LATTICE PATH MATROIDS: ENUMERATIVE ASPECTS AND TUTTE POLYNOMIALS (EXTENDED ABSTRACT)

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Abstract. Fix two lattice paths P and Q from $(0,0)$ to (m,r) that use East and North steps with P never going above Q. We show that the lattice paths that go from $(0,0)$ to (m,r) and that remain in the region bounded by P and Q can be identified with the bases of a particular type of transversal matroid, which we call a lattice path matroid. We consider a variety of enumerative aspects of these matroids and we study three important matroid invariants, namely the Tutte polynomial and, for special types of lattice path matroids, the characteristic polynomial and the β invariant.

Résumé. Soient P et Q deux chemins allant de $(0,0)$ à (m,r) formés par de pas Nord et Est, et tels que P soit toujours en dessous de Q. On montre que les chemins dans la grille qui vont de $(0,0)$ à (m,r) et qui restent dans la région délimitée par P et Q peuvent être identifiés avec les bases d'un certain matroïde transversal, qu'on appelle matroïde de chemins. On considère de nombreux aspects énumératifs de ces matroïdes et on étudie trois invariants importants: le polynôme de Tutte et, pour des classes spéciales de matroïdes de chemins, le polynôme caractéristique et l'invariant β.

1. Introduction

This paper develops a new connection between matroid theory and enumerative combinatorics: with every pair of lattice paths P and Q that have common endpoints we associate a matroid in such a way that the bases of the matroid correspond to the paths that remain in the region bounded by P and Q. These matroids, which we call lattice path matroids, appear to have a wealth of interesting and striking properties. In this paper we focus on the enumerative aspects of lattice path matroids, including the study of important matroid invariants like the Tutte and the characteristic polynomials. Structural aspects of lattice path matroids and their relation with other families of matroids will be the subject of a forthcoming paper [3].

Lattice path matroids provide a bridge between matroid theory and the theory of lattice paths that, as we demonstrate here and in [3], can lead to a mutually enriching relationship between the two subjects. For example, as we show in Section 8, the path interpretation we give for each coefficient of the Tutte polynomial of a lattice path matroid, along with easily computed examples of the Tutte polynomial, can suggest new theorems about lattice paths.

Relatively little matroid theory is required to understand this paper and what is needed is sketched in the first part of the next section.

Due to lack of space, all proofs and several results that appear in the full version of this paper [4] have been omitted. In particular, in the full paper we also show a connection with a problem of current interest in enumerative combinatorics, namely the tennis ball problem [9].

2. Background

In this section we introduce the concepts of matroid theory that are needed in this paper. For a thorough introduction to the subject we refer the reader to Oxley [11]. We conclude this section with the necessary background on the enumerative theory of lattice paths.

Definition 2.1. *A* matroid *is a pair* $\big(E(M), \mathcal{B}(M)\big)$ *consisting of a finite set* $E(M)$ *and a collection* $\mathcal{B}(M)$ *of subsets of* $E(M)$ *that satisfy the following conditions:*

- (B1) $\mathcal{B}(M) \neq \emptyset$,
- (B2) $\mathcal{B}(M)$ *is an antichain, that is, no set in* $\mathcal{B}(M)$ *properly contains another set in* $\mathcal{B}(M)$, *and*
- (B3) *for each pair of distinct sets* B, B' *in* $\mathcal{B}(M)$ *and for each element* $x \in B - B'$, *there is an element* $y \in B' - B$ *such that* $(B - x) \cup y$ *is in* $\mathcal{B}(M)$.

The set $E(M)$ is the *ground set* of M and the sets in $\mathcal{B}(M)$ are the *bases* of M. Subsets of bases are *independent sets*. Sets that are not independent are *dependent*. A *circuit* is a minimal dependent set. If $\{x\}$ is a circuit, then x is a *loop*. Thus, no basis of M can contain a loop. An element that is contained in every basis is an *isthmus*.

It is easy to show that all bases of M have the same cardinality. More generally, for any subset A of $E(M)$ all maximal independent subsets of A have the same cardinality; $r(A)$, the *rank* of A, denotes this common cardinality. In place of $r(E(M))$, we write $r(M)$.

The *closure* of a set $A \subseteq E(M)$ is defined as

$$\mathrm{cl}(A) = \{x \in E(M) : r(A \cup x) = r(A)\}.$$

A set F is a *flat* if $\mathrm{cl}(F) = F$. The flats of a matroid, ordered by inclusion, form a geometric lattice.

It is well-known that matroids can be characterized in terms of each of the following objects: the independent sets, the dependent sets, the circuits, the rank function, the closure operator, and the flats (see Sections 1.1–1.4 of [11]).

This paper investigates a special class of transversal matroids. Let $\mathcal{A} = (A_j : j \in J)$ be a set system, that is, a multiset of subsets of a finite set S. A *transversal* (or system of distinct representatives) of $\mathcal{A}$ is a set $\{x_j : j \in J\}$ of $|J|$ distinct elements such that $x_j \in A_j$ for all j in J. A *partial transversal* of $\mathcal{A}$ is a transversal of a set system of the form $(A_k : k \in K)$ with K a subset of J. The following theorem is a fundamental result due to Edmonds and Fulkerson.

Theorem 2.2. *The partial transversals of a set system* $\mathcal{A} = (A_j : j \in J)$ *are the independent sets of a matroid on* S.

A *transversal matroid* is a matroid whose independent sets are the partial transversals of some set system $\mathcal{A} = (A_j : j \in J)$; we say that $\mathcal{A}$ is a *presentation* of the transversal matroid. The bases of a transversal matroids are the maximal partial transversals of $\mathcal{A}$. For more on transversal matroids see [11, Section 1.6].

The particular matroids of interest in this paper arise from lattice paths, to which we now turn. We consider two kinds of lattice paths, both of which are in the plane. Most of the lattice paths we consider use steps $E = (1, 0)$ and $N = (0, 1)$;

in several cases it is more convenient to use lattice paths with steps $U = (1,1)$ and $D = (1,-1)$. The letters are abbreviations of East, North, Up, and Down. We will often treat lattice paths as words in the alphabets $\{E, N\}$ or $\{U, D\}$, and we will use the notation α^n to denote the concatenation of n letters, or strings of letters, α. If $P = s_1 s_2 \dots s_n$ is a lattice path, then its *reversal* is defined as $P^\rho = s_n s_{n-1} \dots s_1$. The *length* of a lattice path $P = s_1 s_2 \dots s_n$ is n, the number of steps in P.

Here we recall the facts we need about the enumeration of lattice paths; the proofs of the following lemmas can be found in Sections 3 to 5 of the first chapter of [10]. The most basic enumerative results about lattice paths are those in the following lemma.

Lemma 2.3. *For a fixed positive integer k, the number of lattice paths from $(0,0)$ to (kn, n) that use steps E and N and that never pass above the line $y = x/k$ is the n-th k-Catalan number*

$$C_n^k = \frac{1}{kn+1}\binom{(k+1)n}{n}.$$

In particular, the number of paths from $(0,0)$ to (n,n) that never pass above the line $y = x$ is the n-th Catalan number

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

The next lemma treats paths in the alphabet $\{U, D\}$; the first assertion, which concerns what are usually called Dyck paths, is equivalent to the second part of Lemma 2.3 by the obvious identification of the alphabets.

Lemma 2.4. *(i) The number of paths from $(0,0)$ to $(2n,0)$ with steps U and D that never pass below the x-axis is the n-th Catalan number C_n.*
(ii) The number of paths of n steps in the alphabet $\{U, D\}$ that start at $(0,0)$ and never pass below the x-axis (not necessarily ending on the x-axis) is $\binom{n}{\lceil n/2 \rceil}$.

3. Lattice path matroids

In this section we define lattice path matroids as well as several important subclasses. Later sections of this paper develop much of the enumerative theory for lattice path matroids in general and this theory is pushed much further for certain special families of lattice path matroids.

Definition 3.1. *Let $P = p_1 p_2 \dots p_{m+r}$ and $Q = q_1 q_2 \dots q_{m+r}$ be two lattice paths from $(0,0)$ to (m,r) with P never going above Q. Let $\{p_{u_1}, \dots, p_{u_r}\}$ be the set of North steps of P with $u_1 < u_2 < \dots < u_r$; similarly, let $\{q_{l_1}, \dots, q_{l_r}\}$ be the set of North steps of Q with $l_1 < l_2 < \dots < l_r$. Let N_i be the interval $[l_i, u_i]$ of integers. Let $M[P,Q]$ be the transversal matroid that has ground set $[m+r]$ and presentation $(N_i : i \in [r])$; the pair (P,Q) will be called a* presentation *of $M[P,Q]$. A* lattice path matroid *is a matroid M that is isomorphic to $M[P,Q]$ for some such pair of lattice paths P and Q.*

Several examples of lattice path matroids are given after Theorem 3.3, which identifies the bases of these matroids in terms of lattice paths. To avoid needless repetition, throughout the rest of the paper we assume that the lattice paths P and Q are as in Definition 3.1.

We think of $1, 2, \dots, m+r$ as the first step, the second step, etc. Observe that the set N_i contains the steps that can be the i-th North step in a lattice path from

$(0,0)$ to (m,r) that remains in the region bounded by P and Q. When thought of as arising from the particular presentation using bounding paths P and Q, the elements of the matroid are ordered in their natural order, i.e., $1 < 2 < \cdots < m+r$; we will frequently use this order throughout the paper. However, this order is not inherent in the matroid structure; the elements of a lattice path matroid typically can be linearly ordered in many ways so as to correspond to steps in lattice paths.

We associate a lattice path $P(X)$ with each subset X of the ground set of a lattice path matroid as specified in the next definition.

Definition 3.2. *Let X be a subset of the ground set $[m+r]$ of the lattice path matroid $M[P,Q]$. The lattice path $P(X)$ is the word*

$$s_1 s_2 \ldots s_{m+r}$$

in the alphabet $\{E, N\}$ where

$$s_i = \begin{cases} N, & \text{if } i \in X; \\ E, & \text{otherwise.} \end{cases}$$

Thus, the path $P(X)$ is formed by taking the elements of the ground set of $M[P,Q]$ in the natural linear order and replacing each by a North step if the element is in X and by an East step if the element is not in X.

The fundamental connection between the transversal matroid $M[P,Q]$ and the lattice paths that stay in the region bounded by P and Q is the following theorem which says that the bases of $M[P,Q]$ can be identified with such lattice paths.

Theorem 3.3. *A subset B of $[m+r]$ with $|B| = r$ is a basis of $M[P,Q]$ if and only if the associated lattice path $P(B)$ stays in the region bounded by P and Q.*

Corollary 3.4. *The number of basis of $M[P,Q]$ is the number of lattice paths from $(0,0)$ to (m,r) that go neither below P nor above Q.*

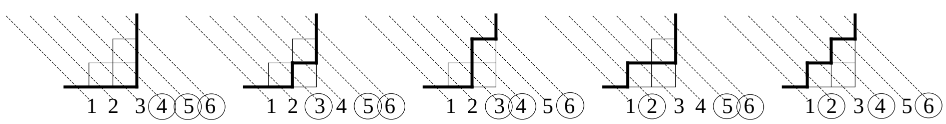

FIGURE 1. The bases $\{4,5,6\}$, $\{3,5,6\}$, $\{3,4,6\}$, $\{2,5,6\}$, and $\{2,4,6\}$ of a lattice path matroid represented as the North steps of lattice paths.

Figure 1 illustrates Theorem 3.3. In this example we have $N_1 = \{2,3,4\}$, $N_2 = \{4,5\}$, and $N_3 = \{6\}$. There are five bases of this transversal matroid. Note that 1 is a loop and 6 is an isthmus.

We now turn to a special class of lattice path matroids, the generalized Catalan matroids, as well as to various subclasses that exhibit a structure that is simpler than that of typical lattice path matroids. Later sections of this paper will give special attention to these classes since the simpler structure allows us to obtain more detailed enumerative results.

Definition 3.5. *A lattice path matroid M is a* generalized Catalan matroid *if there is a presentation (P,Q) of M with $P = E^m N^r$. In this case we simplify the notation $M[P,Q]$ to $M[Q]$. If in addition the upper path Q is $(E^k N^l)^n$ for some positive integers k, l, and n, we say that M is the (k,l)*-Catalan matroid *$M_n^{k,l}$. In place of*

$M_n^{k,1}$ *we write* M_n^k*; such matroids are called* k-Catalan matroids. *In turn, we simplify the notation* M_n^1 *to* M_n*; such matroids are called* Catalan matroids.

Figure 2 gives presentations of a $(2,3)$-Catalan matroid, a 3-Catalan matroid, and a Catalan matroid. These matroids have, respectively, two loops and three isthmuses, three loops and one isthmus, and a single loop and isthmus.

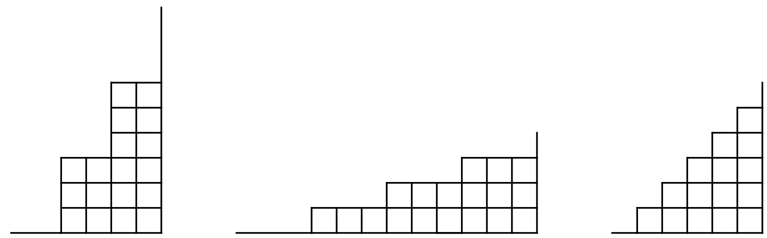

FIGURE 2. Presentations of the rank nine matroid $M_3^{2,3}$, the 3-Catalan matroid M_4^3 of rank four, and the rank six Catalan matroid M_6.

Note that (k,l)-Catalan matroids have isthmuses and loops; specifically, the elements $1,\dots,k$ are the loops and $(k+l)n-l+1,(k+l)n-l+2,\dots,(k+l)n$ are the isthmuses of $M_n^{k,l}$. Also, observe that for the k-Catalan matroid M_n^k, Theorem 3.3 can be restated by saying that an n-element subset B of $[(k+1)n]$ is a basis of M_n^k if and only if its associated lattice path $P(B)$ does not go above the line $y=x/k$.

We next note an immediate consequence of Corollary 3.4 and Lemma 2.3.

Corollary 3.6. *The number of bases of the k-Catalan matroid* M_n^k *is the k-Catalan number* C_n^k*. In particular, the number of bases of the Catalan matroid* M_n *is the Catalan number* C_n*.*

Generalized Catalan matroids have previously appeared in the matroid theory literature under different names and points of view. Welsh [17] introduced them to give a lower bound on the number of matroids and later Oxley, Prendergast, and Row [12] characterized them in several ways. They were recently rediscovered in yet another context in [1], where they are related to a special type of simplicial complex. It can be shown that generalized Catalan matroids are exactly the minors of Catalan matroids [3].

4. Enumeration of lattice path matroids

In this section we give a formula for the number of connected lattice matroids on a given number of elements up to isomorphism; to make the final result slightly more compact, we let the number of elements be $n+1$. Recall that a matroid is called *connected* if every pair of elements are contained in a circuit. A lattice path matroid $M[P,Q]$ is connected if and only if the paths P and Q only intersect at the endpoints. The proof has two main ingredients, the first of which is the following result from [3]. (Recall that P^ρ denotes the reversal $s_{n+1}s_n\dots s_1$ of a lattice path $P=s_1s_2\dots s_ns_{n+1}$.)

Lemma 4.1. *Two connected lattice path matroids* $M[P,Q]$ *and* $M[P',Q']$ *are isomorphic if and only if either* $P'=P$ *and* $Q'=Q$*, or* $P'=Q^\rho$ *and* $Q'=P^\rho$*.*

The second main ingredient is the following bijection, going back at least to Pólya, between the pairs of lattice paths of length $n+1$ that intersect only at their endpoints and the Dyck paths of length $2n$. (See, for example, [8].) A pair (P,Q) of nonintersecting lattice paths from $(0,0)$ to (m,r) can be viewed as the special type of polyomino that in [8] is called a parallelogram polyomino. Associate two sequences $(a_1,\ldots,a_m)$ and $(b_1,\ldots,b_{m-1})$ of integers with such a polyomino: a_i is the number of cells of the i-th column of the polyomino (columns are scanned from left to right) and b_i+1 is the number of cells of column i that are adjacent to cells of column $i+1$. Since the paths are nonintersecting, each b_i is nonnegative. Now associate to (P,Q) the Dyck path π having m peaks at heights $a_1,\ldots,a_m$ and $m-1$ valleys at heights $b_1,\ldots,b_{m-1}$. Figure 3 shows a polyomino and its associated Dyck path; the corresponding sequences for this polyomino are $(1,2,4,2,2)$ and $(0,1,1,0)$. It can be checked that the correspondence $(P,Q)\mapsto\pi$ is indeed a bijection. Hence the number of such pairs (P,Q) of lattice paths of length $n+1$ is the Catalan number C_n.

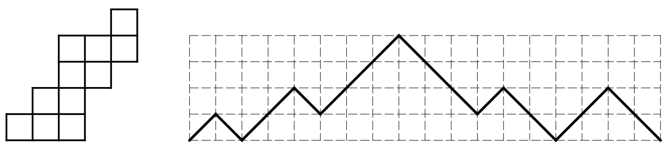

FIGURE 3. A parallelogram polyomino and its associated Dyck path.

Note that C_n is not the number of connected lattice path matroids on $n+1$ elements since different pairs of paths can give the same matroid. According to Lemma 4.1, this happens only for a pair (P,Q) and its reversal (Q^ρ,P^ρ), so we need to find the number of pairs (P,Q) for which $(P,Q)=(Q^\rho,P^\rho)$. It is immediate to check that $(P,Q)=(Q^\rho,P^\rho)$ if and only if the corresponding Dyck path π is symmetric with respect to its center or, in other words, is equal to its reversal. Since a symmetric Dyck path of length $2n$ is determined by its first n steps, the number of such paths is given in part (ii) of Lemma 2.4. From the number C_n we obtained in the last paragraph we must subtract half the number of nonsymmetric Dyck paths, thus giving the following result.

Theorem 4.2. *The number of connected lattice path matroids on $n+1$ elements up to isomorphism is*

$$C_n-\frac{1}{2}\left(C_n-\binom{n}{\lceil n/2\rceil}\right)=\frac{1}{2}C_n+\frac{1}{2}\binom{n}{\lceil n/2\rceil}.$$

This number is asymptotically of order $O(4^n)$. Since it is known that the number of transversal matroids on n elements grows like c^{n^2} for some constant c (see [5]), it follows that the class of lattice path matroids is rather small with respect to the class of all transversal matroids.

From the characterization of connected lattice path matroids at the beginning of this section, it follows that the total number of lattice path matroids (connected or not) on k elements is the number of multisets of connected lattice path matroids, the sum of whose cardinalities is k. A generating function for these numbers can be derived using standard tools; however, the result does not seem to admit a compact form so we omit it.

5. TUTTE POLYNOMIALS

The Tutte polynomial is one of the most widely studied matroid invariants. From the Tutte polynomial one obtains, as special evaluations, many other important polynomials, such as the chromatic and flow polynomials of a graph, the weight enumerator of a linear code, and the Jones polynomial of an alternating knot. (See [6, 18] for many of the numerous occurrences of this polynomial in combinatorics, in other branches of mathematics, and in other sciences.) In this section, after reviewing the definition of the Tutte polynomial, we show that for lattice path matroids this polynomial is the generating function for two basic lattice path statistics. We use this lattice path interpretation of the Tutte polynomial to give a formula for the generating function $\sum_{n\geq 0} t(M_n^k; x, y)z^n$ for the sequence of Tutte polynomials $t(M_n^k; x, y)$ of the k-Catalan matroids. Using this generating function, we then derive a formula for each coefficient of the Tutte polynomial $t(M_n^k; x, y)$.

Fix a linear order $<$ on $E(M)$ and let B be a basis of M. An element $e \notin B$ is *externally active with respect to* B if there is no element y in B with $y < e$ for which $(B - y) \cup e$ is a basis. An element $b \in B$ is *internally active with respect to* B if there is no element y in $E(M) - B$ with $y < b$ for which $(B - b) \cup y$ is a basis. The *internal (external) activity* of a basis is the number of elements that are internally (externally) active with respect to that basis. We denote the activities of a basis B by $i(B)$ and $e(B)$. Note that $i(B)$ and $e(B)$ depend not only on B but also on the order $<$.

The Tutte polynomial can be expressed as follows:

$$t(M; x, y) = \sum_{B \in \mathcal{B}(M)} x^{i(B)} y^{e(B)}. \tag{1}$$

In particular, although $i(B)$ and $e(B)$, for a particular basis B, depend on the order $<$, the multiset of pairs $\big(i(B), e(B)\big)$, as B ranges over the bases of M, does not depend on the order.

Recall that if the bounding lattice paths P and Q go from $(0, 0)$ to (m, r), then the lattice path matroid $M[P, Q]$ has ground set $[m+r]$; the elements in $[m+r]$ represent the first step, the second step, and so on. We use the natural linear order on $[m+r]$, that is, $1 < 2 < \cdots < m + r$. The crux of understanding the Tutte polynomial of a lattice path matroid is the following theorem, which describes internal and external activities of bases in terms of the associated lattice paths.

Theorem 5.1. *Let B be a basis of the lattice path matroid $M[P, Q]$ and let $P(B)$ be the lattice path associated with B. Then $i(B)$ is the number of times $P(B)$ meets the upper path Q in a North step and $e(B)$ is the number of times $P(B)$ meets the lower path P in an East step.*

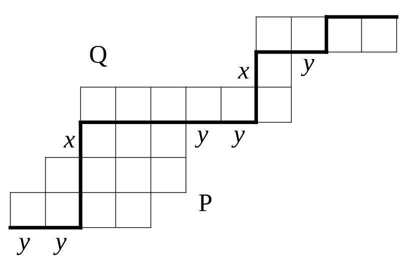

FIGURE 4. The lattice path corresponding to a basis with internal activity 2 and external activity 5, which contributes x^2y^5 to the Tutte polynomial.

Theorem 5.1 is illustrated in Figure 4. It is worth noting the following simpler formulation in the case of k-Catalan matroids.

Corollary 5.2. *Let B be a basis of a k-Catalan matroid and let $P(B)$ be the associated lattice path. Then $i(B)$ is the number of times $P(B)$ returns to the line $y = x/k$ and $e(B)$ is j where $(j, 0)$ is the last point on the x-axis in $P(B)$.*

This lattice path interpretation of basis activities is one of the keys for obtaining the following generating function for the sequence of Tutte polynomials of the k-Catalan matroids.

Theorem 5.3. *Let*

$$C = C(z) = \sum_{n \geq 0} \frac{1}{kn+1} \binom{(k+1)n}{n} z^n$$

be the generating function for the k-Catalan numbers. The generating function for the Tutte polynomials of the k-Catalan matroids is

$$\sum_{n \geq 0} t(M_n^k; x, y) z^n = 1 + \left(\frac{xzy^k}{1 - z \sum_{l=1}^{k} y^l C^{k-l+1}} \right) \frac{1}{1 - xzC^k}. \tag{2}$$

By extracting the coefficients of the expression in Theorem 5.3 we find a formula for the coefficients of the Tutte polynomial of a k-Catalan matroid. To write this formula more compactly, let

$$S(m, s, k) = \sum_{i=0}^{s} (-1)^i \binom{s}{i} \binom{m - ki - 1}{s - 1}.$$

Theorem 5.4. *The coefficient of $x^i y^j$ in the Tutte polynomial $t(M_n^k; x, y)$ of the k-Catalan matroid M_n^k is*

$$\sum_{s=0}^{m} S(m, s, k) \binom{(k+1)(n-1) - i - m}{n - s - i - 1} \frac{s(k+1) - m + k(i-1)}{n - s - i},$$

where $m = j - k$. Equivalently, this is the number of lattice paths that

(i) *go from $(0, 0)$ to (kn, n),*
(ii) *use steps $(1, 0)$ and $(0, 1)$,*
(iii) *do not go above the line $y = x/k$,*
(iv) *have as their last point on the x-axis the point $(j, 0)$, and*
(v) *return to the line $y = x/k$ exactly i times.*

It is an open problem to obtain explicit expressions for the Tutte polynomials of the matroids $M_n^{k,l}$ for values of k and l not covered by the previous theorem, namely $k > 1$ and $l > 1$. The first unsolved case is $k = l = 2$. The sequence $1, 6, 53, 554, 6362, 77580, \ldots$ that gives the number of bases of $M_n^{2,2}$ also arises in the enumeration of certain types of planar trees, and in that context Lou Shapiro gave a nice expression for the corresponding generating function (see entry A066357 in [14]). This sequence also appears in [9].

6. Computing the Tutte polynomial of lattice path matroids

There is no known polynomial-time algorithm for computing the Tutte polynomial of an arbitrary matroid, or even its evaluations at certain points in the plane [18]. There are many evaluations of the Tutte polynomial that are particularly significant;

for instance, it follows from equation (1) that $t(M;1,1)$ is the number of bases of M. Since the bases of a lattice path matroid correspond to paths that stay in a given region and the number of such paths is given by a determinant (see Theorem 1 in Section 2.2 of [10]), the number of bases of a lattice path matroid can be computed in polynomial time. It turns out that other evaluations like $t(M;1,0)$ and $t(M;0,1)$ can also be expressed as determinants. This led us to suspect that the Tutte polynomial of a lattice path matroid could be computed in polynomial time. In this section, we show that this is indeed the case: we give such a polynomial-time algorithm. The results in this section stand in striking contrast to those in [7], where it is shown that for fixed x and y with $(x-1)(y-1) \neq 1$, the problem of computing $t(M;x,y)$ for a transversal matroid M is #P-complete.

By Theorem 5.1, for a lattice path matroid $M = M[P,Q]$, the Tutte polynomial $t(M;x,y)$ is the generating function

$$\sum_{B \in \mathcal{B}(M)} x^{i(B)} y^{e(B)}$$

where $i(B)$ is the number of North steps that the lattice path $P(B)$ corresponding to B shares with the upper bounding path Q and $e(B)$ is the number of East steps that $P(B)$ shares with the lower bounding path P. Any lattice path can be viewed as a sequence of shorter lattice paths. This perspective gives the following algorithm for computing the Tutte polynomial of the lattice path matroid $M = M[P,Q]$ where P and Q go from $(0,0)$ to (m,r). With each lattice point (i,j) in the region R bounded by P and Q, associate the polynomial

$$f(i,j) = \sum_{P'} x^{i(P')} y^{e(P')}$$

where the sum ranges over the lattice paths P' that go from $(0,0)$ to (i,j) and stay in the region R, and where, as for $t(M;x,y)$, the exponent $i(P')$ is the number of North steps that P' shares with Q and $e(P')$ is the number of East steps that P' shares with P. In particular, $f(m,r) = t(M;x,y)$. Note that for a point (i,j) in R other than $(0,0)$, at least one of $(i-1,j)$ or $(i,j-1)$ is in R; furthermore, only $(i-1,j)$ is in R if and only if the step from $(i-1,j)$ to (i,j) is an East step of P, and, similarly, only $(i,j-1)$ is in R if and only if the step from $(i,j-1)$ to (i,j) is a North step of Q. The following rules for computing $f(i,j)$ are evident from these observations and the definition of $f(i,j)$.

(a) $f(0,0) = 1$.
(b) If the lattice points (i,j), $(i-1,j)$ and $(i,j-1)$ are all in the region R, then $f(i,j) = f(i-1,j) + f(i,j-1)$.
(c) If the lattice points (i,j) and $(i-1,j)$ are in R but $(i,j-1)$ is not in R, then $f(i,j) = y\,f(i-1,j)$.
(d) If the lattice points (i,j) and $(i,j-1)$ are in R but $(i-1,j)$ is not in R, then $f(i,j) = x\,f(i,j-1)$.

This algorithm is illustrated in Figure 5 where we apply it to compute the Tutte polynomial of an n-element circuit.

If x and y are set to 1, the algorithm above reduces to a well-known technique for counting lattice paths. This is consistent with the general theory of Tutte polynomials; as noted above, $t(M;1,1)$ is the number of bases of M.

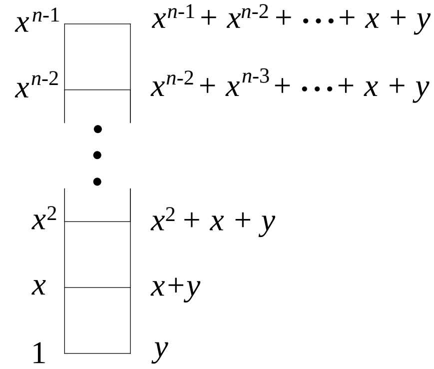

FIGURE 5. The recursive computation of the Tutte polynomial of an n-element circuit via lattice path statistics.

The recurrence above requires at most $(r+1)(m+1)$ steps to compute the Tutte polynomial of a lattice path matroid whose bounding paths go from $(0,0)$ to (m,r). Thus, we have the following corollary.

Corollary 6.1. *The Tutte polynomial of a lattice path matroid can be computed in polynomial time.*

7. The broken circuit complex and the characteristic polynomial

In this section we study two related objects for lattice path matroids, the broken circuit complex and the characteristic polynomial. The second of these is an invariant of the matroid but the first depends on a linear ordering of the elements. We show that under the natural ordering of the elements, the broken circuit complex of any loopless lattice path matroid has a property that is not shared by the broken circuit complexes of arbitrary matroids, namely, the broken circuit complex of a lattice path matroid is the independence complex of another matroid, indeed, of a lattice path matroid. Our study of the characteristic polynomial is more specialized; we focus on the characteristic polynomial $\chi(\widehat{M_n^k};\lambda)$ of the matroid $\widehat{M_n^k}$ obtained from the k-Catalan matroid M_n^k by omitting the loops. Our results on the broken circuit complex lead to a lattice path interpretation of each coefficient of $\chi(\widehat{M_n^k};\lambda)$ from which we obtain a formula for these coefficients. We start by outlining the necessary background on broken circuit complexes; for an extensive account, see [2].

Given a matroid M and a linear order $<$ on the ground set $E(M)$, a *broken circuit* of the resulting ordered matroid is a set of the form $C - x$ where C is a circuit of M and x is the least element of C relative to the linear ordering. A subset of $E(M)$ is an *nbc-set* if it contains no broken circuit. Clearly subsets of nbc-sets are nbc-sets. Thus, $E(M)$ and the collection of nbc-sets of M form a simplicial complex, the *broken circuit complex of M relative to* $<$, which is denoted $BC_<(M)$. Different orderings of $E(M)$ can produce nonisomorphic broken circuit complexes (see, e.g., [2, Example 7.4.4]). The facets of $BC_<(M)$ are the *nbc-basis*, that is, the basis of M that are nbc-sets. The following characterization of nbc-bases is well-known and easy to prove.

Lemma 7.1. *The nbc-basis of M are the bases of M of external activity zero.*

Note that nbc-sets contain no circuits and so are independent. Thus, the broken circuit complex $BC_<(M)$ of M is contained in the *independence complex of* M, that is, the complex with ground set $E(M)$ in which the faces are the independent sets of M. Note also that if M has loops, then the empty set is a broken circuit, so M has no nbc-sets. Thus, throughout this section we consider only matroids with no loops.

As in Section 5, we use the natural ordering on the points of lattice path matroids. The examples in [2] show that the broken circuit complex need not be the independence complex of another matroid. In contrast, Theorem 7.2 shows that the broken circuit complex of a lattice path matroid without loops is the independence complex of another lattice path matroid.

Theorem 7.2. *With the natural order, the broken circuit complex of a lattice path matroid $M[P,Q]$ with no loops is the independence complex of the lattice path matroid $M[P',Q]$ where $NP = P'N$.*

We now turn to the characteristic polynomial, which plays an important role in many enumeration problems in matroid theory (see [13, 19]) and which can be defined in a variety of ways. As mentioned above, the isomorphism type of the broken circuit complex of a matroid M can depend on the ordering of the points. However, it can be shown that the number of nbc-sets of each size is an invariant of the matroid; these numbers are the coefficients of the characteristic polynomial. Specifically, the *characteristic polynomial* $\chi(M;\lambda)$ of a matroid M is

$$\chi(M;\lambda) = \sum_{i=0}^{r(M)} (-1)^i \, \mathbf{nbc}(M;i) \, \lambda^{r(M)-i}, \tag{3}$$

where $\mathbf{nbc}(M;i)$ is the number of nbc-sets of size i. Thus, $(-1)^{r(M)}\chi(M;-\lambda)$ is the face enumerator of the broken circuit complex of M. Alternatively, $\chi(M;\lambda)$ can be expressed in terms of the Tutte polynomial as $\chi(M;\lambda) = (-1)^{r(M)} t(M;1-\lambda,0)$. The characteristic polynomial can also be expressed in the following way in terms of the Möbius function of the lattice of flats:

$$\chi(M;\lambda) = \sum_{\substack{\text{flats } F \\ \text{of } M}} \mu(\emptyset,F)\lambda^{r(M)-r(F)}.$$

(See, e.g., [2, Theorem 7.4.6], for details.) In particular, the absolute value of the constant term of $\chi(M;\lambda)$ is both the number of nbc-bases of M and the absolute value of the Möbius function $\mu(M)$. This and Theorem 7.2 give the following corollary.

Corollary 7.3. *The absolute value of the Möbius function $\mu(M[P,Q])$ of a loopless lattice path matroid is the number of bases of the lattice path matroid $M[P',Q]$ where $NP = P'N$, or, equivalently, of the lattice path matroid $M[P^*,Q^*]$ where $P = P^*N$ and $Q = NQ^*$.*

Our interest is in the characteristic polynomial of a specific type of lattice path matroid. Recall that the elements $1,2,\ldots,k$ are loops of the k-Catalan matroid M_n^k. Thus, the characteristic polynomial of M_n^k is zero. This motivates considering the *loopless Catalan matroid* $\widehat{M_n}$, which we define to be $M[(NE)^{n-1}N]$, and more generally the *loopless k-Catalan matroid* $\widehat{M_n^k}$, which we define to be $M[(NE^k)^{n-1}N]$. Thus, these matroids are formed from almost the same bounding paths as those for the Catalan and k-Catalan matroids except that the initial East steps that give loops have been omitted.

We start with the following consequence of Corollary 7.3.

Corollary 7.4. *The number of nbc-bases of $\widehat{M_n^k}$, that is, $|\mu(\widehat{M_n^k})|$, is the k-Catalan number C_{n-1}^k. In particular, $|\mu(\widehat{M_n})| = C_{n-1}$.*

By using a simple characterization of nbc-sets in terms of lattice paths we obtain the following expression for each coefficient of the characteristic polynomial.

Theorem 7.5. *The absolute value of the coefficient of λ^{n-i} in the characteristic polynomial of the loopless k-Catalan matroid $\widehat{M_n^k}$ is given by the formula*

$$\mathbf{nbc}(\widehat{M_n^k}; i) = \begin{cases} 1, & \text{if } i = 0; \\ \dfrac{(k+1)(n-i-1)+2}{(k+1)(n-1)+2}\dbinom{(k+1)(n-1)+2}{i}, & \text{if } 1 \leq i \leq n-1; \\ C_{n-1}^k, & \text{if } i = n. \end{cases}$$

From the formula in Theorem 7.5 and appropriate manipulation, we see that the linear term in the characteristic polynomial of $\widehat{M_n}$ is also a Catalan number. Note that, however, for the loopless k-Catalan matroid the linear term of the characteristic polynomial is not the corresponding k-Catalan number.

Corollary 7.6. *The linear term in $\chi(\widehat{M_n}, \lambda)$ is C_n.*

8. The β invariant

The β invariant $\beta(M)$ of a matroid M, which was introduced by Crapo, can be defined in several ways; see [19, Section 3] for a variety of perspectives on the β invariant, as well as its applications to connectivity and series-parallel networks. We use the following definition. It can be shown that for any matroid M, the coefficients of x and y in the Tutte polynomial $t(M; x, y)$ are the same; this coefficient is $\beta(M)$. Since loops are externally active with respect to every basis, no basis of a matroid M with loops will have external activity zero, so $\beta(M)$ is zero; analogously, if M has isthmuses, then $\beta(M)$ is zero. Therefore, in this section we focus on matroids with neither loops nor isthmuses.

Let $N_n^{k,k}$ be the generalized Catalan matroid whose upper path is $Q = (N^k E^k)^n$. It is clear from the lattice path presentation that $N_n^{k,k}$ is formed from the (k, k)-Catalan matroid $M_{n+1}^{k,k}$ by deleting the k loops and the k isthmuses. The main result of this section is that $\beta(N_n^{k,k})$ is k times the Catalan number C_{kn-1}. This result was suggested by looking at examples of Tutte polynomials of lattice path matroids, but it can be formulated entirely in terms of lattice paths, which is the perspective we use in the proof. Indeed, the result is most striking when viewed in terms of lattice paths.

The β invariant of $N_n^{k,k}$ is the number of bases with internal activity one and external activity zero; let B be such a basis and let $P(B)$ be its associated lattice path. By Theorem 5.1, the first step of $P(B)$ is N, the second is E, and $P(B)$ does not contain any other North step in Q. It is easy to see that such lattice paths $P(B)$ are in 1-1 correspondence with the paths from $(0, 0)$ to $(kn-1, kn-1)$ that do not go above the path $N^{k-1}(E^k N^k)^{n-1} E^{k-1}$. Recall that the number of paths from $(0, 0)$ to $(kn-1, kn-1)$ that do not go above the line $y = x$ is C_{kn-1}. We can show that the number of paths that do not go above the path $N^{k-1}(E^k N^k)^{n-1} E^{k-1}$ is k times C_{kn-1}. We start with the case $k = 1$.

Theorem 8.1. *The β invariant of $N_n^{1,1}$ is C_{n-1}.*

From here on, we consider only paths that use steps U and D. From the discussion above and the correspondence between the alphabets, we get the following lemma.

Lemma 8.2. *The β invariant of the matroid $N_n^{k,k}$ is the number of paths that*

(i) *go from $(0,0)$ to $\big(2(nk-1),0\big)$,*
(ii) *use steps U and D, and*
(iii) *never go below the path $D^{k-1}(U^kD^k)^{n-1}U^{k-1}$.*

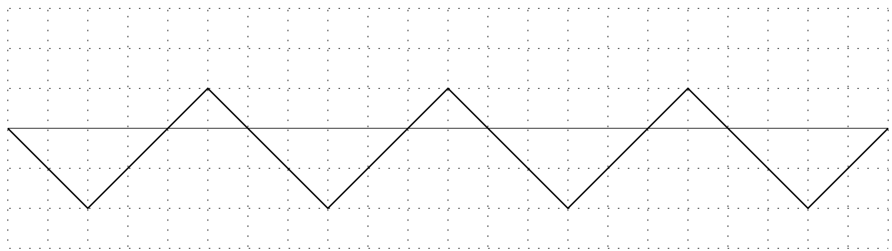

FIGURE 6. The path $D^{k-1}(U^kD^k)^{n-1}U^{k-1}$ for $k=3$ and $n=4$.

A path of the form $D^{k-1}(U^kD^k)^{n-1}U^{k-1}$ is depicted in Figure 6. The next theorem is the main result of this section.

Theorem 8.3. *The number of paths that go from $(0,0)$ to $\big(2(nk-1),0\big)$, use steps U and D, and do not go below the path $D^{k-1}(U^kD^k)^{n-1}U^{k-1}$ is $k\,C_{nk-1}$.*

The proof of the previous theorem is in the spirit of several results generically known as the Cycle Lemma (see the notes at the end of Chapter 5 of [15]). One such result states that among the $2l+1$ possible cyclic permutations of a path from $(0,0)$ to $(2l+1,-1)$, there is exactly one that is a Dyck path. Moreover, the cyclic permutation that leads to a Dyck path is the one that starts after the leftmost minimum of the path (see [16, Theorem 1.1] for more details on this). To prove the theorem, we show that for every Dyck path from $(0,0)$ to $(2kn-1,-1)$, exactly k of its cyclic permutations are paths that do not go below B; conversely, every path that does not go below B can be obtained as one of these k cyclic permutations of a Dyck path.

Let us mention that if we change the bounding path to $(D^kU^k)^n$, the elegance and brevity of the result seem to disappear; currently there is no known comparably simple answer. Indeed, the path $(D^kU^k)^n$ is connected with an open problem in enumeration that is discussed in [9] and in the last section of [4]. The following corollary is an immediate consequence of Lemma 8.2 and Theorem 8.3.

Corollary 8.4. *The β invariant of the matroid $N_n^{k,k}$ is $k\,C_{nk-1}$.*

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