TOTAL NONNEGATIVITY AND $(\mathbf{3}+\mathbf{1})$-FREE POSETS

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ABSTRACT. We prove a factorization theorem for certain totally nonnegative matrices. This factorization implies that the f-vector of a $(\mathbf{3}+\mathbf{1})$-free poset is also the f-vector of a unit interval order.

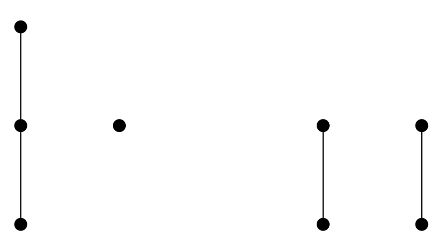
RÉSUMÉ. Nous montrons un théoréme sur la factorisation des certaines maitrices totalement positives. Cette factorisation implique que le vecteur-f d'un ensemble ordonné sans $(\mathbf{3}+\mathbf{1})$ c'est aussi le vecteur-f d'un ordre des intervalles de longeur constant.

1. Introduction

Much current research in algebraic combinatorics concerns the characterization of f-vectors of simplicial complexes, polytopes, and related combinatorial structures. (See [1], [21, Chapters 2,3].) One interesting source of f-vectors is the class of $(\mathbf{3}+\mathbf{1})$-free posets because the generating polynomials for the corresponding f-vectors are known to have only real zeros [17, Corollary 4.1], [22, Corollary 2.9]. (The second of these proofs employs the Schur-positivity of certain chromatic symmetric functions introduced by Stanley [19]. See also [7], [15], [25].) A poset is called $(\mathbf{3}+\mathbf{1})$-free if it contains no induced subposet isomorphic to the poset shown in Figure 1.1 (a).

An interesting subclass of $(\mathbf{3}+\mathbf{1})$-free posets is the class of those which are also $(\mathbf{2}+\mathbf{2})$-free, i.e. which contain no induced subposet isomorphic to the poset shown in Figure 1.1 (b). These are often called *unit interval orders* because a well-known result [16] characterizes them as the posets P for which there exists a map from P to closed intervals of the real line

$$x \mapsto [q_x, q_x + 1],$$



(a) (b)

FIGURE 1.1

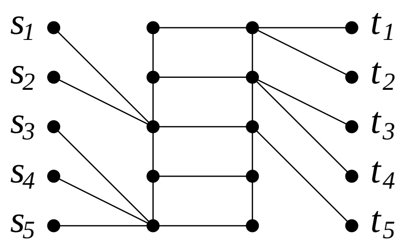


FIGURE 1.2

which satisfies $x <_P y$ if and only if $q_x + 1 < q_y$. No analogous result is known to hold for $(\mathbf{3}+\mathbf{1})$-free posets in general.

Since unit interval orders form a proper subclass of $(\mathbf{3}+\mathbf{1})$-free posets, one might be surprised to learn that the containment of the corresponding two sets of f-vectors is not proper. (See Corollary 4.5.) The proof of this fact relies upon the factorization of a totally nonnegative matrix which we will associate to each $(\mathbf{3}+\mathbf{1})$-free poset.

A matrix is called *totally nonnegative* if the determinant of each of its square submatrices is nonnegative. A result often attributed to Lindström [11] describes the most important example of a totally nonnegative matrix in terms of a *planar network*, a planar acyclic directed graph G with $2n$ distinguished boundary vertices labeled counterclockwise as $s_1, \ldots, s_n, t_n, \ldots, t_1$. (See [10].) Given a planar network G, its *path matrix* $A = [a_{ij}]$, in which a_{ij} counts paths from s_i to t_j, is totally nonnegative. For instance the matrix

$$\begin{bmatrix} 3 & 3 & 2 & 2 & 1 \\ 3 & 3 & 2 & 2 & 1 \\ 5 & 5 & 4 & 4 & 3 \\ 5 & 5 & 4 & 4 & 3 \\ 5 & 5 & 4 & 4 & 3 \end{bmatrix}$$

is easily verified to be totally nonnegative because it is the path matrix of the planar network in Figure 1.2. When drawing planar networks, we will understand vertical edges to be oriented from bottom to top, and other edges to be oriented toward the right. (See also [6].)

We will finish by stating some inequalities which are satisfied by the f-vectors of all $(\mathbf{3}+\mathbf{1})$-free posets and by posing some open questions.

2. ORDER IDEALS AND TOTALLY NONNEGATIVE MATRICES

Given an n-element poset P whose elements are labeled $1, \ldots, n$, we define the *antiadjacency matrix* [20] of P to be the matrix $A = [a_{ij}]$,

$$a_{ij} = \begin{cases} 0 & \text{if } i <_P j, \\ 1 & \text{otherwise.} \end{cases}$$

Clearly, distinct labelings of P can result in distinct antiadjacency matrices. It is easy to see that the antiadjacency matrix of a labeled poset P has no zero entries

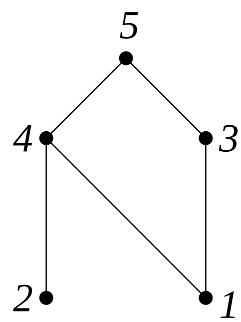


FIGURE 2.1

below the diagonal if and only if P is labeled *naturally* (i.e. each pair i, j of elements satisfying $i <_P j$ also satisfies $i < j$ as integers). Further, it is known that unit interval orders may be labeled so that the corresponding antiadjacency matrices are totally nonnegative [23, Problem 6.19 (ddd)], and that $(\mathbf{3}+\mathbf{1})$-free posets may be labeled so that the corresponding *squared* antiadjacency matrices are totally nonnegative [17, page 238].

To characterize the poset labelings which lead to totally nonnegative (squared) antiadjacency matrices, we will use principal order ideals and dual principal order ideals. More precisely, we will define the *deleted ideals* of an element i to be

$$\Lambda_i^* = \{j \in P \,|\, j <_P i\},$$
$$V_i^* = \{j \in P \,|\, j >_P i\}.$$

For each element i of P, we will define its *altitude* to be the number

$$\alpha(i) = |\Lambda_i^*| - |V_i^*|.$$

(See [5, page 33] for other applications of this function.) Figure 2.1 shows a poset in which we have $\Lambda_1^* = \emptyset$, $V_1^* = \{3, 4, 5\}$ and $\alpha(1) = -3$.

It is easy to verify the following properties of deleted order ideals in $(\mathbf{3}+\mathbf{1})$-free posets and in unit interval orders. (We will use the symbols $\subseteq, \subset$ to denote containment and strict containment, respectively.)

Observation 2.1. *Let i and j be distinct elements of a $(\mathbf{3}+\mathbf{1})$-free poset. The corresponding deleted order ideals satisfy*

$$\Lambda_i^* \subseteq \Lambda_j^* \text{ or } V_i^* \subseteq V_j^*. \tag{2.1}$$

Proof. Left to reader. □

Observation 2.2. *Let i and j be distinct elements of a unit interval order. The corresponding deleted order ideals satisfy*

1. *If $\Lambda_i^* \not\subseteq \Lambda_j^*$, then $\Lambda_j^* \subset \Lambda_i^*$.*
2. *If $V_j^* \not\subseteq V_i^*$, then $V_i^* \subset V_j^*$.*

Proof. Left to reader. □

In $(\mathbf{3}+\mathbf{1})$-free posets, altitude is related to deleted ideals as follows.

Observation 2.3. *Let i and j be elements of a $(\mathbf{3}+\mathbf{1})$-free poset. Then we have $\alpha(i) \leq \alpha(j)$ if and only if we have*

$$|\Lambda_i^*| \leq |\Lambda_j^*| \text{ and } V_i^* \supseteq V_j^*, \tag{2.2}$$

or

$$|V_i^*| \geq |V_j^*| \text{ and } \Lambda_i^* \subseteq \Lambda_j^*. \tag{2.3}$$

Proof. Omitted. □

We will say that a labeling of a poset P *respects altitude* if each pair i, j of poset elements satisfying $\alpha(i) < \alpha(j)$ also satisfies $i < j$ (as integers). Note that a labeling which respects altitude is necessarily natural. The following proposition (essentially stated in [26, Section 8.2]) shows that the antiadjacency matrices of naturally labeled unit interval orders are totally nonnegative precisely when the labelings respect altitude.

Proposition 2.4. *Let P be a labeled n-element unit interval order with antiadjacency matrix $A = [a_{ij}]$. The labeling of P respects altitude if and only if A satisfies*

$$a_{jk} \geq a_{i\ell} \tag{2.4}$$

for $1 \leq i \leq j \leq n$ and $1 \leq k \leq \ell \leq n$.

Proof. Omitted. □

In a 0-1 matrix satisfying the conditions of Proposition 2.4, the zero entries form a Ferrers shape in the upper right corner of the matrix. For instance, the labeling of the poset in Figure 2.1 respects altitude and the antiadjacency matrix of this poset is

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}. \tag{2.5}$$

An easy proof that such a matrix A is totally nonnegative consists of a planar network whose path matrix is A. Figure 2.2 shows a planar network whose path matrix is the antiadjacency matrix (2.5) of the poset in Figure 2.1.

A result analogous to Proposition 2.4 holds for the *squared* antiadjacency matrices of $(\mathbf{3}+\mathbf{1})$-free posets. Before stating this result, let us give one interpretation of the entries of these matrices.

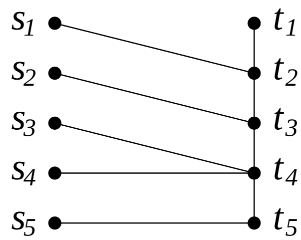


FIGURE 2.2

Lemma 2.5. *Let P be a labeled n-element $(\mathbf{3}+\mathbf{1})$-free poset with antiadjacency matrix A and define the matrix $B = [b_{ij}] = A^2$. Then we have*

$$b_{ij} = \begin{cases} n - |V_i^*| - |\Lambda_j^*| & \text{if } V_i^* \cap \Lambda_j^* \text{ is empty,} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. By the definition of B we have

$$\begin{aligned} b_{ij} &= \sum_{k=1}^{n} a_{ik} a_{kj} \\ &= \#\{k \in [n] \mid a_{ik} = a_{kj} = 1\} \\ &= \#\{k \in P \mid i \not<_P k \not<_P j\} \\ &= n - |V_i^*| - |\Lambda_j^*| + |V_i^* \cap \Lambda_j^*|. \end{aligned}$$

Suppose that b_{ij} is nonzero and let k be an element satisfying $i \not<_P k \not<_P j$. Note that for any element ℓ belonging to the intersection $V_i^* \cap \Lambda_j^*$, the subposet of P induced by $\{i, j, k, \ell\}$ is isomorphic to $\mathbf{3}+\mathbf{1}$. Thus this intersection is empty and we have the desired result. □

Proposition 2.6. *Let P be a labeled n-element $(\mathbf{3}+\mathbf{1})$-free poset with antiadjacency matrix A, and define the matrix $B = [b_{ij}] = A^2$. The labeling of P respects altitude if and only if B satisfies the conditions*

1. $b_{jk} \geq b_{i\ell}$,
2. *If* $b_{ik} - b_{i\ell} \neq b_{jk} - b_{j\ell}$, *then* $b_{i\ell} = 0$ *and* $b_{ik} < b_{jk} - b_{j\ell}$,

for all integers $1 \leq i \leq j \leq n$ and $1 \leq k \leq \ell \leq n$.

Proof. Omitted.

□

As an example of Proposition 2.6, consider the labeled poset in Figure 2.3. This labeling respects altitude. and the corresponding squared antiadjacency matrix is

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}^2 = \begin{bmatrix} 3 & 3 & 2 & 2 & 1 \\ 3 & 3 & 2 & 2 & 1 \\ 5 & 5 & 4 & 4 & 3 \\ 5 & 5 & 4 & 4 & 3 \\ 5 & 5 & 4 & 4 & 3 \end{bmatrix}.$$

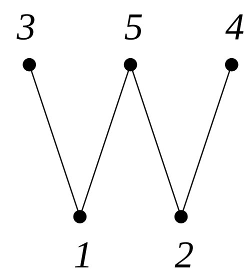


FIGURE 2.3

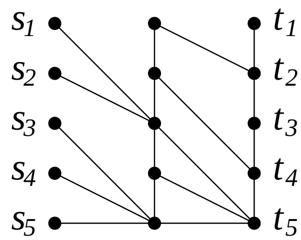


FIGURE 2.4

Again, to prove that such a matrix B is totally nonnegative, it suffices to construct a planar network having path matrix B. Figure 1.2 shows one such planar network which is constructed easily from B. Another possibility in Figure 2.4 is constructed by concatenating two planar networks corresponding to the antiadjacency matrices of unit interval orders [14]. This observation suggests the possibility of factoring the squared antiadjacency matrices of $(\mathbf{3}+\mathbf{1})$-free posets in general. Such a factorization is in fact possible and will be considered further in Sections 3 and 4.

3. A factorization theorem

The planar network in Figure 2.4 is constructed using a factorization of the squared antiadjacency matrix of the poset in Figure 2.3,

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}^2 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}. \tag{3.1}$$

It is clear that each of these factors is the path matrix of part of the planar network. In general, let A be the antiadjacency matrix corresponding to any labeling of a $(\mathbf{3}+\mathbf{1})$-free poset. To obtain a factorization $A^2 = CD$, one constructs C and D from A by "pushing" the zero entries of A to the right and up, respectively.

Theorem 3.1. *Let P be a labeled $(\mathbf{3}+\mathbf{1})$-free poset with antiadjacency matrix A. Let C be the matrix obtained from A by permuting the entries of each row into nonincreasing order, and let D be the matrix obtained from A by permuting the entries of each column into nondecreasing order. Then we have $A^2 = CD$.*

Proof. Let n be the cardinality of P and define the matrices $B = [b_{ij}] = A^2$ and $E = [e_{ij}] = CD$. Since the numbers c_{ij} and d_{ij} are given by

$$c_{ij} = \begin{cases} 1 & \text{if } j \leq n - |V_i^*|, \\ 0 & \text{otherwise,} \end{cases}$$

$$d_{ij} = \begin{cases} 1 & \text{if } i \geq |\Lambda_j^*| + 1, \\ 0 & \text{otherwise,} \end{cases}$$

we have

$$\begin{aligned} e_{ij} &= \sum_{k=1}^{n} c_{ik} d_{kj} \\ &= \#\{k \in [n] \mid c_{ik} = d_{kj} = 1\} \\ &= \#\{|\Lambda_j^*| + 1, \ldots, n - |V_i^*|\} \\ &= \begin{cases} n - |V_i^*| - |\Lambda_j^*| & \text{if } |V_i^*| + |\Lambda_j^*| \leq n, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

We claim that the inequality

$$|V_i^*| + |\Lambda_j^*| \leq n \tag{3.2}$$

holds if and only if the intersection $V_i^* \cap \Lambda_j^*$ is empty. If the intersection is empty, then (3.2) is clear. Suppose therefore that the intersection is not empty. Then P contains some element ℓ which satisfies

$$i <_P \ell <_P j$$

and we have

$$|V_i^* \cup \Lambda_j^*| \leq |V_i^*| + |\Lambda_j^*| - 1. \tag{3.3}$$

If some element lies outside of the union $V_i^* \cup \Lambda_j^*$ above, then it is incomparable to i, ℓ, and j, contradicting the fact that P is $(\mathbf{3}+\mathbf{1})$-free . Thus the cardinality of this union is n and the inequality (3.3) gives

$$n \leq |V_i^*| + |\Lambda_j^*| - 1,$$

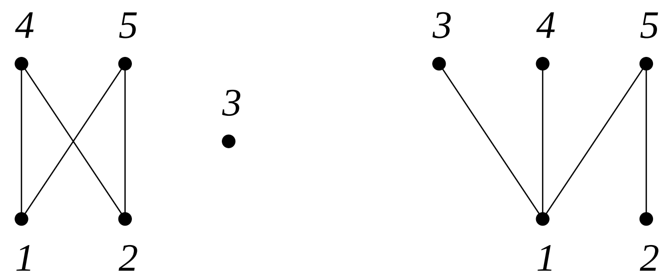
contradicting (3.2).

We therefore obtain the expression

$$e_{ij} = \begin{cases} n - |V_i^*| - |\Lambda_j^*| & \text{if } V_i^* \cap \Lambda_j^* \text{ is empty,} \\ 0 & \text{otherwise,} \end{cases}$$

which is identical to that for b_{ij} given in Lemma 2.5. □

In the event that the labeling of P in Theorem 3.1 respects altitude, the matrices C and D in the theorem are the antiadjacency matrices corresponding to altitude respecting labelings of unit interval orders. Note however that the implied map from $(\mathbf{3}+\mathbf{1})$-free posets to pairs of unit interval orders is neither injective nor surjective.



(a) (b)

FIGURE 3.1

Corollary 3.2. *Let P be a labeled $(\mathbf{3}+\mathbf{1})$-free poset with antiadjacency matrix A. If the labeling of P respects altitude then there are labeled unit interval orders Q_1 and Q_2 whose antiadjacency matrices are the matrices C and D defined in Theorem 3.1. Furthermore the labelings of Q_1 and Q_2 respect altitude.*

Proof. Let P be an n-element $(\mathbf{3}+\mathbf{1})$-free poset with an altitude respecting labeling. By Observation 2.3, the sequence $(|V_1^*|,\ldots,|V_n^*|)$ weakly decreases and the sequence $(|\Lambda_1^*|,\ldots,|\Lambda_n^*|)$ weakly increases. Thus the zero entries of C and D form Ferrers shapes in the upper right corners of these matrices. By Proposition 2.4, the corresponding poset labelings respect altitude. □

Thus the factorization (3.1) associates to the poset in Figure 2.3 the two unit interval orders in Figure 3.1.

4. f-VECTORS OF $(\mathbf{3}+\mathbf{1})$-FREE POSETS

The relationship between the unit interval orders and the $(\mathbf{3}+\mathbf{1})$-free poset in Corollary 3.2 extends beyond the factorization stated in Theorem 3.1. We will show that for any k, there is a bijective correspondence between the k-element chains in any two of these three posets.

A k-element *chain* in a poset P is a sequence of elements $(x_1,\ldots,x_k)$ of P which satisfy $x_1 <_P \cdots <_P x_k$. The *chain polynomial* or *f-polynomial* of P is defined to be

$$f_P(t) = 1 + a_1 t + \cdots + a_m t^m,$$

where a_i is the number of i-element chains in P. The sequence $f_P = (1, a_1, \ldots, a_m)$ is called the *f-vector* of P and is often written as $(1, f_0, \ldots, f_{m-1})$.

To better describe the relationship between the posets mentioned above, we will define a self-map ϕ on the set of all naturally labeled n-element posets. Given such a poset P with antiadjacency matrix A, we define $\phi(P)$ as follows.

1. If all rows of A are weakly decreasing, define $\phi(P) = P$.
2. Otherwise,

(a) Let j be the greatest integer in $[n-1]$ such that there exists an i satisfying

$$0 = a_{i,j} < a_{i,j+1} = 1. \tag{4.1}$$

(b) Define A' to be the matrix obtained from A by exchanging the entries $a_{i,j}$ and $a_{i,j+1}$ for *each* index i satisfying (4.1).

(c) Let $\phi(P)$ be the labeled poset whose antiadjacency matrix is A'.

To see that the map ϕ is well defined, observe that the the matrix $A' = [a'_{g,h}]$ defined in step (2b) satisfies

$$a'_{g,h} = 1 \tag{4.2}$$

whenever $g \geq h$ as integers, and

$$a'_{f,h} = 0 \tag{4.3}$$

whenever $a'_{f,g} = a'_{g,h} = 0$. These are the two defining characteristics of an antiadjacency matrix of a naturally labeled poset. Since it is necessary that P be labeled naturally in order to obtain (4.2), the map ϕ is not defined on arbitrarily labeled posets. Note also that a different choice of j in step (2a) would define a different (but also well defined) map.

In the event that P is a labeled $(\mathbf{3}+\mathbf{1})$-free poset, it is easy to construct the Hasse diagram of $\phi(P)$ from that of P. Let j be the greatest index satisfying (4.1), and let I be the set of elements of P covered by j and not comparable to $j+1$. For each element i in I, replace the edge (i,j) by the edge $(i,j+1)$. For each element h covered by an element in I and not covered by any element in $\Lambda_j^* \cap \Lambda_{j+1}^*$, introduce the new edge (h,j).

While the map ϕ does not in general preserve altitude, it does preserve $|V_x^*|$ for all elements x in P. Further, if P is a $(\mathbf{3}+\mathbf{1})$-free poset which satisfies

$$|V_1^*| \geq \cdots \geq |V_n^*|, \tag{4.4}$$

we can infer several interesting things about P and $\phi(P)$. By Observation 2.3, a labeling of a $(\mathbf{3}+\mathbf{1})$-free poset which respects altitude necessarily satisfies the condition (4.4). Note also that a labeling of a $(\mathbf{3}+\mathbf{1})$-free poset which satisfies the condition (4.4) is necessarily natural.

Lemma 4.1. *Let P be a labeled n-element $(\mathbf{3}+\mathbf{1})$-free poset which satisfies (4.4) and let j be the greatest label which satisfies (4.1) for some $i \leq n$. Then the four deleted ideals $V_j^*(P), V_{j+1}^*(P), V_j^*(\phi(P)), V_{j+1}^*(\phi(P))$ are equal. Furthermore, each element i satisfying (4.1) is covered by j in P and is covered by $j+1$ in $\phi(P)$.*

Proof. Omitted.

□

Furthermore, it is not difficult to see that the f-vectors of P and $\phi(P)$ are equal.

Proposition 4.2. *Let P be a $(\mathbf{3}+\mathbf{1})$-free poset whose labeling satisfies (4.4). Then the f-vector of $\phi(P)$ is equal to that of P.*

Proof. Omitted.

□

In addition to preserving the f-vector of a poset, the map ϕ also preserves $\mathbf{3}+\mathbf{1}$ avoidance.

Proposition 4.3. *Let P be a labeled $(\mathbf{3}+\mathbf{1})$-free poset which satisfies (4.4). Then $\phi(P)$ is $(\mathbf{3}+\mathbf{1})$-free.*

Proof. Omitted.

□

Thus by applying several iterations of the map ϕ to a $(\mathbf{3}+\mathbf{1})$-free poset P whose labeling respects altitude, we obtain the poset Q_1 from Corollary 3.2, and find that the f-vector of Q_1 is equal to that of P.

Theorem 4.4. *Let P be a $(\mathbf{3}+\mathbf{1})$-free poset and let A be the antiadjacency matrix corresponding to an altitude-respecting labeling of P. Define the matrices C and D as in Theorem 3.1, and let Q_1 and Q_2 be the two labeled unit interval orders whose antiadjacency matrices are C and D. Then the f-vectors of all three posets are equal.*

Proof. If P is a unit interval order, then $P = Q_1 = Q_2$ and we are done. Suppose therefore that P is not a unit interval order. Then for some number k we have $\phi^k(P) = Q_1$, which by Proposition 4.2 implies that the f-vectors of P and Q_1 are equal. Applying the same argument to the dual poset of P, we find that the f-vectors of P and Q_2 are equal as well. □

Thus although the set of $(\mathbf{3}+\mathbf{1})$-free posets on n elements strictly contains the set of unit interval orders on n elements (for $n > 3$), the corresponding containment of sets of f-vectors is not strict.

Corollary 4.5. *The set of f-vectors of $(\mathbf{3}+\mathbf{1})$-free posets on n elements is equal to the set of f-vectors of unit interval orders on n elements.*

As is the case with many interesting classes of f-vectors, no characterization of the f-vectors of $(\mathbf{3}+\mathbf{1})$-free posets is known. On the other hand, it is not too difficult to prove some inequalities that must be satisfied by the components of these f-vectors. Somewhat surprisingly, the inequalities below are satisfied also by pure O-sequences [8], by the h-vectors of matroid complexes [2], and by the coefficients of the Poincaré polynomials of universal Coxeter groups [27]. (See also [3], [9], [13], [18, Cor. 2.4].)

Proposition 4.6. *Let $a(t) = a_0 + a_1 t + \cdots + a_m t^m$ be the f-polynomial of a $(\mathbf{3}+\mathbf{1})$-free poset. Then for $i = 0, \ldots, \lfloor \frac{m-1}{2} \rfloor$ we have*

$$a_i \le a_{i+1}, \tag{4.5}$$

$$a_i \le a_{m-i}. \tag{4.6}$$

Proof. Omitted.

□

5. Open problems

A more thorough understanding of the f-vectors of $(\mathbf{3}+\mathbf{1})$-free posets (equivalently, of unit interval orders) would be interesting because this might help to prove conjectures that certain combinatorially defined polynomials have only real zeros. (See for example [12, p. 114] or [24, Prob. 20].)

Problem 5.1. Characterize the f-vectors of unit interval orders.

On the other hand, a better understanding of the factorization in Theorem 3.1 might help to obtain results for $(\mathbf{3}+\mathbf{1})$-free posets analogous to those already known for unit interval orders. For instance the number of nonisomorphic unit interval orders on n elements is well-known to be the nth Catalan number [4], [28], but no such formula is known for $(\mathbf{3}+\mathbf{1})$-free posets.

Problem 5.2. Find a formula for the number of nonisomorphic $(\mathbf{3}+\mathbf{1})$-free posets on n elements.

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