COUNTING OCCURRENCES OF 132 IN A PERMUTATION

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Abstract. We study the generating function for the number of permutations on n letters containing exactly $r \geqslant 0$ occurrences of 132. It is shown that finding this function for a given r amounts to a routine check of all permutations in S_{2r}.

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1. Introduction

Let $\pi \in S_n$ and $\tau \in S_m$ be two permutations. An *occurrence* of τ in π is a subsequence $1 \leqslant i_1 < i_2 < \cdots < i_m \leqslant n$ such that $(\pi(i_1), \dots, \pi(i_m))$ is order-isomorphic to τ; in such a context, τ is usually called a *pattern*.

Recently, much attention has been paid to the problem of counting the number $\psi_r^\tau(n)$ of permutations of length n containing a given number $r \geqslant 0$ of occurrences of a certain pattern τ. Most of the authors consider only the case $r = 0$, thus studying permutations *avoiding* a given pattern. Only a few papers consider the case $r > 0$, usually restricting themselves to the patterns of length 3. In fact, simple algebraic considerations show that there are only two essentially different cases for $\tau \in S_3$, namely, $\tau = 123$ and $\tau = 132$. Noonan [No] has proved that $\psi_1^{123}(n) = \frac{3}{n}\binom{2n}{n-3}$. A general approach to the problem was suggested by Noonan and Zeilberger [NZ]; they gave another proof of Noonan's result, and conjectured that

$$\psi_2^{123}(n) = \frac{59n^2 + 117n + 100}{2n(2n-1)(n+5)}\binom{2n}{n-4}$$

and $\psi_1^{132}(n) = \binom{2n-3}{n-3}$. The latter conjecture was proved by Bóna in [B2]. A general conjecture of Noonan and Zeilberger states that $\psi_r^\tau(n)$ is P-recursive in n for any r and τ. It was proved by Bóna [B1] for $\tau = 132$. However, as stated in [B1], a challenging question is to describe $\psi_r^\tau(n)$, $\tau \in S_3$, explicitly for any given r.

In this note we suggest a new approach to this problem in the case $\tau = 132$, which allows to get an explicit expression for $\psi_r(n) = \psi_r^{132}(n)$ for any given r. More precisely, we present an algorithm that computes the generating function $\Psi_r(x) = \sum_{n \geqslant 0} \psi_r(n) x^n$ for any $r \geqslant 0$. To get the result for a given r, the algorithm performs certain routine checks for each element of the symmetric group S_{2r}. The algorithm has been implemented in C, and yielded explicit results for $1 \leqslant r \leqslant 6$.

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2. Preliminary results

To any $\pi \in S_n$ we assign a bipartite graph G_π in the following way. The vertices in one part of G_π, denoted V_1, are the entries of π, and the vertices of the second part, denoted V_3, are the occurrences of 132 in π. Entry $i \in V_1$ is connected by an edge to occurrence $j \in V_3$ if i enters j. For example, let $\pi = 57614283$, then π contains 5 occurrences of 132, and the graph G_π is presented on Figure 1.

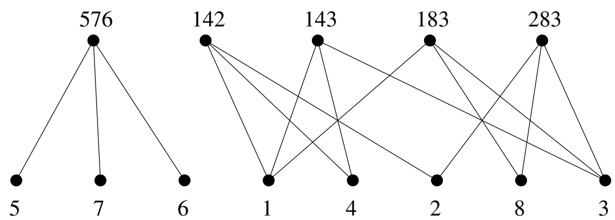

Fig. 1. Graph G_π for $\pi = 57614283$

Let $\widetilde{G}$ be an arbitrary connected component of G_π, and let $\widetilde{V}$ be its vertex set. We denote $\widetilde{V}_1 = \widetilde{V} \cap V_1$, $\widetilde{V}_3 = \widetilde{V} \cap V_3$, $t_1 = |\widetilde{V}_1|$, $t_3 = |\widetilde{V}_3|$.

Lemma 1. *For any connected component $\widetilde{G}$ of G_π one has $t_1 \leqslant 2t_3 + 1$.*

Proof. Assume to the contrary that the above statement is not true. Consider the smallest n for which there exists $\pi \in S_n$ such that for some connected component $\widetilde{G}$ of G_π one has

$$t_1 > 2t_3 + 1. \tag{$*$}$$

Evidently, $\widetilde{G}$ contains more than one vertex, since otherwise $t_1 = 1$, $t_3 = 0$, which contradicts $(*)$. Let l be the number of leaves in $\widetilde{G}$ (recall that a leaf is a vertex of degree 1). Clearly, all the leaves belong to $\widetilde{V}_1$; the degree of any other vertex in $\widetilde{V}_1$ is at least 2, while the degree of any vertex in $\widetilde{V}_3$ equals 3. Calculating the number of edges in $\widetilde{G}$ by two different ways, we get $l + 2(t_1 - l) \leqslant 3t_3$, which together with $(*)$ gives $l > t_3 + 2$. This means that there exist two leaves $u, v \in \widetilde{V}_1$ incident to the same vertex $a \in \widetilde{V}_3$.

Let $w \in \widetilde{V}_1$ be the third vertex incident to a. If w is a leaf, then $\widetilde{G}$ contains only four vertices a, u, v, w, and hence $t_1 = 3$, $t_3 = 1$, which contradicts $(*)$. Hence, the degree of w is at least 2. Delete the entries u, v from π and consider the corresponding permutation $\pi' \in S_{n-2}$. Denote by $\widetilde{G}'$ the connected component of $G_{\pi'}$ containing w. Since the degree of w in $\widetilde{G}$ was at least 2, we see that $\widetilde{G}'$ is obtained from $\widetilde{G}$ by deleting vertices u, v, and a. Therefore, $t_1' = t_1 - 2$, $t_3' = t_3 - 1$, and hence $t_1' > 2t_3' + 1$, a contradiction to the minimality of n. $\square$

Denote by G_π^n the connected component of G_π containing entry n. Let $\pi(i_1), \dots, \pi(i_s)$ be the entries of π belonging to G_π^n, and let $\sigma = \sigma_\pi \in S_s$ be the corresponding permutation. We say that $\pi(i_1), \dots, \pi(i_s)$ is the *kernel* of π and denote it $\ker \pi$; σ is called the *shape* of the kernel, or the *kernel shape*, s is called the *size* of the kernel, and the number of occurrences of 132 in $\ker \pi$ is called the *capacity* of the kernel. For example, for $\pi = 57614283$ as above, the kernel equals 14283, its shape is 14253, the size equals 5, and the capacity equals 4.

The following statement is implied immediately by Lemma 1.

Theorem 1. *Let $\pi \in S_n$ contain exactly r occurrences of 132, then the size of the kernel of π is at most $2r+1$.*

We say that ρ is a *kernel permutation* if it is the kernel shape for some permutation π. Evidently ρ is a kernel permutation if and only if $\sigma_\rho = \rho$.

Let $\rho \in S_s$ be an arbitrary kernel permutation. We denote by $S(\rho)$ the set of all the permutations of all possible sizes whose kernel shape equals ρ. For any $\pi \in S(\rho)$ we define the *kernel cell decomposition* as follows. The number of cells in the decomposition equals $s(s+1)$. Let $\ker \pi = \pi(i_1), \dots, \pi(i_s)$; the *cell* $C_{ml} = C_{ml}(\pi)$ for $1 \leqslant l \leqslant s+1$ and $1 \leqslant m \leqslant s$ is defined by

$$C_{ml}(\pi) = \{\pi(j)\colon i_{l-1} < j < i_l,\ \pi(i_{\rho^{-1}(m-1)}) < \pi(j) < \pi(i_{\rho^{-1}(m)})\},$$

where $i_0 = 0$, $i_{s+1} = n+1$, and $\alpha(0) = 0$ for any α. If π coincides with ρ itself, then all the cells in the decomposition are empty. An arbitrary permutation in $S(\rho)$ is obtained by filling in some of the cells in the cell decomposition. A cell C is called *infeasible* if the existence of an entry $a \in C$ would imply an occurrence of 132 that contains a and two other entries $x, y \in \ker \pi$. Clearly, all infeasible cells are empty for any $\pi \in S(\rho)$. All the remaining cells are called *feasible*; a feasible cell may, or may not, be empty. Consider the permutation $\pi = 67382451$. The kernel of π equals 3845, its shape is 1423. The cell decomposition of π contains four feasible cells: $C_{13} = \{2\}$, $C_{14} = \varnothing$, $C_{15} = \{1\}$, and $C_{41} = \{6, 7\}$, see Figure 2. All the other cells are infeasible; for example, C_{32} is infeasible, since if $a \in C_{32}$, then $a\pi'(i_2)\pi'(i_4)$ is an occurrence of 132 for any π' whose kernel is of shape 1423.

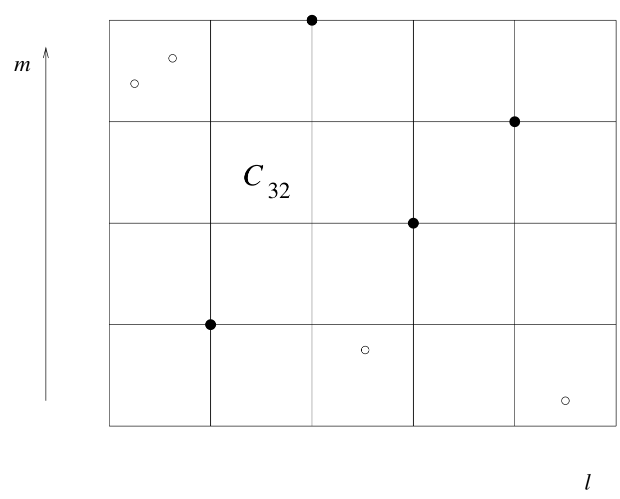

FIG. 2. KERNEL CELL DECOMPOSITION FOR $\pi \in S(1423)$

As another example, permutation $\widetilde{\pi} = 10\,11\,7\,12\,4\,6\,5\,8\,3\,9\,2\,1$ belongs to the same class $S(1423)$. Its kernel is $7\,12\,8\,9$, and the feasible cells are $C_{13} = \{4, 6, 5\}$, $C_{14} = \{3\}$, $C_{15} = \{2, 1\}$, $C_{41} = \{10, 11\}$.

Given a cell C_{ij} in the kernel cell decomposition, all the kernel entries can be positioned with respect to C_{ij}. We say that $x = \pi(i_k) \in \ker\pi$ lies *below* C_{ij} if $\rho(k) < i$, and *above* C_{ij} if $\rho(k) \geqslant i$. Similarly, x lies to the *left* of C_{ij} if $k < j$, and to the *right* of C_{ij} if $k \geqslant j$. As usual, we say that x lies to the *southwest* of C_{ij} if it lies below C_{ij} and to the left of it; the other three directions, northwest, southeast, and northeast, are defined similarly.

The following statement plays a crucial role in our considerations.

Lemma 2. *Let $\pi \in S(\rho)$ and $\pi(i_k) \in \ker\pi$, then any cell C_{ml} such that $l > k$ and $m > \rho(k)$ is infeasible.*

Proof. Assume to the contrary that there exist l and m as above such that C_{ml} is feasible, and consider the partition of the entries in $\ker\pi$ into two subsets: $\varkappa_1$ containing the entries of $\ker\pi$ that lie to the southwest of C_{ml} and $\varkappa_2$ containing the rest of the entries. Observe that $\pi(i_k) \neq n$, since $n \geqslant m > \rho(k) = \pi(i_k)$. Moreover, $\pi(i_k) \in \varkappa_1$ and $n \in \varkappa_2$. Since $\pi(i_k)$ and n belong to the same connected component of G_π, there exists at least one occurrence of 132 whose elements are distributed between $\varkappa_1$ and $\varkappa_2$. Let a denote the minimal entry in this occurrence, let c denote its maximal entry, and let b denote the remaining entry.

Evidently, $a \in \varkappa_1$. Assume first that $c \in \varkappa_2$.

If c lies to the left of C_{ml}, then the existence of $z \in C_{ml}$ would imply that acz is an occurrence of 132, and hence C_{ml} is infeasible.

If c lies to the northeast of C_{ml} and b lies above C_{ml}, then the existence of $z \in C_{ml}$ would imply that zcb is an occurrence of 132, and hence C_{ml} is infeasible. If b lies below C_{ml}, then the existence of $z \in C_{ml}$ would imply that azb is an occurrence of 132, and hence C_{ml} is infeasible.

If c lies to the southeast of C_{ml}, then the existence of $z \in C_{ml}$ would imply that azc is an occurrence of 132, and hence C_{ml} is infeasible.

It remains to consider the case $c \in \varkappa_1$, which means that b belongs to $\varkappa_2$ and lies to the southeast of C_{ml}. Hence, the existence of $z \in C_{ml}$ would imply that azb is an occurrence of 132, and hence C_{ml} is infeasible. □

As an easy corollary of Lemma 2, we get the following proposition. Let us define a partial order $\prec$ on the set of all feasible cells by saying that $C_{ml} \prec C_{m'l'} \neq C_{ml}$ if $m \geqslant m'$ and $l \leqslant l'$.

Lemma 3. *$\prec$ is a linear order.*

Proof. Assume to the contrary that there exist two feasible cells C_{ml} and $C_{m'l'}$ such that $l < l'$ and $m < m'$, and consider the entry $x = \pi(i_l) \in \ker\pi$. By Lemma 2, $x > \pi(i_{\rho^{-1}(m'-1)})$, that is, x lies above the cell $C_{m'l'}$, since otherwise $C_{m'l'}$ would be infeasible. For the same reason, $y = \pi(i_{\rho^{-1}(m'-1)})$ lies to the right of $C_{m'l'}$, and hence to the right of x. Therefore, the existence of $z \in C_{ml}$ would imply that zxy is an occurrence of 132, and hence C_{ml} is infeasible, a contradiction. □

Consider now the dependence between two nonempty feasible cells lying on the same horizontal or vertical level.

Lemma 4. *Let C_{ml} and $C_{ml'}$ be two nonempty feasible cells such that $l < l'$. Then for any pair of entries $a \in C_{ml}$, $b \in C_{ml'}$, one has $a > b$.*

Proof. Assume to the contrary that there exists a pair $a \in C_{ml}$, $b \in C_{ml'}$ such that $a < b$. Consider the entry $x = \pi(i_l) \in \ker \pi$. By Lemma 2, $x > b$, since otherwise $C_{ml'}$ would be infeasible. Hence axb is an occurrence of 132, which means that both a and b belong to $\ker \pi$, a contradiction. □

Lemma 5. *Let C_{ml} and $C_{m'l}$ be two nonempty feasible cells such that $m < m'$. Then any entry $a \in C_{ml}$ lies to the right of any entry $b \in C_{m'l}$.*

Proof. Assume to the contrary that there exists a pair $a \in C_{ml}$, $b \in C_{m'l}$ such that a lies to the left of b. Consider the entry $y = \pi(i_{\rho^{-1}(m'-1)}) \in \ker \pi$. By Lemma 2, y lies to the right of b, since otherwise $C_{m'l}$ would be infeasible. Hence aby is an occurrence of 132, which means that both a and b belong to $\ker \pi$, a contradiction. □

Lemmas 3–5 yield immediately the following two results.

Theorem 2. *Let $\widetilde{G}$ be a connected component of G_π distinct from G_π^n. Then all the vertices in $\widetilde{V}_1$ belong to the same feasible cell in the kernel cell decomposition of π.*

Let $F(\rho)$ be the set of all feasible cells in the kernel cell decomposition corresponding to permutations in $S(\rho)$, and let $f(\rho) = |F(\rho)|$. We denote the cells in $F(\rho)$ by $C^1, \dots, C^{f(\rho)}$ in such a way that $C^i \prec C^j$ whenever $i < j$.

Theorem 3. *For any given sequence $\alpha_1, \dots, \alpha_{f(\rho)}$ of arbitrary permutations there exists $\pi \in S(\rho)$ such that the content of C^i is order-isomorphic to α_i.*

3. Main Theorem and explicit results

Let ρ be a kernel permutation, and let $s(\rho)$, $c(\rho)$, and $f(\rho)$ be the size of ρ, the capacity of ρ, and the number of feasible cells in the cell decomposition associated with ρ, respectively. Denote by K the set of all kernel permutations, and by K_t the set of all kernel shapes for permutations in S_t. The main result of this note can be formulated as follows.

Theorem 4. *For any $r \geqslant 1$,*

$$\Psi_r(x) = \sum_{\rho \in K_{2r+1}} \left(x^{s(\rho)} \sum_{r_1 + \cdots + r_{f(\rho)} = r - c(\rho)} \prod_{j=1}^{f(\rho)} \Psi_{r_j}(x) \right), \qquad (**)$$

where $r_j \geqslant 0$ for $1 \leqslant j \leqslant f(\rho)$.

Proof. For any $\rho \in K$, denote by $\Psi_r^\rho(x)$ the generating function for the number of permutations in $\pi \in S_n \cap S(\rho)$ containing exactly r occurrences of 132. Evidently, $\Psi_r(x) = \sum_{\rho \in K} \Psi_r^\rho(x)$. To find $\Psi_r^\rho(x)$, recall that the kernel of any π as above contains exactly $c(\rho)$ occurrences of 132. The remaining $r - c(\rho)$ occurrences of 132 are distributed between the feasible cells of the kernel cell decomposition of π. By Theorem 2, each occurrence of 132 belongs entirely to one feasible cell. Besides, it follows from Theorem 3, that occurrences of 132 in different cells do not influence one another. Therefore,

$$\Psi_r^\rho(x) = x^{s(\rho)} \sum_{r_1+\dots+r_{f(\rho)}=r-c(\rho)} \prod_{j=1}^{f(\rho)} \Psi_{r_j}(x),$$

and we get the expression similar to $(**)$ with the only difference that the outer sum is over all $\rho \in K$. However, if $\rho \in K_t$ for $t > 2r+1$, then by Theorem 1, $c(\rho) > r$, and hence $\Psi_r^\rho(x) \equiv 0$. $\square$

Theorem 4 provides a finite algorithm for finding $\Psi_r(x)$ for any given $r > 0$, since we have to consider all permutations in S_{2r+1}, and to perform certain routine operations with all shapes found so far. Moreover, the amount of search can be decreased substantially due to the following proposition.

Proposition. *The only kernel permutation of capacity $r \geqslant 1$ and size $2r+1$ is $2r-1\, 2r+1\, 2r-3\, 2r \dots 2r-2j-3\, 2r-2j \dots 1\, 4\, 2$. Its contribution to $\Psi_r(x)$ equals $x^{2r+1}\Psi_0^{r+2}(x)$.*

This proposition is proved easily by induction, similarly to Lemma 1. The feasible cells in the corresponding cell decomposition are $C_{2r-2j+1,2j+1}$, $j = 0,\dots,r$, and $C_{1,2r+2}$, hence the contribution to $\Psi_r(x)$ is as described.

By the above proposition, it suffices to search only permutations in S_{2r}. Below we present several explicit calculations.

Let us start from the case $r = 0$. Observe that $(**)$ remains valid for $r = 0$, provided the left hand side is replaced by $\Psi_r(x)-1$; subtracting 1 here accounts for the empty permutation. So, we begin with finding kernel shapes for all permutations in S_1. The only shape obtained is $\rho_1 = 1$, and it is easy to see that $s(\rho_1) = 1$, $c(\rho_1) = 0$, and $f(\rho_1) = 2$ (since both cells C_{11} and C_{12} are feasible). Therefore, we get $\Psi_0(x) - 1 = x\Psi_0^2(x)$, which means that

$$\Psi_0(x) = \frac{1-\sqrt{1-4x}}{2x},$$

the generating function of Catalan numbers.

Let now $r = 1$. Since permutations in S_2 do not exhibit kernel shapes distinct from ρ_1, the only possible new shape is the exceptional one, $\rho_2 = 132$, whose contribution equals $x^3\Psi_0^3(x)$. Therefore, $(**)$ amounts to

$$\Psi_1(x) = 2x\Psi_0(x)\Psi_1(x) + x^3\Psi_0^3(x),$$

and we get the following result.

Corollary 1 (Bóna [B2, Theorem 5]**).**

$$\Psi_1(x) = \frac{1}{2}\left(x - 1 + (1-3x)(1-4x)^{-1/2}\right);$$

equivalently,

$$\psi_1(n) = \binom{2n-3}{n-3}$$

for $n \geqslant 3$.

Let $r = 2$. We have to check the kernel shapes of permutations in S_4. Exhaustive search adds four new shapes to the previous list; these are 1243, 1342, 1423, and 2143; besides, there is the exceptional $35142 \in S_5$. Calculation of the parameters s, c, f is straightforward, and we get

Corollary 2.

$$\Psi_2(x) = \frac{1}{2}\left(x^2 + 3x - 2 + (2x^4 - 4x^3 + 29x^2 - 15x + 2)(1-4x)^{-3/2}\right);$$

equivalently,

$$\psi_2(n) = \frac{n^3 + 17n^2 - 80n + 80}{2n(n-1)}\binom{2n-6}{n-2}$$

for $n \geqslant 4$.

Let $r = 3, 4, 5, 6$; exhaustive search in S_6, S_8, S_{10}, and S_{12} reveals 20, 104, 503, and 2576 new nonexceptional kernel shapes, respectively, and we get

Corollary 3. *Let* $3 \leqslant r \leqslant 6$*, then*

$$\Psi_r(x) = \frac{1}{2}\left(P_r(x) + Q_r(x)(1-4x)^{-r+1/2}\right),$$

where

$$\begin{aligned}
P_3(x) &= 2x^3 - 5x^2 + 7x - 2,\\
P_4(x) &= 5x^4 - 7x^3 + 2x^2 + 8x - 3,\\
P_5(x) &= 14x^5 - 17x^4 + x^3 - 16x^2 + 14x - 2
\end{aligned}$$

and

$$\begin{aligned}
Q_3(x) &= -22x^6 - 106x^5 + 292x^4 - 302x^3 + 135x^2 - 27x + 2,\\
Q_4(x) &= 2x^9 + 218x^8 + 1074x^7 - 1754x^6 + 388x^5 + 1087x^4\\
&\qquad - 945x^3 + 320x^2 - 50x + 3,\\
Q_5(x) &= -50x^{11} - 2568x^{10} - 10826x^9 + 16252x^8 - 12466x^7 + 16184x^6 - 16480x^5\\
&\qquad + 9191x^4 - 2893x^3 + 520x^2 - 50x + 2.
\end{aligned}$$

Equivalently,

$$\psi_r(n) = R_r(n)\frac{(2n-3r)!}{n!r!(n-r-2)!},$$

for $n \geqslant r + 2$*, where*

$$\begin{aligned}
R_3(n) &= n^6 + 51n^5 - 407n^4 - 99n^3 + 7750n^2 - 22416n + 20160,\\
R_4(n) &= n^9 + 102n^8 - 282n^7 - 12264n^6 + 32589n^5 + 891978n^4\\
&\qquad - 7589428n^3 + 25452024n^2 - 39821760n + 23950080,\\
R_5(n) &= n^{12} + 170n^{11} + 1861n^{10} - 88090n^9 - 307617n^8 + 27882510n^7\\
&\qquad - 348117457n^6 + 2119611370n^5 - 6970280884n^4\\
&\qquad + 10530947320n^3 + 2614396896n^2 - 30327454080n + 29059430400.
\end{aligned}$$

The expressions for $P_6(x)$*,* $Q_6(x)$*, and* $R_6(n)$ *are too long to be presented here.*

4. Further results and open questions

As an easy consequence of Theorem 4 we get the following results due to Bóna [B1].

Corollary 4. *Let $r \geqslant 0$, then $\Psi_r(x)$ is a rational function in the variables x and $\sqrt{1-4x}$.*

In fact, Bóna has proved a stronger result, claiming that

$$\Psi_r(x) = P_r(x) + Q_r(x)(1-4x)^{-r+1/2}, \qquad (***)$$

where $P_r(x)$ and $Q_r(x)$ are polynomials and $1-4x$ does not divide $Q_r(x)$. We were unable to prove this result; however, it stems almost immediately from the following conjecture.

Conjecture 1. *For any kernel permutation $\rho \neq 1$,*

$$s(\rho) \geqslant f(\rho).$$

Indeed, it is easy to see that $\Psi_r(x)$ enters the right hand side of $(**)$ with the coefficient $2x\Psi_0(x)$, which is a partial contribution of the kernel shape $\rho_1 = 1$. Since $1 - 2x\Psi_0(x) = \sqrt{1-4x}$, we get by induction from $(**)$ that $\sqrt{1-4x}\Psi_r(x)$ equals the sum of fractions whose denominators are of the form $x^d(1-4x)^{r-c(\rho)-f(\rho)/2}$, where $d \leqslant f(\rho)$. On the other hand, each fraction is multiplied by $x^{s(\rho)}$, hence if $s(\rho) \geqslant f(\rho)$ as conjectured, then x^d in the denominator is cancelled. The maximal degree of $(1-4x)$ is attained for $\rho = \rho_1$, and is equal to $r-1$, and we thus arrive at $(***)$.

In view of our explicit results, we have even a stronger conjecture.

Conjecture 2. *The polynomials $P_r(x)$ and $Q_r(x)$ in $(***)$ have halfinteger coefficients.*

Another direction would be to match the approach of this note with the previous results on restricted 132-avoiding permutations. Let $\Phi_r(x;k)$ be the generating function for the number of permutations in S_n containing r occurrences of 132 and avoiding the pattern $12\dots k \in S_k$. It was shown previously that $\Phi_r(x;k)$ can be expressed via Chebyshev polynomials of the second kind for $r = 0$ ([CW]) and $r = 1$ ([MV]). Our new approach allows to get a recursion for $\Phi_r(x;k)$ for any given $r \geqslant 0$.

Let ρ be a kernel permutation, and assume that the feasible cells of the kernel cell decomposition associated with ρ are ordered linearly according to $\prec$. We denote by $l_j(\rho)$ the length of the longest increasing subsequence of ρ that lies to the north-east from C^j. For example, let $\rho = 1423$, as on Figure 2. Then $l_1(\rho) = 1$, $l_2(\rho) = 2$, $l_3(\rho) = 1$, $l_4(\rho) = 0$.

Theorem 5. *For any $r \geqslant 1$ and $k \geqslant 3$,*

$$\Phi_r(x;k) = \sum_{\rho \in K_{2r+1}} \left(x^{s(\rho)} \sum_{r_1+\cdots+r_{f(\rho)}=r-c(\rho)} \prod_{j=1}^{f(\rho)} \Phi_{r_j}(x;k-l_j(\rho)) \right),$$

where $r_j \geqslant 0$ *for* $1 \leqslant j \leqslant f(\rho)$ *and* $\Phi_r(x;m) \equiv 0$ *for* $m \leqslant 0$.

As in the case of $\Psi_r(x)$, the statement of the theorem remains valid for $r = 0$, provided the left hand side is replaced by $\Phi_r(x;k) - 1$. This allows to recover known explicit expressions for $\Phi_r(x;k)$ for $r = 0, 1$, and to get an expression for $r = 2$, which is too long to be presented here.

This approach can be extended even further, to cover also permutations containing r occurrences of 132 and avoiding other permutations in S_k, for example, $23 \dots k1$.

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