kühnel.

In the talk I will give a survey of what is known on the subject including the outline of the proof of the UBT for several classes of manifolds and pseudomanifolds and of the (partial) proof of Kühnel's conjecture.

References

[1] A. Björner and G. Kalai, An extended Euler-Poincarè theorem, Acta Math. 161 (1988) 279–303.

[2] P. Hersh and I. Novik, A short simplicial h-vector and the Upper Bound Theorem, *Discrete Comput. Geom.* **28**(2002), 283–289.

[3] G. Kalai, The diameter of graphs of convex polytopes and f-vector theory, in: "Applied Geometry and Discrete Mathematics, The Klee Festschrift," *DIMACS Series in Discrete Mathematics and Computer Science* **4** (1991), 387– 411.

[4] V. Klee, The number of vertices of a convex polytope, *Canadian J. Math* **16** (1964), 702–720.

[5] P. McMullen The maximum numbers of faces of a convex polytope, *Mathematika* **17** (1970), 179–184.

[6] I. Novik, Upper Bound Theorems for homology manifolds, *Israel J. Math.*, **108** (1998), 45–82.

[7] R. Stanley, The Upper Bound Conjecture and Cohen-Macaulay rings, *Studies in Applied Math.* **54** (1975), 135–142.

[8] R. Stanley, *Combinatorics and Commutative algebra*, Boston Basel Berlin: Birkhäuser, 1996.

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